1 Introduction

Many common materials used in engineering systems, including rock, concrete, ceramic, and certain composites, are considered quasi-brittle. Understanding their failure mechanisms is vital for using and processing these materials across different size scales. Quasi-brittle materials lack the energy dissipative mechanisms that more ductile materials have [1–3], resulting in a greater importance to understand the distribution of flaws in the micro-scale, such as microcracks or inclusions. Crack pattern variability for similar sized samples with equivalent loading [4] has been due to these microstructural flaws. The size effect, wherein the fracture strength decreases as the size of the material sample increases [3,5], is a direct result of these microstructural defects. Therefore, characterizing these networks of microcracks is important to understand the behavior of these materials under loading, as very different fracture patterns [4], ultimate strengths [6,7], and other heterogeneities can be seen for different material samples. Understanding the microstructure of these materials, and its influence on macroscopic fracture, greatly decreases the risk and uncertainty inherent in designs that involve quasi-brittle materials.

Certain material types, such as rock with bedding planes, can contain microstructural defects that are angularly biased, i.e., the microstructure is anisotropic. Depending on the direction of loading, the material strength can be very different. A major increase in risk can be assumed if the engineer only uses a homogeneous fracture strength for a given material when designing and modeling an engineering system. To capture these flaws, implicit or explicit methods may be used to introduce these defects into a given model. Explicit methods directly incorporate the microstructure, above certain length scale, into the solution scheme. Some examples are lattice modeling [8] and lattice discrete particle modeling [9], which model the flaws as a particle network connected by springs. Because of the small space and time scales required to properly resolve the microstructural defects in larger structures, explicit methods can be very computationally expensive.

Less computationally intensive implicit methods incorporate the overall effect of microstructure in the analysis, but do not directly represent them at the macroscale. An example of an implicit method is Weibull’s weakest link method [10,11], which provides a statistical phenomenological characterization of the fracture strength, and has been used successfully by the authors to capture the statistical fracture response of rock in hydraulic fracturing [12], fracture under dynamic compressive loading [13], and in fragmentation studies [14]. While these methods are generally successful in capturing statistical variation for specific properties, they lack the direct connection from the distribution of microstructural defects to macroscopic fracture response.

Therefore, a general method is desired which can link the material microstructure to the meso- and macrostructure that does not require the exorbitant computational costs of explicit methods. To resolve this issue, homogenization methods are used to link the material microstructure to the macrostructure by averaging the effect of the microstructure in a volume element (VE). Similar approaches have been used to calibrate certain fracture models, see, for example, Refs. [15–19]. A representative volume element (RVE), also known as a representative elementary volume in rock mechanics, is used in homogenization theories to compute bulk material properties of a composite at the continuum level. The
RVE should be much smaller than the macroscopic domain size so that the homogenized properties of the RVE can be assigned to a point of the continuum domain. Yet, RVEs should be large enough to be representative; that is, for a given homogenized property there is not much variation if larger VEs are used for homogenization. For a macroscopically homogeneous material with ergodic properties, an RVE is defined [20] such that the homogenized properties only vary within a specific range across different realizations of the RVE [21–24]; see also [25,26] for the statistical models that underlie the determination of RVE size in preceding references.

A volume element which is smaller than the given size requirements to form a representative volume element is known as a statistical volume element (SVE). These elements may be used to capture the statistical variation in material properties caused by the material microstructure. Multiple SVEs may be used to properly characterize the randomness in material properties. Many studies have been performed to determine how the SVE observation window [27,28], boundary conditions [29], and microstructure defect clustering [30] all effect the given material property statistics. SVEs have been used to capture both the material elastic and fracture properties.

We have used SVEs to homogenize microcracked rocks in Ref. [31] to derive an inhomogeneous yet isotropic field for tensile strength. By computing angle-dependent fracture strengths at the SVE level, anisotropic rocks were homogenized in Ref. [32]. Herein, we extend the work presented in Ref. [32] by using the microcrack statistics of a real-world Yuen-Long marble sample from Ref. [33]. In addition, an index is introduced to determine the anisotropy of homogenized fracture strength fields. Finally, following Ref. [34], spatial and angular covariance functions of the strength fields are used to analyze how the strengths vary in space and as a function of loading direction, respectively.

Once the anisotropy of fracture strength is considered, there are two main approaches to incorporate the anisotropy into a macroscopic continuum model. First, a second-order microstructure tensor [35,36] in addition to commonly used invariants of the stress tensor is used to define a general bulk failure criterion. Second, in contrast to the aforementioned bulk failure models, well-known interfacial models such as Mohr–Coulomb or Hoek and Brown [37] are made angle-dependent [38–40] by assigning different tensile strengths or friction coefficients for different angles of loading.

In this work, the latter approach is adopted, such that for any potential angle of loading we characterize different fracture strength parameters. Linear elastic fracture mechanics (LEFM) theory is used to derive angle-dependent uniaxial tensile, uniaxial compressive, and shear strengths. The SVE analysis process and angle-dependent strength formulation are derived in Sec. 2. Then, isotropic and anisotropic microcrack-filled domains are generated using real microcrack distribution statistics from Yuen-Long marble [33] and the statistics of the strength fields are analyzed for isotropic and anisotropic domains. Finally, the asynchronous spacetime Galerkin finite element method [41] is used to analyze the fracture response of these domains with the applied random fracture strength fields.

2 Formulation

This section defines the statistical volume element approach for defining fracture strength with respect to varying loading angle in a quasi-brittle domain containing microcracks. Section 2.1 provides an overview for the anisotropic fracture analysis process in a domain with distributed microcracks using SVEs. The microcrack length and angle distribution will be detailed in Sec. 2.2. Then, Sec. 2.3 describes the process to calculate the mesoscopic angular-dependent fracture strength field. Some useful angle-independent strength measures are introduced in Sec. 2.4. Afterward, a description of the covariance function and its use to determine the correlation of the fracture strength random fields is provided in Sec. 2.5. Finally, the asynchronous spacetime discontinuous Galerkin (aSDG) method is detailed in Sec. 2.7 for performing macroscopic dynamic fracture simulations within the given quasi-brittle domains.

2.1 Multiscale Anisotropic Fracture Analysis. Figure 1 details the multiscale model that is described in this work. In step one and at the macroscale, an RVE of the given material is considered. In Ref. [33], many samples of Yuen-Long marble were scanned to calculate the statistics of the material microstructure. A 32 mm by 32 mm square domain of Yuen-Long marble was determined to contain a sufficient number of microcracks to be considered representative. The statistics of microcracks demonstrate very low sample to sample variability at this scale. As the size of the RVE decreases, the RVE approaches the SVE regime. The characteristic size of the SVE, $l_{SVE}$, is smaller than the overall size of the domain of interest and the ratio of the SVE size to the average microcrack length must be small enough that SVE does not approach the RVE limit. As the ratio of $l_{SVE}$ to the average microcrack length approaches infinity, randomness is lost, resulting in a homogenous material that does not accurately represent the inhomogeneity at the mesoscale due to the existence of microcracks. Several SVE sizes will be studied in Sec. 3 to show this size effect.

