A Discontinuous Galerkin Method for the Solution of One Dimensional Radiative Transfer Equation

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***Abstract*—The radiative transfer equation (RTE) for a plane-parallel problem involving scattering, absorption and radiation is solved using the discontinuous Galerkin (DG) finite element method (FEM). Both space and angle directions are discretized by the DG method. The problem is formulated for nonzero phase function. The method is validated against exact solutions, and compared with other space-angle and hybrid FEMs for a few benchmark problems. The performance of the method is also studies for the solution of problems with discontinuous solution.**

I. INTRODUCTION

The radiative transfer equation (RTE) describes the interaction of radiation in an absorbing, scattering medium. Two of the most popular solution methods for these transfer equations are the Discrete Ordinate (S_N) method and Spherical Harmonic (P_N) method. With both SN and PN methods, one popular method for discretizing the spatial domain is the Galerkin finite element method (FEM), see for example [1]. One disadvantage of Galerkin FEMs in solving the radiative transfer equation in that it cannot handle discontinuities in the solution. Discontinuous Galerkin (DG) methods relax the continuity constraint of continuous FEMs, in that jump in solution between elements not only is enforced weakly but also is based on wave propagation direction. Hybrid methods with DG discretization in space are presented for S_N [2] and P_N [3] discretization in angle direction. We present a formulation wherein both space and angle directions are discretized by a DG formulation of a 1D plane-parallel RTE problem.

II. FORMULATION

For the 1D scattering problem, the plane-parallel radiative transfer equation can be written as,

$$\mu\frac{\mathrm{d}I}{\mathrm{d}z} + \beta I - \frac{\sigma_s}{2}\int_{-1}^{1}\Phi(\mu,\mu')I(z,\mu')\mathrm{d}\mu' = S \tag{1}$$

where μ is the direction cosine, $I(z,\mu)$ is the radiative intensity, S is the source term, and $\Phi(\mu,\mu')$ is the anisotropic scattering phase function. The values $\beta = \kappa_\eta + \sigma_s$, κ_η, and σ_s, are extinction, absorption, and scattering coefficients.

The partial differential equation (PDE) is enforced on the square domain $(z,\mu) \in \Omega = (\underline{z},\bar{z}) \times (-1,1)$, where $\underline{z}$ and $\underline{z}$ are the minimum and maximum values of the spatial coordinate z. The boundary conditions are specified on inflow boundaries for I which are comprised of $\mu > 0$ and $\mu < 0$ for $z = \underline{z}$ and $z = \bar{z}$, respectively.

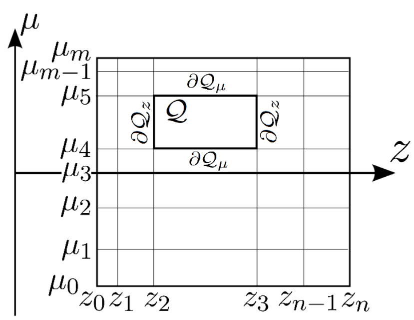


Fig. 1. Demonstration of region of applicability of residuals in $z,\mu)$ domain.

Figure 1 shows an m by n tensor product discretization of Ω, where $(\underline{z},\bar{z})$ and $(-1,1)$ are discretized into $\{z_0, z_1, \cdots, \underline{z}\}$ and $\{\mu_0, \mu_1, \cdots, \mu_n\}$, respectively, for $z_0 = \underline{z}$, $z_m = \bar{z}$, $\mu_0 = -1$, and $\mu_n = 1$. In a DG formulation, residuals (errors) must be specified both in the interior and on the boundary of elements. For an element $\mathcal{Q} \in \Omega$, these residuals respectively are,

$$\mathcal{R}_{\mathcal{Q}} = \mu\frac{\mathrm{d}I}{\mathrm{d}z} + \beta I - \frac{\sigma_s}{2}\int_{-1}^{1}\Phi(\mu,\mu')I(z,\mu')\mathrm{d}\mu' - S \tag{2a}$$

$$\mathcal{R}_{\partial\mathcal{Q}_z} = \mu(I^* - I) \tag{2b}$$

where (2a) is enforced inside $\mathcal{Q}$ and (2b) only on vertical boundaries of $\mathcal{Q}$ in fig 1 since the PDE (1) only has partial derivatives with respect to z. Finally, the target value I^* is always specified from the upstream value of the wave equation (1) in that in the upper half $(\mu > 0)$ I^* comes from the left boundary condition or for an interior element such as $\mathcal{Q}$ is specified from the trace of the neighboring element on its left edge. For all upper half elements $I^* = I$ on their right edge. That is, (2b) is only non-trivially enforced on the left edge of elements. For the lower half elements the same argument hold with waves moving from right to left, thus enforcing (2b) only on the right edge of elements. Finally, the discrete weak form is obtained by weighted residual formulation of (2),

