CFEM

$$\boxed{\int_{\mathcal{D}} w\left(\frac{d c_v T}{dt} + \nabla\cdot q - Q\right) dV + \int_{\partial\mathcal{D}_f} w\left(\bar{q} - q\cdot n\right) dS = 0}$$ CFEM WRS

cancel this

weak form

$$(ab)' = (ab)' - a'b$$

$$w(\nabla\cdot q) = \nabla\cdot(wq) - (\nabla w) q$$

$$\int_{\mathcal{D}} w(\nabla\cdot q)\, dV = \underbrace{\int_{\mathcal{D}} \nabla\cdot(wq)\, dV} - \int_{\mathcal{D}} (\nabla w) q\, dV$$

$$\int_{\mathcal{D}} w(\nabla\cdot q)\, dV = \int_{\partial\mathcal{D}} (wq)\cdot n\, dS - \int_{\mathcal{D}} (\nabla w) q\, dV$$

$$\int w\left(\frac{dc_v T}{dt} - Q\right) dV + \int_{\partial\mathcal{D}_f \cup \partial\mathcal{D}_u} (wq)\cdot n\, dS - \int_{\mathcal{D}} \nabla w \cdot q\, dV$$

$$+ \int_{\partial\mathcal{D}_f} w\bar{q}\, dS - \int_{\partial\mathcal{D}_f} w\, q\cdot n\, dS = 0$$

$\partial\mathcal{D}_f$

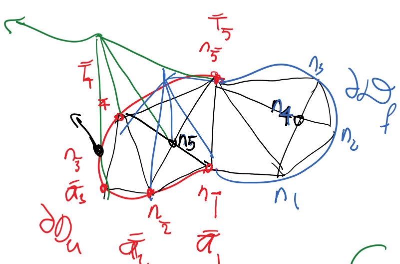
$T = \bar{T}$

$\partial\mathcal{D}_u$

$$\int_{\mathcal{D}} w\left(\frac{dc_v T}{dt} - Q\right) dV - \int_{\mathcal{D}} \nabla w\cdot q\, dV + \int_{\partial\mathcal{D}_f} w\bar{q}\, dS + \cancel{\int_{\partial\mathcal{D}_u} wq\cdot n\, dS} = 0$$

$= 0$ on $\partial\mathcal{D}_u$

0



$n_p = 5$

$n_f = 5$

$$T = \begin{bmatrix} N_1 & N_2 & \cdots & N_{n_f} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_{n_f} \end{bmatrix} + \begin{bmatrix} N_{\bar{1}} & N_{\bar{2}} & \cdots & N_{\bar{n_p}} \end{bmatrix} \begin{bmatrix} \bar{a}_1 \\ \vdots \\ \bar{a}_{n_p} \end{bmatrix}$$

we have n_f dofs

$\longrightarrow$ n_f weights $\begin{bmatrix} w_1 \\ \vdots \\ w_{n_f} \end{bmatrix}$

Galerkin: $w = N_f$

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{n_f} \end{bmatrix} = \begin{bmatrix} N_1 \\ \vdots \\ N_{n_f} \end{bmatrix}$$

as can be seen $\begin{bmatrix} w_1 \\ \vdots \\ w_{n_f} \end{bmatrix} = 0 \quad \forall x \in \partial\mathcal{D}_f$

CFEM weak statement

$$\int_{\mathcal{D}} w\left(\frac{d c_v T}{dt} - Q\right) dV - \int_{\mathcal{D}} \nabla w\cdot q\, dV + \int_{\partial\mathcal{D}_f} w\bar{q}\, dS = 0$$

$q = \ldots f(T, \nabla T) \ldots$ constitutive eqn

Fourier heat law $q = -\underset{\sim}{k}\nabla T$

symmetric, positive definite 2nd order tensor

$$\int w\left(\frac{d c_v T}{dt} - Q\right) dV + \int (\nabla \ldots) (k \nabla T)\, dV + \int w\bar{q}\, dS = 0$$

$$\boxed{\int_{\mathcal{D}} w\left(\frac{d\, c_v T}{dt} - Q\right) dV + \int \nabla w \cdot \underbrace{k \nabla T}_{L_m}\, dV + \int_{\partial \mathcal{D}_p} w \bar{q}\, dS = 0} \quad (1)$$