Circular SVEs of diameter $l_{SVE}$ are used to transverse the RVE with spacing $S = L_{SVE}/n$, where $n$ is a grid line spacing variable which was chosen such that the entire sample is contained in sufficient resolution regardless of SVE size. For this paper, $n$ is equal to 5. Within each SVE, the contained microcracks are processed using LEFM principles, as described in Sec. 2.3. A loading $P$ is applied at discrete loading angles, $\theta$, and a field of mesoscopic fracture strength is calculated for each angle; cf. step 2 in Fig. 1. In step 3, the calculated stress strength values are assigned at the center points of the SVEs. In Fig. 1, sample uniaxial tensile strength fields are shown for four different angles at load $P$. As the SVE size is at an intermediate level between the representative crack and macroscopic domain sizes, the constructed fields are called mesoscopic strength fields. Finally, if desired, dynamic fracture analysis can be performed on the same macroscopic domain shown in step 1 without the explicit representation of microcracks. Instead, by using the mesoscopic fracture strength fields, the computational cost is significantly reduced; still, some level of material inhomogeneity is maintained, which is important for more realistic fracture simulations.

2.2 Microcrack Length and Angle Distributions. Since extensive sampling of rock is difficult or expensive, and detailed microcrack measurements for large domains are scarce, statistical methods are often used to realize microcracked samples that are consistent with the microcrack statistics of the original rock mass. Depending on the method used for realizing samples, different types of statistics of a microcracked domain are used; see Ref. [42] for a few examples. In the pick-and-place algorithm, cracks are modeled as discontinuity lines. The distributions of crack center location, length, and angle are sampled for each crack, and a crack with a specific location, length, and angle is inserted in the domain. This process continues until the target crack density $\rho_c$, which is the sum of squares of the microcrack lengths per unit area of the domain, is reached.

While the pick-and-place algorithm can be employed for different microcrack statistics, in this manuscript we use the specific microcrack statistics provided for a Yuen-Long marble sample in Ref. [33], given that all specific descriptors required by this method are experimentally measured and reported therein. Specifically, we employ the statistics of sample S19. The spatial location of the centers of microcracks follows a uniform distribution, as the examined Yuen-Long marble is macroscopically homogeneous. A power-law distribution is used, and experimentally verified, to represent the distribution of crack length in Ref. [33].
fact, as discussed in Ref. [43], field observations indicate that crack length often follows a power law distribution in rock. For the power-law distribution a minimum value (crack half-length) $a_0$ is assigned to ensure the cumulative distribution function to be finite and take the value of one for infinite crack half-length; cf. Eq. (2) below. The cumulative distribution for half-cracks longer than $a_0$ for an elemental volume $V_0$ is defined as

$$g(a) = \left(\frac{q^*}{a}\right)^z$$

where $q^*$ and Cauchy distribution exponent $z > 1$ are the parameters of the power distribution, subject to the condition

$$V_0 \int_0^\infty g(a) da = 1$$

To ensure this equality, the lower integration limit, $a'$ is defined by

$$a' = \left[\left(\frac{q^*}{a_0}\right)^z - 1\right]^{1/z}$$

The specific model values for sample S19 are reported in Sec. 3 and used to generate new random crack fields consistent with S19 microcrack length statistics. The S19 rock sample from Ref. [33] is isotropic and crack angle follows a uniform distribution. One objective of this study is relating the angle distribution of microcracks to anisotropy of the mesoscopically homogenized fracture strength fields, we create a second macroscopically anisotropic rock domain where the angle distribution is intentionally nonuniform. A triangular distribution with a peak at $\theta_c = 30$ deg and a range of $\theta_c = 25$–35 deg is used for this synthetic crack angle distribution. The final parameter for defining the microcrack domain is the crack density, where for both rock domains, the crack density of $e_0 = 0.243$ of sample S19 is used. It is noted that the statistics of microcracks plays a crucial role in the homogenized strengths at the mesoscale and macroscopic fracture properties. For example, we have demonstrated that by changing the shape of microcrack length distribution, while keeping the mean length fixed, macroscopic fracture strength can change by a factor of three [44]. Our focus in the remainder of the manuscript will be on the effect of statistics of microcracks on mesoscopic and macroscopic response, rather than validation of the macroscopic response with experimental results. Specific attention is directed to angular dependency of properties at different scales.

2.3 Fracture Strength Calculations. Within the SVE, the fracture strength of every microcrack intersecting the element is calculated. As detailed previously, a circular SVE observation window was selected. Cracks that are both completely encircled or only partially intersecting an SVE are considered. For SVEs which contain no microcracks, a maximum fracture strength is assigned based on a crack of minimum length. As all of the intersecting cracks fracture strengths are calculated, the minimum fracture strength is retained and assigned to the center point of the SVE, resulting in a mesh of fracture strengths with uniform spacing in both horizontal and vertical directions. Each mesh point contains a vector of angular-dependent fracture strengths, i.e., the fracture strength calculated for each discrete loading angle.

It is noted that a few SVEs may contain no cracks. For these SVEs, a maximum fracture strength is assigned based on a crack
of minimum length to avoid having an infinite strength. This minimum length is twice the value of minimum allowable half-length crack $a'$. For the power law distribution in Eq. (3). For the homogenization to be valid, the SVE size should be chosen large enough such that the majority of SVEs are representative of rock microstructure, in that they contain a sufficient number of microcracks. This size depends on the (length) distribution of microcracks and crack density. If the SVE size is too small, the maximum strength based on the minimum crack length is assigned to many SVEs. However, in this paper, the SVE sizes are chosen large enough that even for the smallest SVE size, very few SVEs contain no cracks.

For a crack of half-length $a$, the strength is calculated using LEFM principles. There are three assumptions in calculating fracture strengths. First, the interaction of microcracks is not taken into account. Second, for the microcracks that intersect the boundary of the SVE, the geometric mean of the total microcrack length minus the half-length $a$ of stress tensor for which the crack with $2a$ length is twice the value of minimum allowable half-length crack $a'$, is used to compute fracture strengths at angle $\theta$. For computing fracture strengths at angle $\theta$, in-plane stresses are applied in the $(x, y)$ coordinate system. To use LEFM theory, we need to transfer these stresses to the $(x', y')$ coordinate system. This is achieved by Mohr circle transformation of the in-plane components of the stress tensor between the two coordinate systems

$$
\sigma_{x'} = \sigma_{xx} \cos^2 \theta + \sigma_{yy} \sin^2 \theta + \sigma_{xy} \sin 2\theta
$$

$$
\sigma_{y'} = \sigma_{xx} \sin^2 \theta + \sigma_{yy} \cos^2 \theta - \sigma_{xy} \sin 2\theta
$$

$$
\sigma_{xy} = -\sin 2\theta \left( \frac{\sigma_{xx} - \sigma_{yy}}{2} + \sigma_{xy} \cos 2\theta \right)
$$

