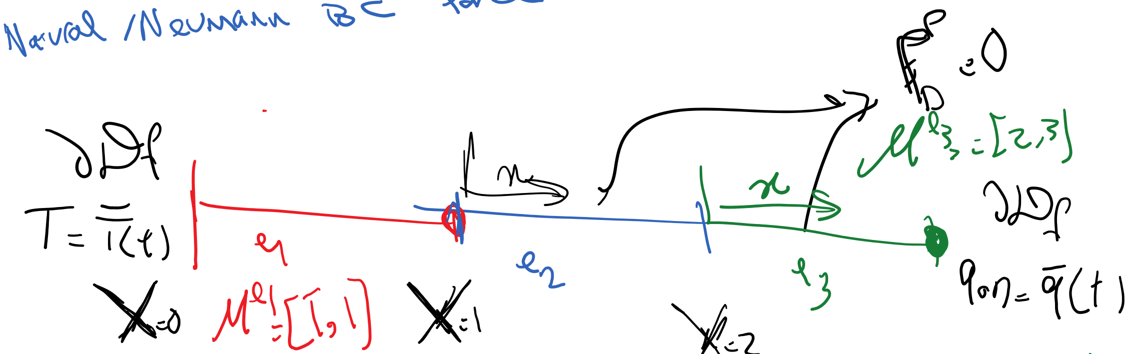
2022/09/15

Thursday, September 15, 2022 9:48 AM

Natural / Neumann BC force




$f_N^{e_1} = 0$ $\quad$ $f_N^{e_2} = 0$ $\quad$ $f_N^{e_3} = \int_{\partial e_N} N^{e^T} \bar{q}\, ds \overset{1D}{=} N^{e^T} \bar{q} \Big|_{x=1}$

$$= \begin{bmatrix} N_1^{e_3} \\ N_2^{e_3} \end{bmatrix}_{x=1} \bar{q} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \bar{q}$$

let's assume $\bar{q} = 1$

$$f_N^{e_3} = \begin{array}{c} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{array} \begin{array}{c} 2 \\ 3 \end{array}$$

$$F = -F_D + f_N + \cancel{f_r}^{\,0} = \begin{bmatrix} \bar{T} - \frac{1}{6}\dot{\bar{T}} \\ 0 \\ 0 + 1 \end{bmatrix}$$

$M\dot{a} + Ka = F$ semi-discrete form → we'll descretize this in time

---

Discontinuous Gakerkin formulation for heat conduction problem (one of many possible DG formulations)

The domain $D$ has 3 residuals $\quad$ CFEM

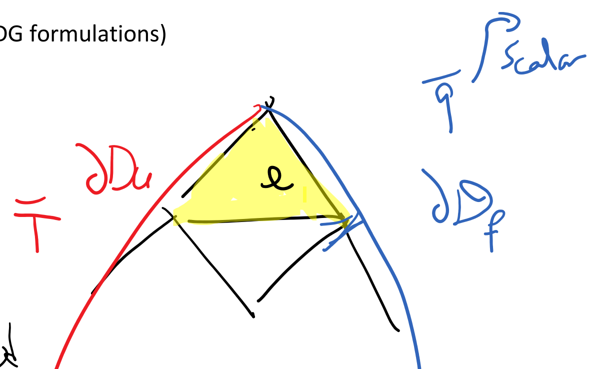
a) $R_i = C\dot{T} + \nabla \cdot q - Q$ $\quad$ inside $D$

b) $R_f = \bar{q} - q \cdot n$ $\quad$ on $\partial D_f$

c) $R_u = \bar{T} - T$ $\quad$ on $\partial D_u$

(b), (c) are enforced weakly in CFEM

→ satisfied strongly & a priori in CFEM




---

DG Residual is formed for an element

For element e

a) $R_i = C\dot{T} + \nabla \cdot q - Q$ $\quad$ inside e

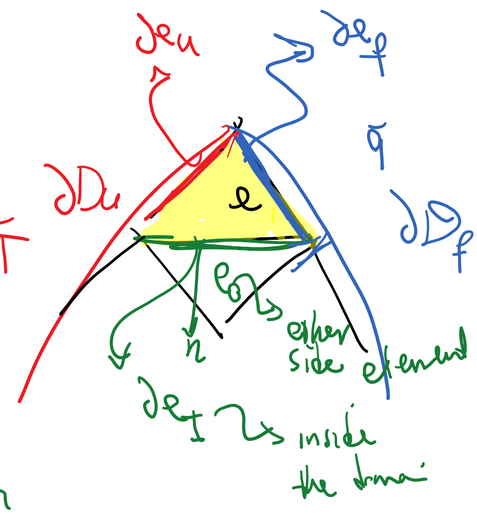
b) $R_f = q_n^* - q \cdot n$ $\quad$ on $\partial e$ $\quad$ the whole boundary of element