$w \to N_f^T$, $\nabla w \to (\nabla N_f)^T$

weak statement

Discretization: $T = \underbrace{N_p a_p}_{\Phi_p} + N_f a_f$

$N_f = [N_1 \; \cdots \; N_{nf}]$, $a_f = \begin{bmatrix} a_1 \\ \vdots \\ a_{nf} \end{bmatrix}$

Galerkin

$$w = \begin{bmatrix} w_1 \\ \vdots \\ w_{nf} \end{bmatrix} = \begin{bmatrix} N_1 \\ \vdots \\ N_{nf} \end{bmatrix} = N_f^T$$

$$T = N_p a_p + N_f a_f \to \nabla T = \underbrace{\begin{bmatrix} \frac{\partial T}{\partial x_1} \\ \frac{\partial T}{\partial x_2} \end{bmatrix} = \overbrace{\begin{bmatrix} \frac{\partial N_1}{\partial x_1} & \cdots & \frac{\partial N_{np}}{\partial x_1} \\ \frac{\partial N_1}{\partial x_2} & \cdots & \frac{\partial N_{np}}{\partial x_2} \end{bmatrix}}^{B_p} \begin{bmatrix} a_T \\ \vdots \\ a_{np} \end{bmatrix}}_{\text{red}}$$

$$+ \underbrace{\overbrace{\begin{bmatrix} \frac{\partial N_1}{\partial x_1} & \cdots & \frac{\partial N_{nf}}{\partial x_1} \\ \frac{\partial N_1}{\partial x_2} & \cdots & \frac{\partial N_{nf}}{\partial x_2} \end{bmatrix}}^{B_f}}_{\text{blue}} \begin{bmatrix} a_1 \\ \vdots \\ a_{nf} \end{bmatrix}$$

$B = L_m N$ (here $L_m = \nabla$)

$\nabla w = B_f^T$

$$\int_{\mathcal{D}} w\left(\frac{d\, c_v T}{dt} - Q\right) dV + \int \nabla w \cdot k \nabla T\, dV + \int_{\partial \mathcal{D}_p} w \bar{q}\, dS = 0$$

$w \to N_f^T$, $T \to (N_p a_p + N_f a_f)$, $\nabla T \to (B_p a_p + B_f a_f)$, $w \to N_f^T$

$$\int_{\mathcal{D}} N_f^T \frac{d\, c_v (\overbrace{N_p a_p + N_f a_f}^{T})}{dt} + \int_{\mathcal{D}} B_f^T k (B_p a_p + B_f a_f)\, dV = \int_{\mathcal{D}} N_f^T Q\, dV - \int_{\partial \mathcal{D}_p} N_f^T \bar{q}\, dS$$

1) $\frac{dT(x,t)}{dt} = \frac{d\, N_p(\vec{x}) a_p(t) + N_f(x) a_f(t)}{dt} = N_p \dot{a}_p + N_f \dot{a}_f$

2) take a_p terms to the RHS

$$\underbrace{\left(\int_{\mathcal{D}} (N_f)^T c_v N_f\, dV\right)}_{\text{Mass matrix}} \dot{a}_f + \underbrace{\left(\int_{\mathcal{D}} (B_f)^T k B_f\, dV\right)}_{K_{ff} := K} a_f = \underbrace{\int_{\mathcal{D}} N_f^T Q\, dV}_{F_r} \underbrace{- \int_{\partial \mathcal{D}_p} N_f^T \bar{q}\, dS}_{F_N} \underbrace{- \left(\int_{\mathcal{D}} N_f^T c_v N_p\, dV\right)\dot{a}_p + \left(\int B_f^T k B_p\, dV\right) a_p}_{F_D}$$