The strengths for angle $\theta$ correspond to the following stress tensors $\sigma = (\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$ in $(x, y)$ coordinate system:

\[\sigma := (\sigma_{N}, \sigma_{H}, 0) \quad \text{hydrostatic tension stress} \]

\[\sigma := (0, \sigma_{N}(\theta), 0) \quad \text{uniaxial tension stress} \]

\[\sigma := (0, -\sigma_{C}(\theta), 0) \quad \text{uniaxial compressive stress} \]

\[\sigma := (0, 0, \sigma_{S}(\theta)) \quad \text{shear stress} \]

where as mentioned $\sigma_{N}(\theta), \sigma_{C}(\theta)$, and $\sigma_{S}(\theta)$ are positive uniaxial tensile, uniaxial compressive, and shear stresses at angle $\theta$. Note that there is no argument of loading direction for hydrostatic tensile strength since it corresponds to equal principle stresses $\sigma_{N}$ for all angles of loading. We also observe that uniaxial tensile strength at angle $\theta$ corresponds to $\sigma = (0, \sigma_{N}(\theta), 0)$ rather than $(\sigma_{N}(\theta), 0, 0)$; for this strength at angle $\theta$, we apply a tensile stress normal to the direction of a hypothetical fracture line with angle $\theta$. This corresponds to only $\sigma_{N}$ nonzero in Fig. 2. Finally in Eq. (5), the form of any of the loadings for angle $\theta$ is known, but the factor of stress tensor for which the crack with $2a_{\text{eff}}$ at angle $\theta$, satisfies an LEFM fracture criterion corresponds to the magnitude of the strength.

The resolvable components of traction $t = (t_n, t_d)$ acting on the crack plane are

\[t_n = \sigma_{eff} \cos \theta, \quad t_d = \sigma_{eff} \sin \theta \]

where $t_n$ and $t_d$ are the normal and shear components of $t$, $\sigma$ is the shear coefficient, and $\theta$ is the Macaulay positive operator. The reason that in Eq. (6) the positive operator acts on $-t_d$, is that for the shear stress to induce mode II fracture, the far field shear stress $\sigma_{eff}$ must be greater than the friction traction $k(-t_d)$. When under tensile loading, i.e., when $t_d \geq 0$, friction is zero $k(-t_d) = 0$. We model the crack in Fig. 2 in isolation undergoing far field normal stress $t_n$ corresponding to mode I fracture if $t_n > 0$. The resolvent shear stress $t_d$ results in a mode II stress intensity factor (SIF) if acting shear stress $\sigma_{eff}$ can overcome friction $k(-t_d)$, i.e.,

\[t_d > k(-t_d) \]

There are several criteria such as maximum circumferential stress criterion, MERR, and minimum strain energy density that can be used to determine whether $K_I$ and $K_{II}$ are large enough to result in propagation at the tips of the crack and at what direction relative to the crack the extension will propagate. For the MERR criterion, to investigate crack propagation at a potential kink angle $\psi$ relative to crack direction, stress intensity factors at the tip of the kink crack extension, $K_I(\psi)$ and $K_{II}(\psi)$, need to be computed from $K_I$ and $K_{II}$ in Eq. (7) [46]. The crack extends in a direction $\psi$ for which the energy release rate in terms of $K_I(\psi)$ and $K_{II}(\psi)$ is maximum and exceeds the fracture toughness of material [45]. One can show that the corresponding failure criterion in terms of $K_I$ and $K_{II}$ can very accurately be approximated by

\[K_I^2 + K_{II}^2 = K_T^2 \]

where $K_T$ is the critical mode I SIF. For plane stress condition, i.e., for samples with small thickness $B$, tri-axial stress state is less pronounced than plane strain condition. Thus, $K_T$ is larger than $K_{T,\text{c}}$, its corresponding value for plane stress condition. In general, $K_T$ can be obtained from $B$ and $K_{T,\text{c}}$ [47]. While MERR criterion is slightly less conservative than maximum circumferential stress criterion and minimum strain energy density criteria, the form of the approximate relation (8) greatly simplifies the determination.
of fracture strengths for a crack with relative angle \( \theta_s \) to the loading direction.

The process for computing any of the fracture strengths \( SNH \), \( s_y(\theta) \), \( s_c(\theta) \), and \( s_d(\theta) \) is summarized as follows. First, relative crack angle is computed from \( \theta_S = \theta - \theta_c \). As will be discussed later, for certain strengths and angles \( \theta_c \), one or both of \( t_y \) and \( t_c \) are negative and Eq. (8) cannot be satisfied for any magnitude of loading; that is, the given angle-dependent strength is infinite.

Following this process for the four different strengths, the angle-dependent strengths are obtained as:

\[
s_N H \sqrt{\pi \alpha_{eff}} = \frac{1}{K_c} \tag{9a}
\]

\[
s_y(\theta) \sqrt{\pi \alpha_{eff}} = \frac{1}{K_c} \cos \theta_s \tag{9b}
\]

\[
s_c(\theta) \sqrt{\pi \alpha_{eff}} = \begin{cases} 
1, & -\frac{\pi}{2} < \theta_s < 0 \\
\cos \theta_s (\sin \theta_s + k \cos \theta_s), & 0 < \theta_s < \frac{\pi}{4} \\
\infty, & \frac{\pi}{4} < \theta_s < \frac{\pi}{2} \\
\cos \theta_s (\sin \theta_s - k \cos \theta_s), & \frac{\pi}{2} < \theta_s \leq \frac{\pi}{4} \\
1, & \frac{\pi}{4} < \theta_s \leq \frac{\pi}{2}
\end{cases} \tag{9c}
\]

\[
s_d(\theta) \sqrt{\pi \alpha_{eff}} = \begin{cases} 
1, & -\frac{\pi}{2} < \theta_s < 0 \\
\frac{1}{\cos 2\theta_s - k \sin 2\theta_s}, & 0 < \theta_s < \frac{\pi}{4} \\
\frac{\sin \theta_s}{\cos \theta_s}, & \frac{\pi}{4} < \theta_s < \frac{\pi}{2} \\
1, & \frac{\pi}{4} < \theta_s < \frac{\pi}{2}
\end{cases} \tag{9d}
\]

where \( \phi := \tan^{-1}(k) \) is the friction angle. Also, note that the relative crack angle \( \theta_s \) is taken in \((-\pi/2, \pi/2] \), given that shifts of \( \pi \) in crack angle do not change the problem description in Fig. 2. Based on Eq. (9a), the maximum (hydrostatic tensile strength, \( NH \), corresponding on minimum half-length crack angle \( \theta_s = 0 \).