c) $R_u = T^* - T$ $\quad$ on $\partial e$

b) $n_T = q_n^* - q\cdot n$ on $\partial e$

c) $R_u = T^* - T$ on $\partial e$ → this will be satisfied WEAKLY

$(\ )^* =$ target or star value




$$q_n^* = \begin{cases} \bar q & \partial e_f \\ q\cdot n & \partial e_u \\ q_n^*(q, q_o, T, T_o, \vec n) & \text{on } \partial e_I \end{cases}$$

( $R_f = q\cdot n - q\cdot n = 0$ it's free, no real eqn for $q\cdot n$ here

one choice we'll use is the average $\frac{1}{2}(q+q_o)\cdot n$

$$T^* = \begin{cases} \bar T & \partial e_f \quad R_u = T - T = 0 \text{ no effective condition here} \\ \bar T & \\ T^*(q, q_o, T, T_o, n) & \text{on } \partial e_I \end{cases}$$

one choice is the average flux $T^* = \frac{1}{2}(T + T^o)$

---

WRS instead of $w$ I'll use $(\hat{\ })$ notation

$T \longrightarrow$ weight of $T = \hat T$

$\forall e$ find $T, q \ni \forall \hat T$

$$\int_e \hat T\,(\overbrace{c\dot T + \nabla\cdot q - Q}^{R_i})\,dV$$

$$+\int_{\partial e} \hat T\,(\underbrace{q_n^* - q_n}_{R_f})\,ds + \int_{\partial e} f(\hat T)\,(\underbrace{T^* - T}_{R_u})\,ds$$

$[T][q]$ → dimension of the integral

→ new term added for DG

$f(\hat T)$ some function of $\hat T$

a) $f = \hat T$ $\quad \lambda\,[f(\hat T)(T^* - T)] = [T]^2\lambda$ $\quad \frac{k}{[L]}$ → $h$ $\quad q = -k\nabla T$ $\quad [q] = \frac{[k][T]}{[L]}$

$\lambda = \frac{k}{h}$ make physical dimension consistent

$\lambda = \frac{k}{h}$ make physical dimension consistent $\hookrightarrow h$

b) $f(\hat{T}) = \hat{q} = -k\nabla\hat{T}$ ✓ I'll use this now

ARS: ① $\forall e$ $\int_e \hat{T}(C\dot{T} + \nabla\cdot q - Q)\,dv + \int_{\partial e} \hat{T}(q_n^* - q\cdot n)\,ds + \int_{\partial e} \hat{q}(T^* - T)\,ds = 0$ $\hookrightarrow -k\nabla\hat{T}$

Weak $\int_e \hat{T}\nabla q - \int_{\partial e} \hat{T} q\cdot n\,ds = \cancel{\int_e \nabla\cdot(\hat{T}q)\,ds} - \int_e \nabla\hat{T}\cdot q - \cancel{\int_{\partial e} \hat{T}q\cdot n\,ds} = -\int_e \nabla\hat{T}\cdot q\,dv$

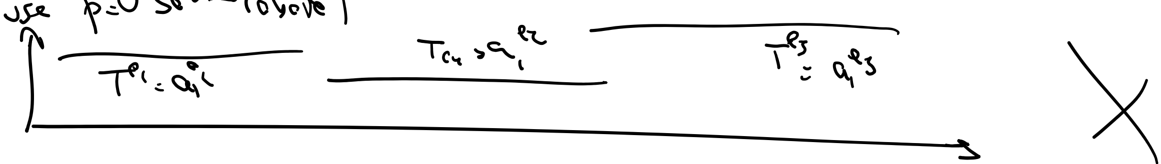
$$\boxed{\text{Weak} \int_e (\hat{T}C\dot{T} - \nabla\hat{T}q - \hat{T}Q)\,dv + \int_{\partial e} \hat{T} q_n^*\,ds + \int_{\partial e} \hat{q}(T^* - T)\,ds = 0}$$

