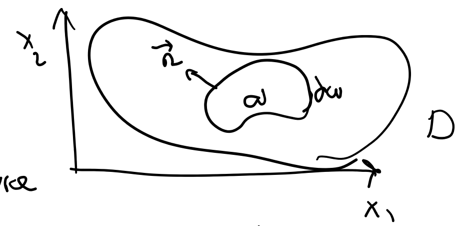
2022/09/08

Thursday, September 8, 2022 9:40 AM

heat conduction problem



Balance of energy

$$\frac{D}{Dt}E_{\omega} = \frac{D}{Dt}\int_{\omega} e \, dV = \int_{\omega} Q \, dV - \int_{\partial\omega} q \cdot n \, ds$$

Q: heat source

e: energy density (= energy per unit volume)

q: heat flux

Ignoring Mechanical effects ($\varepsilon, \sigma, \ldots$)
Electromagnetic effects
⋮

For thermal effects only

$e = C_V T$

T: temperature

C_V: volumetric heat capacity

$q \cdot n > 0$ means that energy goes out

$f_x \cdot n > 0$ mean balance quantity (E here) goes n

$$\boxed{\frac{D}{Dt}\int_{\omega} e \, dV = \int_{\omega} Q \, dV - \int_{\partial\omega} q \cdot n \, ds} \quad ①$$

Balance law

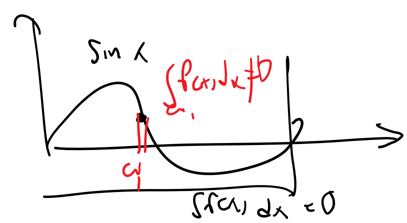
ω

$$\int_{\omega} \frac{de}{dt} dV = \int_{\omega} Q \, dV - \int_{\omega} \nabla \cdot q \, dV$$

General balance law

$$\forall \omega \subseteq D \int_{\omega} \Big(\underbrace{\frac{de}{dt} + \nabla \cdot q - Q}_{\text{integrand}} \Big) dV = 0 \quad \equiv \quad \int \left(\frac{\partial f_t}{\partial t} + \nabla \cdot f_x - r \right) dV = 0$$

since ω is arbitrary $\xrightarrow{\text{integral}=0}$ integrand $= 0$

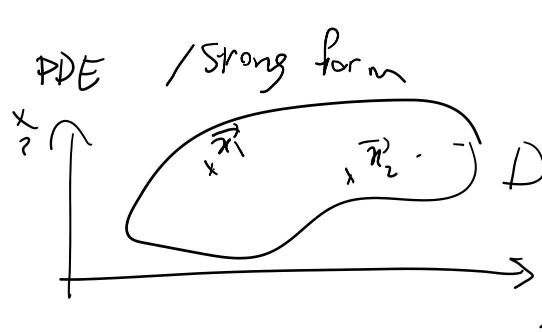


using Localizaion theorem $\longrightarrow$

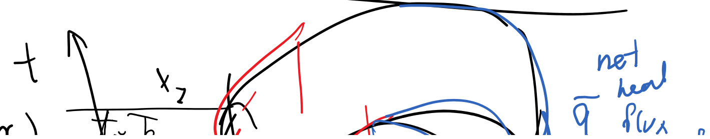
Localization theorem $\forall \omega \int_{\omega} f(x)\,dx = 0 \rightarrow f(x) = 0$

$$\forall x \in D \quad \boxed{\frac{dC_V T}{dt} + \nabla \cdot q - Q = 0} \quad ②$$

PDE / Strong form



IC & BC



IC & BC

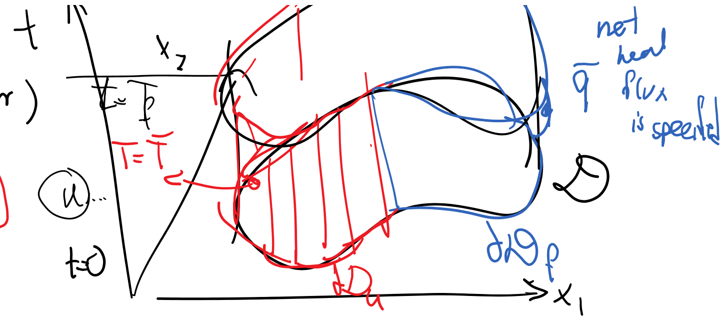
IC:) $T(x,0) = T_0(x)$ (1 time der)

BC) $\left\{ \begin{array}{l} T(x,t) = \bar{T}(x,t) \quad \forall x \in \partial\mathcal{D}_u \times [0,T_f] \\ \text{Dirichlet / Essential} \\ q \cdot n = \bar{q} \quad \forall x \in \partial\mathcal{D}_p \times [0,T] \\ \text{Neumann / Natural} \end{array} \right.$

(u)...