The fracture strengths in Eq. (9) are shown in Fig. 3 for \( k = 0.3 \). As expected, the hydrostatic tensile strength is angle-dependent and takes the value \( K_c/\sqrt{\pi \alpha_{eff}} \). The uniaxial tensile strength takes the lowest value when the loading is normal to the crack, that is, for \( \theta_s = 0 \). As \( \theta_s \rightarrow \pi/2 \), a much higher normal stress in \( y \) is needed to result in crack propagation through Eq. (8). For the compressive strength, the resolvent \( t_y \) is always negative and the friction term in Eq. (6b) is nonzero. Under this loading, crack can propagate only in mode II if \( |t_c| \geq 0 \); cf. Eq. (7b). For relative crack angles \( |\theta_s| \leq \phi \), the friction traction \( k(-t_y) \) is higher than \( |m_{y'}| \). Consequently, crack surfaces cannot slip and have the excess shear stress resulting in mode II crack propagation. For \( |\theta_s| > \phi \), slip can occur. The maximum uniaxial compressive strength of \( 2 \cos \phi K_c/(1 - \sin \phi) \sqrt{\pi \alpha_{eff}} \) is achieved for \( |\theta_s| = \phi/2 + \pi/4 \). As \( |\theta_s| \rightarrow \pi/2 \), \( s_c(\theta) \) tends to infinity.

The only strength whose dependency is not even with respect to \( \theta_s \) is \( s_d(\theta) \). When \( \theta_s = -\pi/2 \), the stress state \( 5d \) results in \( \sigma_{y'} = s_N (\theta_s = -\pi/4) \) and \( \sigma_{c'} = 0 \) in Eqs. (4b) and (4c). Thus, \( t_y = s_N (\theta_s = -\pi/4) \) and \( t_c = 0 \) in Eq. (6) and the crack is in pure mode II; cf. Eq. (7). On the other hand, for \( \theta_s = \pi/4 \) we obtain \( t_y = -s_N (\theta_s = \pi/4) \) and \( t_c = 0 \). That is, \( \theta_s = \pi/4 \) and \( \pi/4 \) correspond to pure tension and compression on crack surfaces. Clearly, in the former case, the strength is equal to uniaxial strength, and in the latter case, the crack cannot propagate. This explains the nonsymmetric dependency of \( s_y \) on \( \theta_s \). In short, for \( \theta_s < 0 \), the resultant \( t_y \) is positive and crack can propagate in mixed mode. For \( \theta_s > 0 \), \( t_y < 0 \) and crack can only propagate when resolvent shear stress is positive, that is tangential stress \( \sigma_{c'} \) is larger than friction term \( k(-t_y) \), in Eq. (6b). This occurs for \( \theta_s > \pi/2 \).

For a given SVE, the hydrostatic tensile strength \( S_{NH} \), and angle-dependent uniaxial normal \( S_y(\theta) \), uniaxial compressive \( S_c(\theta) \), and shear \( S_d(\theta) \) strengths are defined as

\[
S_{\theta}(\theta) = \min_{\theta \in \mathbb{R}^+} S_i(\theta) \tag{10}
\]

where \( \theta \) refers to one of the strength modes \( NH, N, C, \) and \( S \). The minimum of strength is taken over microcracks with index \( i \) over \( T_{SVE} \), the set of all microcracks that are in or intersect the SVE; cf. Fig. 1(b). That is, for a given loading direction and strength type, the SVE takes the minimum of the corresponding strength of all the cracks that interact with the SVE.

There are three assumptions in using Eq. (10) as follows. First, the value of \( S_{\theta}(\theta) \) is capped by \( S_{\text{max}} \), the maximum fracture strength based on the value of \( \theta_s \). Since in Eq. (9) some of these strengths are infinite or tend to infinity, depending on the angle \( \theta_s \), for some SVEs \( S_{\theta}(\theta) \) may take a very large value even if it contains some microcracks. Using the maximum value \( S_{\text{max}} \) basically denotes an infinite value for \( S_{\theta}(\theta) \), corresponding to a situation that none of the microcracks can propagate for the given angle of loading. This is specifically relevant to the anisotropic rock domain as discussed in Sec. 3.2. Second, it is assumed that all these microcracks are effectively in an infinite domain. This is a reasonable assumption for the majority of SVEs, if the SVE size is sufficiently smaller than the macroscopic domain size. Third, microcrack interaction is not taken into account. Generally, microcrack interaction reduces the strength. In many studies, crack interaction is taken into account by reducing the strength through a nonincreasing function of crack density in the form \( p(y) \gamma(S) \); see, for example, Refs. [33] and [48]. No microcrack interaction model is employed in this study. This is because we are more interested in spatial variation of strengths, averaged by using SVEs, rather than their absolute values. Moreover, these crack interaction models often can analytically be computed only for simplistic microcrack distributions such as periodic array of parallel cracks with the same size. We believe that full-scale finite element analysis should be employed to accurately model crack

\[\text{Fig. 3 Different fracture strengths for the crack in Fig. 2 as a function of relative crack angle } \theta_s \text{ for } k = 0.3\]
interaction for more general distribution of microcracks. However, the computational cost of this approach is prohibitive herein, given that an extremely large number of SVEs are analyzed to study the statistics of homogenized strengths.

### 2.4 Angle-Dependence of Fracture Strength

For an SVE, fracture strengths $S_x(\theta)$, for $x \in \{N, H, S\}$, are angle-dependent. Accordingly, we can define the following angle-independent measures for an SVE:

$$m(S_x) = \min_{\theta \in [0, \pi]} S_x(\theta)$$ (11a)

$$\bar{S}_x = \frac{\sum_{\theta \in [0, \pi]} S_x(\theta)}{\sin(\pi)}$$ (11b)

$$A_x = \frac{\bar{S}_x}{m(S_x)}$$ (11c)

where $\zeta$ stands for standard deviation. If angular dependency of fracture strength is not very high and there is not a specific bias with respect to a given angle, for example, rocks with bedding plane, the minimum SVE strength $m(S_x)$ over all angles of loading, can be considered as a reasonable angle-independent fracture strength for the SVE. The mean SVE strength, $\bar{S}_x$, is another (less conservative) alternative for this purpose. The anisotropy index, $A_x$, corresponds to the coefficient of variance of the angle-dependent strength $S_x$. For an SVE with isotropic strength $A_x = 0$. Conversely, it takes higher values as the angular variation of strength increases. In subsequent sections, we use $A_x$ as a measure of fracture strength anisotropy.

### 2.5 Covariance and Correlation Functions

We are interested in studying how the strength fields change spatially and angularly. The strength field $S_x$, $x \in \{N, H, S\}$, is a function of space-angle coordinate $\xi = (X, Y; \theta)$. The covariance function between two space-angle coordinates $\xi_a$ and $\xi_b$ for strength $S_x$ is denoted by $\text{cov}_{S_x}$ and is defined as

$$\text{cov}_{S_x}(\xi_a, \xi_b) = E[(S_x(\xi_a) - \mu_x)(S_x(\xi_b) - \mu_b)]$$

$$= E[(S_x(\xi_a)S_x(\xi_b) - \mu_x \mu_b)]$$ (12)

where $E$ is the mean value operator and $\mu_x, \mu_b$ are shorthands for means of $S_x$ at $\xi_a$ and $\xi_b$, that is, $\mu_x = E(S_x(\xi_a))$ and $\mu_b = E(S_x(\xi_b))$.