① I'll use this

- In DG methods we can use p = 0 (polynomial order = 0) for the primary fields in general.
- However, for this particular formulation, p = 0 -> $\hat{T} = 1\,a(t)$ ; $\hat{T} = 1 \rightarrow \nabla\hat{T} = 0 \rightarrow$ we cannot enforce $T^* - T \longrightarrow$ the minimum order is p=1

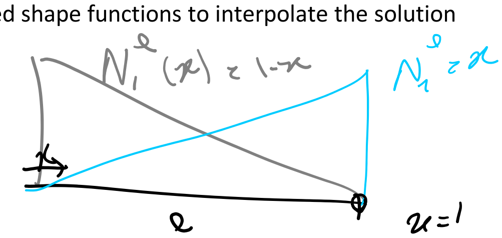
Let's solve our sample 1D problem using the DG method


we cannot use p=0 solution (above)




At least p=1

We can use Continuous Finite Element-based shape functions to interpolate the solution




This approach is called "Nodal DG". I shared the book with you.
Some advantages:

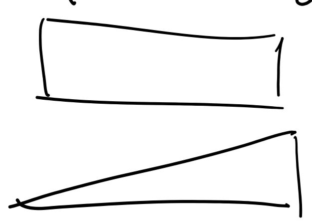
- Generally good conditioning of element matrices
- DOFs have physical meaning
- ... (simplifying some element boundary computation)

However, since there is no strong continuity constraint, we can use any basis for linear space

$$T(x) = a_1^e(t)\,\phi_1(x) + a_2^e(t)\,\phi_2(x) = \Phi\, a = \underbrace{[1 \quad x]}_{N}\begin{bmatrix} a_1^e \\ a_2^e \end{bmatrix}$$

e.g. $\phi_1(x) = 1$

$\phi_2(x) = x$



I'm going to use this

---

Calculations for a sample element:

$T_L = T(x=0) = a_1$ | $n_L = -1$

$q_L = q(x=0) = -k a_2$

$x=0$ $\qquad$ $x=L$

$e$

$n_R = 1$

$T_R = a_1 + L a_2$

$q_R = -k a_2$

$x$

$$\underline{T = a_1 + a_2 x} = [1 \quad x]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$B = \nabla T = T_{,x} = [0 \quad 1]$$

$$q = -kBa = -k[0 \quad 1]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \underline{-k a_2}$$

$$\int_e (\hat{T} c \dot{T} - \nabla \hat{T}\, \underbrace{q}_{-k\nabla T} - \hat{T} Q)\, dv + \int_{\partial e} \hat{T} q_n^{*}\, ds + \int_{\partial e} \underbrace{(-k \nabla \hat{T}) \cdot n}_{\hat{q}\cdot n} (T^{*} - T)\, ds = 0$$

$$\left(\int_e N^T c N\, dV\right)\dot{a} + \underbrace{\left(\int_e B^t k B\, dV\right)}_{k^e} a - \underbrace{\int_e N^T Q\, dv}_{Fr} + \left(N^T q_{n_L}^{*}\right) + \left(N^T q_{n_R}^{*}\right)$$

$$+ \left[\underbrace{-kB^T \cdot n}_{-k\nabla \hat{T} \cdot n}(T^{*} - T)\right]_L + \left[-kB^T n (T^{*} - T)\right]_R$$

We're going to plug in the values for T left and right and q left and right:

$$m^e \dot{a} + k^e a - Fr + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{N_L^T} q_L^{*} + \underbrace{\begin{bmatrix} 1 \\ L \end{bmatrix}}_{N_R^T} q_R^{*} + \left(-k \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B^T} \underbrace{(-1)}_{n_L} \cdot \left(T_L^{*} - \underbrace{[1 \quad 0]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}_{T}\right)\right)$$

$$N_L^T \qquad N_R^T \qquad \qquad T$$

$$+\left(-k \begin{bmatrix} 0 \\ 1 \end{bmatrix} (1) \left( \overset{*}{T_R} - [1 \quad L] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right)\right) = 0$$