(f or n) ....

3



Closing the system

Unknowns are T & $q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ 2D

$T \to$ scalar, $q \to$ vector

How many eqns 1

eqn (1) ⟺ (2)

Constitutive eqn, example Fourier heat law

4 $\boxed{q = -k\nabla T}$ empirical eqn / material dependent

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All equations STRONG 5 continuum Initial Boundary Value Problem (IBVP)

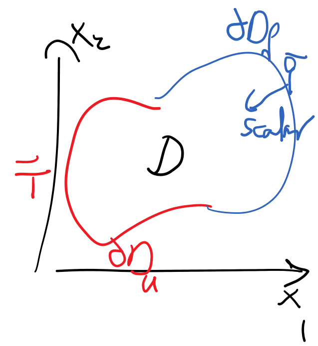
$\forall x \in \mathcal{D}$ $\quad R_i = \dfrac{d c_v T}{dt} + \nabla \cdot q - Q = 0$ Residual inside (PDE)

IC $\quad \forall x \in \mathcal{D} \quad T(x,0) = T_0(x)$

BCs $\left\{ \begin{array}{ll} \forall x \in \partial\mathcal{D}_u & R_u = \bar{T} - T = 0 \\ \forall x \in \partial\mathcal{D}_p & R_p = \bar{q} - q \cdot n = 0 \end{array} \right.$

Const. eqn

$q = -k\nabla T$



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we can solve this now

Weighted Residual Statement (WRS)

choose weight function w and multiply residuals by w

~~weight~~

choose weight function w and multiply residuals by it

$$\forall w \quad \underbrace{\int_{\Omega} w R_i \, dV + \int_{\partial\Omega_f} w R_f \, dS}_{\substack{R_i = 0 \;\&\; R_f = 0 \\ \Omega \qquad \partial\Omega_f \\ \text{satisfied weakly}}} + \int_{\partial\Omega_u} f(w) R_u \, dS = 0 \qquad (6)$$

$f(w)$: some function of w

No such term for continuous FEM

in DG we have a similar term as we'll see shortly

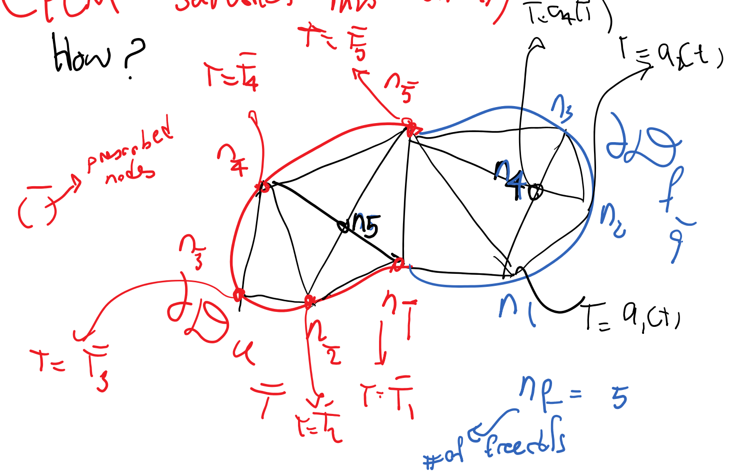
CFEM satisfies this strongly

How?

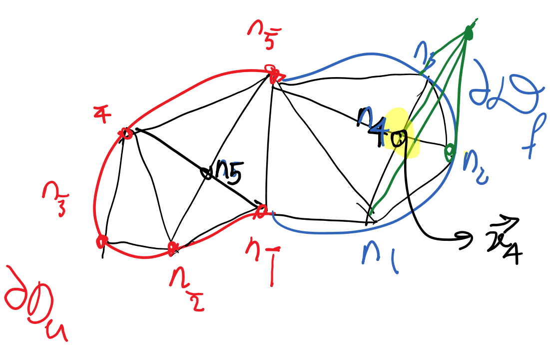
unknowns of the problem are "free" dof's, i.e. temperature of nodes inside or on natural BC:

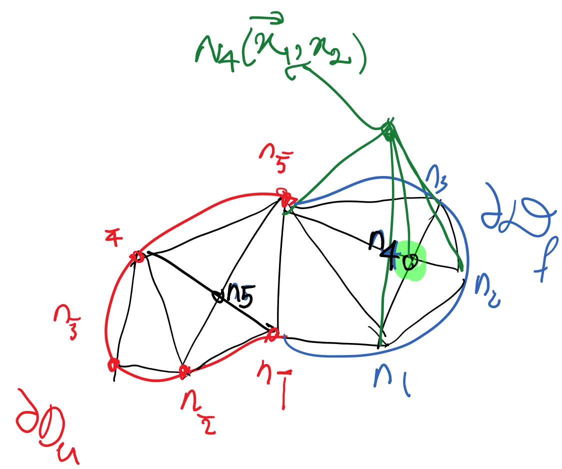
$$T^h(x,t) = \phi_p(x,t) + \overbrace{\sum_{i=1}^{n_f} N_i(x)\, \underbrace{a_i(t)}_{\text{unknowns}}}^{T_f}$$

we have discretized the problem (finite # of unknowns)



N's are shape functions





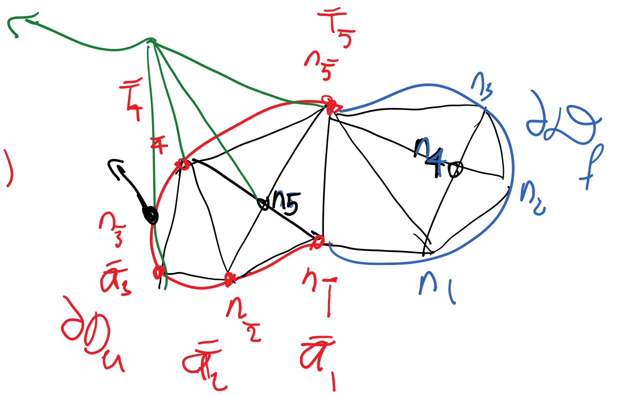
$$T_f(x) = N_1(\vec{x})\, a_1(t) + N_2(x)\, a_2(t) + N_3(x)\, a_3(t) + N_4(x)\, a_4(t) + N_5(x)\, a_5$$

$$T_f(\vec{x}_4) = \underbrace{N_1(\vec{x}_4)}_{0} a_1(t) + \underbrace{N_2(\vec{x}_4)}_{0} a_2(t) + \underbrace{N_3(x_4)}_{0} a_3(t) + \underbrace{N_4(x_4)}_{1} a_4(t) + \underbrace{N_5(x_4)}_{0} a_5(t)$$

$$T_f(x_4) = a_4(t)$$

basically $\boxed{T(x_i) = a_i \quad \text{for free dofs}}$ (7)

Particular function $\phi_p(\vec{x})$ satisfies the essential BC @ essential nodes



$\phi_p(\vec{x}) = \sum_{i=1}^{n_p} N_i(\vec{x})\, \bar{a}_i(t)$

$T @ n_i$

$n_f = 5$

$n_p = 5$

(8)

$$\boxed{\begin{aligned} T^h(x) &= \phi_p(\vec{x},t) + T_f(x,t) \\ &= \underbrace{\left[N_1(\vec{x}) \;\; N_2(\vec{x}) \; \cdots \; N_{n_p}(\vec{x})\right]}_{N_p(x)} \underbrace{\begin{bmatrix} \bar{a}_1(t) \\ \bar{a}_2(t) \\ \vdots \\ \bar{a}_{n_p}(t) \end{bmatrix}}_{a_p(t)} + \underbrace{\left[N_1(x) \;\; N_2(x) \; \cdots \; N_{n_f}(\vec{x})\right]}_{N_f(x)} \underbrace{\begin{bmatrix} a_1(t) \\ \vdots \\ a_{n_f}(t) \end{bmatrix}}_{a_f(t)} \\ &= N_p(\vec{x})\, a_p(t) + N_f(x)\, a_f(t) \end{aligned}}$$

$$\underbrace{\int_\Omega w R_i \, dV + \int_{\partial\Omega_f} w R_f \, dS}_{\partial\Omega_f} + \cancel{\int_{\partial\Omega_u} f(w)\, R_u \, dS} = 0$$

$R_i = 0$ & $R_f = 0$

CFEM this is satisfied strongly @ prescribed dof's

CFEM

$$\boxed{\int_\Omega w\left(\frac{d C_v T}{dt} + \nabla \cdot q - Q\right) dV + \int_{\partial\Omega_f} \underline{w}\left(\bar{q} - \underline{q \cdot n}\right) dS = 0}$$

CFEM WRS

∂D_ℓ

cancel this

weak form

$$w(\nabla\cdot q) = \nabla\cdot(wq) - (\nabla w) q$$

$$\int_D w(\nabla\cdot q)\,dV = \underbrace{\int_D \nabla\cdot(wq)\,dV} - \int_D (\nabla w) q\,dV$$

$$\int_D w(\nabla\cdot q)\,dV = \int_{\partial D} (wq)\cdot n\,dS - \int_D (\nabla w)\cdot q\,dV$$