After the covariance function is calculated, the correlation of the two variables can then be calculated. The Pearson correlation coefficient, referred to as Pearson’s $r$, is used to calculate how correlated the two variables are. If $r$ is equal to $+1$, then the two variables are completely positively linearly correlated, while if equal to $-1$ the variables are completely negatively linearly correlated. An $r$ value equal to $0$ means that the two variables are not linearly correlated at all. The correlation function for strength $S_x$ between its values at space-angle coordinates $\xi_a$ and $\xi_b$ is denoted by $\text{corr}_{S_x}(\xi_a, \xi_b)$ and is defined by

$$\text{corr}_{S_x}(\xi_a, \xi_b) = \frac{\text{cov}_{S_x}(\xi_a, \xi_b)}{\sqrt{\text{var}(S_x(\xi_a)) \text{var}(S_x(\xi_b))}}$$ (13)

where $\zeta$ refers to the standard deviation and $\zeta_a, \zeta_b$ are shorthands for the standard deviation of $S_x$ at points $\xi_a$ and $\xi_b$, respectively. That is, $\zeta_a = \sqrt{\text{var}(S_x(\xi_a))}$ and $\zeta_b = \sqrt{\text{var}(S_x(\xi_b))}$.

These two covariance and correlation functions are then used to determine how fracture strength fields are related in space and in angle. Homogenizing random fields that are consistent with material microstructure, in this context, is of crucial importance for the resulting stochastic partial differential equations for elastodynamic fracture problem. Investigating how these fields are related provide an interesting look at understanding the fracture strength fields for a given material.

### 2.6 Calculation of Mesoscopic Field Statistics

The statistics of fracture strengths, homogenized at the mesoscale, depend on the statistics of microcracks at the microscale and the size of SVE. As mentioned in Sec. 2.2, two distinct isotropic and anisotropic microcrack statistics are used at the microscale. We assume that the mesoscopically homogenized fields are strongly stationary and that the probability density function (PDF) of $S_x(\xi), \xi = (X, Y; \theta)$, does not depend on its spatial coordinate $(X, Y)$ for all $x \in \{N, H, S\}$; this condition is also referred to as weakly homogeneous in homogenization field. This implies that pointwise statistics such as mean $E(S_x(\xi))$ and variance also only depend on $(X, Y)$. Furthermore, higher-order moments such as covariance function will depend on relative spatial distance of the points, not their absolute spatial position. Thus, in Sec. 3 when the PDFs of $S_x$ are presented, only their angular argument is maintained. Moreover, the spatial dependence of the covariance function in Eq. (12) is only through $\Delta X = X_b - X_a$ and $\Delta Y = Y_b - Y_a$, the spatial difference of $\xi_a$ and $\xi_b$, and the function can be expressed as

$$\text{cov}_S(\Delta X, \Delta Y; \theta_a, \theta_b) = \text{cov}_S(0, 0; \xi_a, \xi_b)$$

$$= \text{cov}_S(X_a, Y_a, \theta_a; X_b, Y_b, \theta_b)$$ (14)

This will be used in reported covariance functions in Sec. 3.3.

The second assumption for the mesoscopic fields is their ergodicity, in that the statistical properties can be deduced from a single, sufficiently large realization of these fields. That is, for either of the two isotropic and anisotropic microcrack models, only one rock domain is realized and subsequently homogenized by the SVEs. This simplifies the computation of PDFs of $S_x(0)$ in Secs. 3.1 and 3.2, as the spatial domain PDF of $S_x$ over that single realization is computed and used in lieu of ensemble PDF of $S_x(0)$ at a fixed spatial location over many realizations. Moreover, the assumption of the ergodicity for the first moments is used to similarly compute covariance function $\text{cov}$ in Eq. (14) by moving the base point in one realization, rather than computing the function across many realizations. This assumption is used to compute the covariance (and correlation) functions in Sec. 3.3.

The spatial stationary condition assumption is justified by the fact that the spatial location of microcracks in sample S19 is reported to follow a uniform distribution [33]; as discussed in Sec. 2.2, this would result in a uniform spatial distribution of crack center-points in the pick-and-place algorithm. In addition, if the randomized rock domain is large enough, based on the details of domain realization scheme and statistics of microcracks, the statistics of mesoscopic fields homogenized by this realization should be representative of the ensemble statistics; this has been the rationale for assuming ergodicity condition for the pointwise PDF and covariance function. Stationarity and ergodicity assumptions are often used in geo-statistics [49] and homogenization [22] fields. It is noted that the weaker wide-sense stationarity, i.e., second-order stationarity, is also used in homogenization, e.g., Refs. [20] and [23], where only the mean value and covariance function are stationary. While strong stationarity and the aforementioned ergodicity conditions are used and justified for this study, we emphasize these conditions can be violated in many practical applications; see, for example, the discussion on the stationary condition for geo-masses in Ref. [50].

### 2.7 Asynchronous Spacetime Discontinuous Galerkin Method

The aSDG finite element method formulated for elastodynamic [41] is used for the analysis of domains with mesosopic inhomogeneous and anisotropic fracture strength fields. The method utilizes discontinuous basis functions across all element boundaries and directly discretizes spacetime using nonuniform grids that satisfy a special causality constraint; the Tent Pitcher
The isotropic domain has a uniform crack distribution of $\theta$, in $[0, 360]^{\circ}$ deg. The SVE analysis process was used to process the microcrack domain to develop mesoscopic strength fields using multiple SVE sizes: $L_{SVE} = 1, 2, 4, \text{ and } 8 \text{ mm}$. Since fracture strengths are assigned at the centers of the SVEs, the edge size of the square region for which fracture strength is assigned is $40 - L_{SVE} \text{ mm}$, with the smallest value of $32 \text{ mm}$ for $L_{SVE} = 8 \text{ mm}$. This explains why larger $40 \times 40 \text{ mm}^2$ domains are considered in Fig. 4, so that the smallest edge size of the mesoscopic fracture strength fields matches the $32 \text{ mm}$ edge size of the VEs in Ref. [33]. From here on, all homogenized mesoscopic strength fields are shown in a square with $32 \text{ mm}$ edge size.

The loading angle $\theta$ was varied between $[0, 180]^{\circ}$ deg with a discrete angular spacing of $2^{\circ}$ within this range, resulting in 90 discrete loading angle strength fields. The minimum uniaxial tensile, uniaxial compressive, shear, and hydrostatic tensile fracture strengths were calculated for every crack in the domain, with the minimum strength retained for each SVE.

At a glance at Fig. 6, there is no discernable trend showing a change of strength depending on the loading angle for an isotropic domain. By calculating the statistics of the mesoscopic strength field, the PDF of the strength field at each discrete loading angle can be compared. From Fig. 7, the PDF for each loading angle confirms that general strength of the domain does not change depending on the loading angle, as there is no bias in the direction of microcracks.