$nf \times 1$, 1×1, $1 \times nf$; 2×2; $nf \times 2$, $2 \times nf$

F_r source term / force

Force from Natural/Neumann BC F_N

$M_{ff} := M$ = mass matrix

$K_{ff} := K$ stiffness, $nf \times nf$

prescribed BC

Dirichlet F_D

$\partial \mathcal{D}_q$, $\partial \mathcal{D}_p$

(2a)

$$\boxed{M \dot{a} + K a = F_r + F_N - F_D}$$

Solid Mechanics

$M_{ff} = M = \int_{\mathcal{D}} N_f^T c_v N_f\, dV$ $\quad a = a_f$

$M \ddot{a} + C \dot{a} + K a = \text{—} \quad \text{—}$

$M = \int (N_f^T \rho N_f)\, dV$

$$M_{ff} = M = \int_{\Omega} N_f^t c_v N_p \, dV \qquad a = a_f$$

$$K_{ff} = K = \int_{\Omega} B_f^t D B_f \, dV$$

D: material / section property here $= k$

$$F_r = \int_{\Omega} N_p^t Q \, dV, \quad F_N = -\int_{\partial\Omega_f} N_p^t \bar{q} \, dS$$

$$F_D = \underbrace{\left(\int B_f^t D B_p \, dV\right)}_{K_{fp}} a_p + \int \underbrace{\left(N_f^t c_v N_p\right)}_{M_{fp}} \dot{a}_p$$

$F_D = M_{fp}\dot{a}_p + K_{fp} a_p$

$$M\dot{a} + C\dot{a} + K a = \ldots$$

$$M = \int (N_p^t \rho N_p) \, dV$$

$$C = \int N_p^t d N_f \, dV$$

damping

$$K = \int B_p^t D B_f \, dV$$

$L_m = \varepsilon(\ )$ $\qquad \varepsilon(N)$

$D = C$ stiffness matrix $\quad \sigma = C\varepsilon$

2b

system of ODEs (semi-discrete form)

space is discretized

but time still is not

Global approach OR node/dof centered

Local approach or element centered

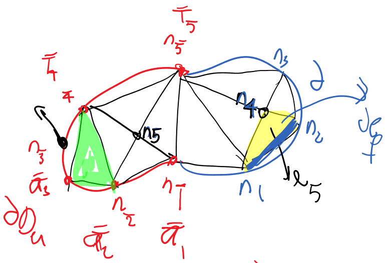
We take care of the contributions of each element to M, K, and the right hand side

$$K^e = \int_e B_e^t D B \, dv \qquad M^e = \int_e N_e^t c_v N_e \, dv$$

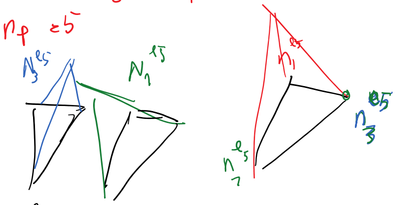
$$f_r^e = \int N_e^t Q \, dV \qquad f_N^e = -\int_{\partial e_f} N_e^t \bar{q} \, dS \quad / \partial e_f = e \cap \partial\Omega_f$$

$$f_D^e = M^e \dot{a}^e + K^e a^e \quad \text{local}$$

$$f_D = M_{fp} \dot{a}_p + K_{fp} a_p \quad \text{global}$$



$\partial\Omega_f \qquad n_f = 5$



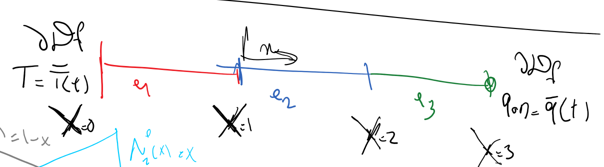
$$N^e = [N_1^e \ N_2^e \ N_3^e]$$

$$B^e = \nabla N^e = \begin{bmatrix} N_{1,1} & N_{2,1} & N_{3,1} \\ N_{1,2} & N_{2,2} & N_{3,2} \end{bmatrix}$$