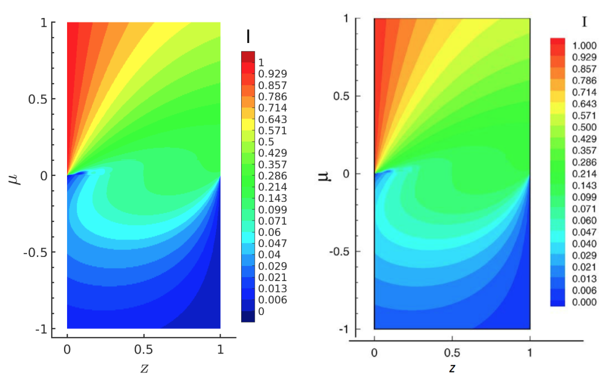
$$\int_{\mathcal{Q}} \check{I}.\left\{\mu\tfrac{\mathrm{d}I}{\mathrm{d}z} + \beta I - \tfrac{\sigma_s}{2}\int_{-1}^{1}\Phi(\mu,\mu')I(z,\mu')\mathrm{d}\mu' - S\right\}\mathrm{d}z\mathrm{d}\mu + \int_{\partial\mathcal{Q}_z}\check{I}.\mu(I^* - I)\mathrm{d}\mu = 0 \tag{3}$$

where the weight functions $\check{I}$ and trial solution I are polynomials of order p in both space and angle, interpolated with respect to a local coordinate system.

III. Numerical Examples

Consider a slab of thickness L with space dependence scattering coefficient, $\sigma_s(z) = z/L$, and a unit extinction coefficient, $\beta = 1$. The anisotropic phase function is given by $\Phi(\mu, \mu') = \sum_{m=0}^{M} a_m P_m(\mu) P_m(\mu')$, where a_m are specified constants, P_m are Legendre polynomials. The number of terms $M = 7$ and coefficients as chosen the same as those in [4]. Constant boundary conditions of $\bar{I} = 1$ and $\bar{I} = 0$ are enforced on the top-left and bottom-right boundaries, respectively.

The basis order in both space and angle is $p = 4$. The domain is discretized with a coarse and non-uniform 6×8 grid with finer elements near $\mu = 0$ to resolve the strong gradients. The distribution of radiative intensity in spacial and angular domain is shown in fig. 2 obtained by the DG and least-square finite element method [4] methods, respectively. It shows that DG results are in agreement with LS results.



(a) DG (b) LS [4]

Fig. 2. Contour plot of radiative intensity, $I(z, \mu)$, in space-angle domain.

The second problem demonstrate the capability of a DG formulation in resolving strong discontinuities in solution. Herein, a delta source term, $S(z = 0.375, \mu = 0.75) = 10$, is applied in a domain with Rayleigh scattering. Vacuum and reflective boundaries conditions are used on the top-left and bottom-right boundaries of the domain.

The exact solution (propagation of a delta ray from $z = 0.375^+, \mu = 0.75$ and its reflection along $\mu = -0.75$) cannot be exactly captured in discrete setting, given that the solution is not a function and in addition FEMs are often discretized by polynomials. With continuous FEMs there is an additional challenge where due to continuity constraint between elements the intensity propagates leftward from $z = 0, \mu = 0.75$.

Our results, not reported here, demonstrate that direct integration of delta point source within an element results in negative intensity when $p > 0$ is used to interpolate I. To remedy this a constant average value for S is used over the elements over which the point source is applied.

Figure 3 shows the solution for a 32×64 grid in space and angle, interpolated with $p = 3$ elements. As can be observed the intensity propagated to the right from its source at $z = 0.375, \mu = 0.75$ while it decays. As expected, the ray reflects and continues in the bottom have around $\mu = -0.75$. Due to Rayleigh scattering, small values of I are observed at angles other than those covered by the source term and its reflection.

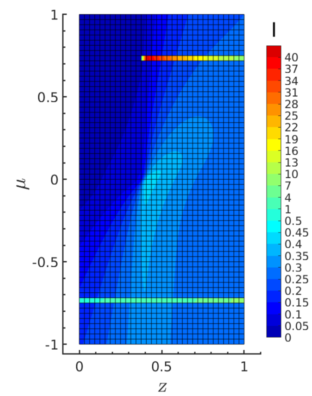


Fig. 3. Scattering from a point source in a Rayleigh scattering medium.

IV. Conclusions

The cleanness of the solution, *e.g.,* the absence of nonphysical backscattering, overshoot or undershoot for the source term problem, is the testament to the appropriateness of the DG method for the RTE. We plan to employ the other advantages of the DG method in upcoming works where solution details, *e.g.,* point sources, can be efficiently captured by h-adaptivity and p-enrichment; unlike continuum FEMs no transition element are required.

Acknowledgement

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