The effect of SVE size on homogenized strengths is shown in Fig. 8. As the SVE size increases, the (spatial) variation of homogenized $S_N$ and its corresponding mean value decrease. This is the well-known size effect for quasi-brittle materials [3, 5]. As $L_{SVE}$ increases, the SVE tends to the RVE limit, explaining the decrease in the variation of homogenized strengths. Also, the likelihood of containing larger cracks increases as $L_{SVE}$ increases. This explains the decrease in the overall values of $S_N$. Again, the PDF of the strength field shown in Fig. 9 agrees with this analysis. As the SVE size increases, the mean strength of the field decreases and the standard deviation of the response becomes smaller. It is noted that if $L_{SVE}$ were chosen sufficiently smaller than $1 \text{ mm}$, many SVEs would contain no cracks and would be assigned the strength $S_{N\text{max}}$, as described in Sec. 2.3. This would result in a bimodal PDF for $S_N$, see, for example, Fig. 9 in Ref. [32].

We use $A_N$ to study the anisotropy of the homogenized strength field $S_N$. A field of anisotropy index is constructed by assigning the value of $A_N$ for each SVE at its centroid. While the measure of anisotropy is defined for all angle-dependent strengths in Eq. (11c), the results are only presented for $S_N$.

Figure 10 shows the spatial distribution of $A_N$ for different SVE sizes. While for easier comparison of results, the maximum limit of 1 is used in all contour plots, $A_N$ can be arbitrarily high. Since the distribution of crack angle is uniform in $[0, 360]^{\circ}$ deg, we expect the macroscopic response of the domain to be isotropic. That is, $A_N$ should tend to zero at all points as $L_{SVE}$ increases toward infinity. This aspect can be observed in the figure. For the smallest SVE size in Fig. 10(a), $A_N$ varies with almost the same high spatial frequency that $S_N$ fields varies in Fig. 6. There are several islands of high anisotropy measures that are caused by highly anisotropic distribution of microcrack angles in these regions. Some of the zones with the highest values of $A_N$ are framed in the figure. For larger SVE sizes in Figs. 10(b)–10(d), the intensity of the anisotropy index at the peak points decreases and the field is generally smoothened. The same argument that is
used in the decrease in fracture strength in Figs. 8 and 9 for larger SVE sizes applies here; for this macroscopically isotropic material, the angle distribution of the most critical (longest) cracks within an SVE tends to its macroscopic uniform distribution of [0, 360] deg. The spatial variation of this field also decreases, again, because the SVEs tend to the RVE limit. However, for $L_{SVE} = 8\text{ mm}$, the same islands of high $A_N$ persist, even though their intensity and sharpness have significantly decreased compared to those in Fig. 10(a).

### 3.2 Anisotropic Domain

The anisotropic domain was generated with a triangular distribution in crack angle, with a peak of $\theta_c = 30\deg$ and a range of $[25, 35]\deg$; cf. Fig. 5(b). Therefore, the lowest fracture strengths should be for loading angles close to $\theta = 30\deg$, since this loading angle will effectively be pulling the cracks open. The maximum strength of SVEs should be close to $\theta = 120\deg$, since this is pulling the cracks almost parallel to their direction.

By viewing Fig. 11, the differing trend in strength based on loading angle can be seen. Loading angles of $\theta = 120\deg$ creates a uniform, homogeneous high strength field equal to $S_{\text{max}}$, corresponding to minimum crack length $2a' = 0.015\text{ mm}$; since most cracks are almost parallel to loading angle, i.e., $\theta_d \approx \pi/2$, $s_N(\theta)$ tends to infinity in Eq. (9b). At 90 deg apart from this, $\theta = 30\deg$, a nearly uniform low-strength field is observed in Fig. 11(a). This trend is confirmed by the PDF of the strength field by varying the loading angle, as shown in Fig. 12. As the strength gets closer to the $\theta$ of minimum strength for the anisotropic domain, the mean strength steadily decreases. When comparing the isotropic PDF shown in Fig. 7 to the anisotropic PDF shown in Fig. 12, the effect of anisotropic crack distributions can be shown clearly. An interesting note to this is that for anisotropic domains, the actual strength of the field can be much higher over a wide range of loading angles compared to isotropic domains. Understanding the anisotropy of a given material can be crucial to whether a given material could be used for a specific application and loading condition.

Figure 13 shows the spatial distribution of the anisotropy measure for the anisotropic domain for different SVE sizes. Since microcrack angles are biased around $\theta_c = 30\deg$, this domain is anisotropic even when the VE size tends to infinity; the lowest and highest strengths are expected at angles $\theta_d = 30\deg$ and $\theta_d = 120\deg$, respectively. Overall, as the SVE size increases, the spatial frequency of variations of $A_N$ decreases and the field becomes more uniform. This is because the SVE tends to the RVE size limit.

Another observation is that the averaged fracture strength field $S_{\text{av}}$ is less anisotropic for smaller SVE sizes. As the $L_{SVE}$ increases, there are more microcracks contained in an SVE and very low strengths are expected for $\theta_d$ in $[25, 35]\deg$, particularly for the mode of the crack angle distribution $\theta_d = 30\deg$. This results in a highly angle-dependent, yet more spatially uniform distribution for $S_{\text{av}}$, reflected in more uniform and higher values for $A_N$ in Figs. 13(c)–13(d). In contrast, for smaller SVEs, there is

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**Fig. 4** Generated microcrack domains: (a) isotropic domain and (b) anisotropic domain

**Fig. 5** Generated microcrack domain crack PDFs: (a) length $2a$ (mm) and (b) $\theta_c$ (deg)
a higher variability in angles within [25,35] deg that have the longest cracks, thus the lowest strengths, within the SVEs. Thus, the averaged strength field is less anisotropic, yet more spatially variable due to higher variability of the statistics of microcracks within the SVEs. The frames regions in Fig. 13 demonstrate that, similar to Fig. 10, the regions of higher anisotropy are preserved and averaged as the SVE size increases.

To represent this difference in the isotropic and anisotropic domains, four SVEs were selected based on the measure of anisotropy, \( \kappa_N \). Four points, chosen from the lowest measure of anisotropy to highest in the isotropic domain, are shown in Figs. 10(a) and 13(a). While the two domains do not have the same microcrack distribution and are unrelated, the same points were used for both isotropic and anisotropic domain. P2 in the isotropic domain was selected such that it corresponded to the same coordinates as the location of the highest anisotropy in the anisotropic domain. The coordinates of each point, measure of anisotropy \( \kappa_N \), and mean strength \( S_N \) are shown in Table 1, and how the respective uniaxial strength field \( S_N(\theta) \) changes in angle is shown in Fig. 14.

For the isotropic domain, as \( \kappa_N \) increases, the strength field becomes more anisotropic, dominated by one angle of the highest and lowest strength. The standard deviation increases such that the peak and minimum strength are further away from the given mean strength. For the isotropic domain, each SVE contains a different angle of maximum and minimum strength. The measure of anisotropy is a good indicator for the relative dominance of one crack angle contained in that SVE.

The anisotropic domain contains a dominant angle of maximum strength no matter the level of anisotropy. In the anisotropic domain, P2 contains the highest measure of anisotropy. As is expected, the width of the angular band of maximum strength for P2 is also the narrowest, as shown in Fig. 14(b). Generally, while the angle of maximum and minimum strength is roughly the same for all SVEs in the anisotropic domain, the measure of anisotropy indicates how narrow the band of maximum and minimum strength is, which is caused by the SVE only containing cracks within a narrow angular distribution.