$n_R$

$$m\dot{a} + \tilde{K}^e a - F_r + \begin{bmatrix} q_L^* - q_R^{**} \\ L q_R^{**} \end{bmatrix} + k \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(\overset{*}{T_L} - \overset{*}{T_R}\right) + k \begin{bmatrix} 0 & 0 \\ 1 & L \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$- k \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\longrightarrow \boxed{m\dot{a} + \tilde{K} a - F_r + \begin{bmatrix} \overset{*}{q_L} + \overset{*}{q_R} \\ L \overset{*}{q_R} + k\left(\overset{*}{T_L} - \overset{*}{T_R}\right) \end{bmatrix} + k \begin{bmatrix} 0 & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

(2i)

Equation for 1 element

$$2\text{ii} \quad m = \int N^T c_v N \, dv = \int_0^L \begin{bmatrix} 1 \\ x \end{bmatrix} c_v [1 \quad x] \, dx = c_v L \begin{bmatrix} 1 & L/2 \\ L/2 & L^2/3 \end{bmatrix}$$

$$2\text{iii} \quad \tilde{K} = \int B^T k B \, dv = \int_0^L \begin{bmatrix} 0 \\ 1 \end{bmatrix} k [0 \quad 1] \, dx = kL \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$2 \rightarrow \boxed{cL \begin{bmatrix} 1 & L/2 \\ L/2 & L^2/3 \end{bmatrix} \dot{a} + 2kL \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} a + \begin{bmatrix} \overset{*}{q_L} + \overset{*}{q_R} \\ L \overset{*}{q_R} + k\left(\overset{*}{T_L} - \overset{*}{T_R}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

(3)

element level formula

System eqn

$e_1 \qquad e_2 \qquad e_3$

element level dofs $\quad T^{e_1} = a_1^{e_1} + a_2^{e_1} x \qquad T^{e_2} = a_1^{e_2} + a_2^{e_2} x^{e_2} \qquad T^{e_3} = a_1^{e_3} + a_2^{e_3} x^{e_3}$

Global dofs $\quad = a_1 + a_2 x^{e_1} \qquad a_3 + a_4 x^{e_2} \qquad a_5 + a_6 x^{e_3}$

System equation

$$cL \begin{bmatrix} 1 & L/2 \\ L/2 & L^2/3 \end{bmatrix} \begin{bmatrix} \dot{a}_1 \\ \dot{a}_2 \end{bmatrix} + 2kL \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} (q_L)_{e_1} + (q_R)_{e_1} \\ L \overset{*}{q}_{R_{e_1}} + k\left(\overset{*}{T}_{L_{e_1}} - \overset{*}{T}_{R_{e_1}}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad e_1$$

$$CL\begin{bmatrix}1 & L/2\\ L/2 & L^2/3\end{bmatrix}\begin{bmatrix}\ddot q_1\\ \ddot q_2\end{bmatrix} + 2KL\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix} + \begin{bmatrix} q_{L_{e1}} + q_{R_{e1}} \\ L\dot q_{R_{e1}} + K(\dot T_{L_{e1}} - \dot T_{R_{e1}})\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \qquad e_1$$

$$CL\begin{bmatrix}1 & L/2\\ L/2 & L^2/3\end{bmatrix}\begin{bmatrix}\ddot q_3\\ \ddot q_4\end{bmatrix} + 2KL\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}q_3\\ q_4\end{bmatrix} + \begin{bmatrix} q_{L_{e2}} + q_{R_{e2}} \\ L\dot q_{R_{e2}} + K(\dot T_{L_{e2}} - \dot T_{R_{e2}})\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$CL\begin{bmatrix}1 & L/2\\ L/2 & L^2/3\end{bmatrix}\begin{bmatrix}\ddot q_5\\ \ddot q_6\end{bmatrix} + 2KL\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}q_5\\ q_6\end{bmatrix} + \begin{bmatrix} q_{L_{e3}} + q_{R_{e3}} \\ L\dot q_{R_{e3}} + K(\dot T_{L_{e3}} - \dot T_{R_{e3}})\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \qquad e_3$$

8 unknowns    6 equations