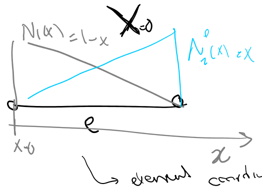
Example of 1D heat conduction

Color coding is done based on the element not the BC type

For simplicity all elements are identical



$k = 1, \quad c_v = 1$

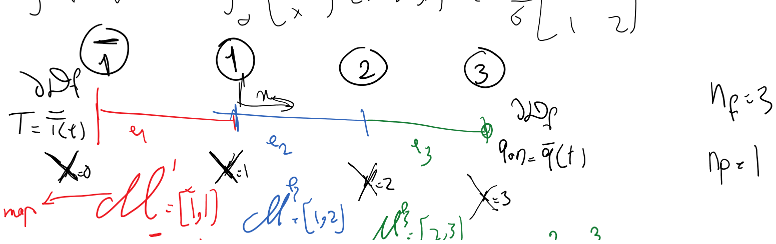


$$N = [N_1, N_2] = [1-x, x]$$

$$B = \nabla N = N_{,x} = [-1, 1]$$

$$K^e = \int_e B^t D B \, dx = \int_0^1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} k [-1 \ 1] \, dx = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$M^e = \int N^{e^T} c_v N^e \, dx = \int_0^1 \begin{bmatrix} 1-x \\ x \end{bmatrix} [1-x, x] = \frac{1}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$



$n_f = 3$

$n_p = 1$

$X=0$ map $\leftarrow \mathcal{M}^{1} = [1,1]$ $X=1$ $\mathcal{M}^{e_2} = [1,2]$ $X=2$ $\mathcal{M}^{e_3} = [2,3]$ $X=3$ $q_{on} = \bar{q}(t)$ $n_p = 1$

$$K^{e_1} = \begin{array}{c|cc} & 1 & 1 \\ \hline 1 & 1 & -1 \\ 1 & -1 & 1 \end{array} \qquad K^{e_2} = \begin{array}{c|cc} & 1 & 2 \\ \hline 1) & 1 & -1 \\ 2) & -1 & 1 \end{array} \qquad K^{e_3} = \begin{array}{c|cc} & 2 & 3 \\ \hline 2) & 1 & -1 \\ 3) & -1 & 1 \end{array}$$

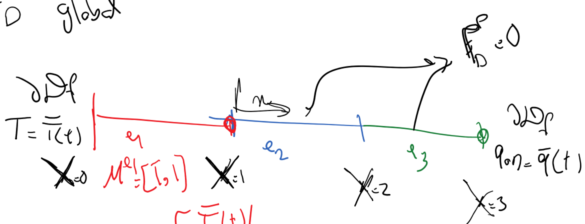
$$M^{e_1} = \frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad M^{e_2} = \frac{1}{6}\begin{array}{c|cc} & 1 & 2 \\ \hline 1) & 2 & 1 \\ 2) & 1 & 2 \end{array} \qquad M^{e_3} = \frac{1}{6}\begin{array}{c|cc} & 2 & 3 \\ \hline 2) & 2 & 1 \\ 3) & 1 & 2 \end{array}$$

$$K_{3\times3} = \begin{bmatrix} 1 & -1 & \\ -1 & 1+1 & -1 \\ & -1 & 1 \end{bmatrix} \qquad M = \frac{1}{6}\begin{bmatrix} 2+2 & 1 & \\ 1 & 2+2 & 1 \\ & 1 & 2 \end{bmatrix}$$

$$K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad M = \begin{bmatrix} 4 & 1 & \\ 1 & 4 & 1 \\ & 1 & 2 \end{bmatrix}$$

$F_r = 0$ no source term

F_D global



$$a_{e_1} = \begin{bmatrix} \bar{T}(t) \\ ? \end{bmatrix} \quad ? \rightarrow \text{replace by zero}$$

$$f_D^{e_1} = M^{e_1}\dot{a}_{e_1} + K^{e_1} a_{e_1} = \frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \dot{\bar{T}} \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \bar{T} \\ 0 \end{bmatrix}$$

$$= \begin{array}{c|c} \begin{bmatrix} \frac{1}{3}\dot{\bar{T}} + \bar{T} \\ \frac{1}{6}\dot{\bar{T}} - \bar{T} \end{bmatrix} & \begin{matrix} \bar{1} \\ 1 \end{matrix} \end{array}$$

$$F_D = \begin{bmatrix} \frac{1}{6}\dot{\bar{T}} - \bar{T} \\ 0 \\ 0 \end{bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$