In short, for the isotropic domain, measure of anisotropy is higher for smaller SVEs and it tends to zero for larger SVEs, as shown in Fig. 10. In contrast, for the anisotropic domain, \( \kappa_N \) tends to a finite high value for larger SVEs and it takes lower values for smaller SVEs. For both cases, the homogenized strength fields are more inhomogeneous for smaller SVE sizes.

![Fig. 6 S_N(\theta) mesoscopic isotropic strength fields (L_SVE = 1 mm) at various \( \theta \): (a) \( \theta = 30 \) deg, (b) \( \theta = 60 \) deg, (c) \( \theta = 90 \) deg, and (d) \( \theta = 120 \) deg](image-url)

![Fig. 7 Isotropic S_N(\theta) PDF for varying \( \theta \)](image-url)
material inhomogeneity is desirable in fracture analysis, the smaller SVE size
$L_{SVE} = 1$ mm is used to produce mesoscopic strength fields for the macroscopic simulations in Sec. 3.4.

3.3 Covariance and Correlation. We first study the dependence of the covariance and correlation functions on space and angle coordinates for $S_N$. Next, the correlation between different angle-independent strengths derived from $S_N, \pi \in \{NH, N, C, S\}$ is investigated. In the following, we use the assumed stationarity and ergodicity discussed in Sec. 2.6 to compute the covariance and correlation functions based on only the single realization for each rock model, and to express the covariance function as a function of spatial distance of the two points; cf. Eq. (14).

3.3.1 Spatial Correlation. We first analyze the form of covariance function in space. That is, $\xi_A$ and $\xi_B$ are chosen such that the same loading angle is chosen for both, meaning that $\theta_A = \theta_B$. For this analysis, we use the strength field $S_N$ homogenized by $L_{SVE} = 1$. Moreover, $\theta_A = \theta_B$ is set to zero to focus on spatial correlation of uniaxial tensile strength for zero degree loading direction. This spatial covariance function of $S_N$ for zero loading direction is only a function of change of spatial coordinate $(D_X, D_Y)$ and is denoted and defined by $\text{cov}_X \equiv \text{cov}_N(D_X, D_Y; 0, 0)$. This function can be evaluated from a single homogenized strength field by moving the space position of $n_A$ and keeping the spatial difference of $n_B$ and $n_A$ fixed and equal to $(D_X, D_Y)$.

Figure 9 shows $\text{cov}_X$ for both isotropic and anisotropic microcrack statistics. Several observations can be made. First, from the circular contour lines, it is evident that the covariance function is only a function of the relative distance of the two points and not their relative angle. That is the spatial covariance beyond being homogeneous (depending on only on relative spatial difference of $\xi_A$ and $\xi_B$) is also isotropic (not being a function of relative spatial angle between $\xi_A$ and $\xi_B$). Second, we observe that the spatial covariance function quickly tends to zero in a spatial range roughly equal to the SVE size. This corresponding to traversing about 5 SVEs as $n$ is chosen equal to 5; cf. Sec. 2.1. Third, the spatial covariance function has an almost identical form for both isotropic and anisotropic microcrack distribution models.

3.3.2 Angular Correlation. Next, we analyze the form of the covariance function in angle. That is, $\xi_A$ and $\xi_B$ share the same spatial location $(X, Y)$, but have arbitrary angles $0 \leq \theta_A, \theta_B < \pi$. Again, considering $S_N$ for $L_{SVE} = 1$, this covariance function is...
defined and denoted by $\text{cov}_{\theta}(\theta_A, \theta_B) = \text{cov}_{\theta}(0, 0; \theta_A, \theta_B)$. Similar to the calculation of $\text{cov}_{X,Y}$, the spatial stationarity of homogenized strength fields is used to populate the statistics for $\text{cov}_{\theta}$ by moving $(X, Y)$ in a realization of $S_N$. Several observations are made for $\text{cov}_{\theta}$ of the isotropic domain in Fig. 16(a). First, the value of the function along the diagonal line $\theta_A = \theta_B$ corresponds to $\text{cov}_{\theta}(0, 0)$, the variance of the uniaxial tensile strength for angle $\theta$. Since the orientation of microcracks is isotropic (uniform in $[0, \pi]$), we expect the statistics of $S_N(X, Y, \theta)$ to be stationary in $\theta$ as well, that is, independent of the angle of loading. Consequently, the variance of $S_N(\theta)$ too should be independent of $\theta$, meaning that the value of $\text{cov}_{\theta}(0, 0)$ on diagonal $\theta_A = \theta_B = \theta$ should be constant. The small variations of $\text{cov}_{\theta}$ on this diagonal in Fig. 16(a) are contributed to numerical errors associated with the finite set of values used in evaluating the covariance function.

Second, again from the stationary condition of $S_N$ in $\theta$, $\text{cov}_{\theta}(\theta_A, \theta_B)$ should only be a function of the relative difference of the angles $\theta_B - \theta_A$ rather than their individual values. Since $S_N(X, Y, \theta)$ is periodic in $\theta$ with period $\pi$, $\text{cov}_{\theta}(0, 0)$ is a periodic function with the solution in $(0, \pi) \times (0, \pi)$ (the part shown in the figure) repeating in 2D $(\theta_A, \theta_B) \in \mathbb{R}^2$ angle space. Considering these facts, the value of the function only depends on the distance of the point $(\theta_A, \theta_B)$ from the diagonal $\theta_A = \theta_B$ line. Again, the small deviation of $\text{cov}_{\theta}$ from this condition in Fig. 16(a) is a consequence of the finite number of points used for computing the covariance function. Finally, we observe that the covariance function tends to zero for an angle difference of about $|\theta_B - \theta_A| = 30$ deg.

The covariance function in angle for the anisotropic domain is shown in Fig. 16(b). Again, the value on the diagonal $\theta_A = \theta_B = \theta$ is the variance of $S_N(\theta)$. For the anisotropic domain, the highest variances of $S_N(\theta)$ are observed at angles about 10 deg outside the angles corresponding to the highest strength around 120 deg. The observed high variance is a consequence of stipulating a maximum strength corresponding to minimum allowable crack length $2a$; cf. Sec. 2.2. Overall, we observe a much more complex form for the covariance function of anisotropic domain compared to that in Fig. 16(a); specifically, due to the anisotropy of $S_N(0)$, $\text{cov}_{\theta}(\theta_A, \theta_B)$ is no longer only a function of the angle difference $|\theta_A - \theta_B|$. The angular correlation function, defined by $\text{corr}_{\theta}(\theta_A, \theta_B) = \text{corr}_{\theta}(X_A = X, Y_A = Y, \theta_A; X_B = X, Y_B = Y, \theta_B)$, is shown for isotropic and anisotropic rock domains in Fig. 17. As expected, strengths $S_N(X, Y, \theta_A)$ and $S_N(X, Y, \theta_B)$ are highly correlated as the angle difference tends to zero. In fact, for $\theta_A = \theta_B = 0$, by definition the correlation is one, given that the two strengths coincide. This corresponds to the value of one on diagonal line $\theta_A = \theta_B$ for both cases in Fig. 17. For isotropic rock, fracture strengths are highly correlated for an angle difference $|\theta_A - \theta_B|$ roughly less than 30 deg, and the correlation is almost zero for higher angle differences. The zero correlation of fracture strength for this high of an angle difference can be compared to zero correlation between the strength of two points that are roughly 1 mm apart in Fig. 15.

The angular correlation of $S_N$ for the anisotropic domain is vastly different; aside for a 10 deg range centered around 120 deg, the angle corresponding to the highest strengths, fracture strength of any two angle of loading are highly correlated. This is due to the fact that the majority of cracks are aligned close to 30 deg, and in approximate sense, $1/\cos(\theta - 30)$ is the factor that relates strength at angle $\theta$ to that for the 30 deg, i.e., the weakest strength; cf. Eq. (9b) and note that $\theta_B \approx \theta - 30$. Clearly due to the local variation of crack angle in the range $[25, 35]$ deg, there is not this perfect correlation between $S_N$ at two arbitrary angles; however, correlations close to one are observed in Fig. 17(b). The zero

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**Figure 10** Measure of anisotropy, $A_{SN}$, for the macroscopically isotropic domain for different SVE sizes: (a) $L_{SVE} = 1$ mm, (b) $L_{SVE} = 2$ mm, (c) $L_{SVE} = 4$ mm, and (d) $L_{SVE} = 8$ mm. The observed high variance is a consequence of stipulating a maximum strength corresponding to minimum allowable crack length $2a$; cf. Sec. 2.2. Overall, we observe a much more complex form for the covariance function of anisotropic domain compared to that in Fig. 16(a); specifically, due to the anisotropy of $S_N(0)$, $\text{cov}_{\theta}(\theta_A, \theta_B)$ is no longer only a function of the angle difference $|\theta_A - \theta_B|$. The angular correlation function, defined by $\text{corr}_{\theta}(\theta_A, \theta_B) = \text{corr}_{\theta}(X_A = X, Y_A = Y, \theta_A; X_B = X, Y_B = Y, \theta_B)$, is shown for isotropic and anisotropic rock domains in Fig. 17. As expected, strengths $S_N(X, Y, \theta_A)$ and $S_N(X, Y, \theta_B)$ are highly correlated as the angle difference tends to zero. In fact, for $\theta_A = \theta_B = 0$, by definition the correlation is one, given that the two strengths coincide. This corresponds to the value of one on diagonal line $\theta_A = \theta_B$ for both cases in Fig. 17. For isotropic rock, fracture strengths are highly correlated for an angle difference $|\theta_A - \theta_B|$ roughly less than 30 deg, and the correlation is almost zero for higher angle differences. The zero correlation of fracture strength for this high of an angle difference can be compared to zero correlation between the strength of two points that are roughly 1 mm apart in Fig. 15.

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correlation of high strength angle range and other angles is contributed to assigning $S_{\text{max}}$ for angles around 120 deg.

3.3.3 Correlation Between Different Strength Types.

Finally, we study the correlation between different fracture strength fields. To more effectively focus on the type of strength, rather than angle-dependent strengths $S_a(\theta)$, we use the two alternatives of SVE angle-independent strengths proposed in Sec. 11. The $\bar{S}_N$, $\bar{S}_S$, and $\bar{S}_C$ in Eq. (11a) provide an overall average angle-independent strength for an SVE. In contrast, $m(S_N)$, $m(S_S)$, and $m(S_C)$ are the most conservative choices by taking the minimum of strengths over all angles of loading in Eq. (11b). When for a given $L_{\text{SVE}}$, i.e., observation size, strength anisotropy is low, either of these two choices of strength can be used to define a unique isotropic strength for a given spatial location. For example, we have used the mean and min of uniaxial normal strengths for two different types of composites in Refs. [62] and [63], respectively. However, in these works only macroscopically isotropic materials were considered and the correlation between different strength types was not studied.

The Pearson correlation coefficient between these six derived angle-independent and inhomogeneous strength fields and the already angle-independent strength $S_{\text{NH}}$ are provided in Table 2 for the isotropic domain. Very high correlations are observed between $S_N$, $\bar{S}_N$, $\bar{S}_S$, and $\bar{S}_C$. The lowest value corresponds to a correlation of 0.847 between $S_{\text{NH}}$ and $\bar{S}_C$. This can be contributed to pure mode I and mode II fracture for hydraulic tensile and uniaxial compressive loadings, respectively. In addition, for a single crack $S_C$ is highly anisotropic, and even in the zone for which $S_C$ is finite, its value quickly tends to infinity; cf. Eq. (9c) and Fig. 3. The highly different fracture modes for a single crack explains the lower correlation between $S_{\text{NH}}$ and $S_C$ even after the operations (10) and (11b) are taken into account to derive these angle-independent SVE strengths.

However, if instead of using angular mean of strengths in an SVE, their minimum value is used, and a perfect correlation is observed between $S_N$, $m(S_N)$, $m(S_S)$, and $m(S_C)$. By inspecting Fig. 3 and Eq. (9), we observe that the angular minimum of all strengths $s_N$, $s_N$, $s_S$, and $s_C$ is the same. Thus, from Eqs. (10) and (11a), for all $\alpha \in \{N, H, C, S\}$, $m(s_{\alpha}) = K_c/\sqrt{a_{\text{max}}}$, where $a_{\text{max}}$ is the maximum half-length of a crack within an SVE. That is, $S_N$, $m(S_N)$, $m(S_S)$, and $m(S_C)$ are all equal, explaining their perfect correlation.

**Fig. 11** $S_N$ mesoscopic anisotropic strength fields ($L_{\text{SVE}} = 1$ mm) at various loading angles: (a) $\theta = 30$ deg, (b) $\theta = 60$ deg, (c) $\theta = 90$ deg, and (d) $\theta = 120$ deg

**Fig. 12** Anisotropic $S_N$ PDF for varying $\theta$
From the preceding comparison of $S_a$ and $m(S_a)$, it appears that if one aims to simplify the fracture strength fields by eliminating their angular-dependency, the mean value is a more appropriate choice for microcracked domains as it maintains the fundamental differences between different types of strength. Also, as elaborated in Ref. [62], the use of angular minimum strengths can be too conservative.

The correlation coefficients for the anisotropic domain are provided in Table 3. As explained, the correlation between all minimum strengths and $S_{NH}$ is one. For the mean strengths, the correlations are considerably higher than those for the isotropic domain, with a minimum value of 0.967. Similar to high correlation of strength between different angles of loading in Fig. 17, this can be contributed to large percentage of cracks being oriented close to 30 deg and more or less constant factors between different fracture types in Eq. (9) for $h_{c/30}$.

From the comparison of the results for isotropic and anisotropic domains, it is suggested that eliminating fracture strengths $S_S$ and $S_C$ in a failure criterion and maintaining only $S_N$ is a more sensible choice for anisotropic domains. Under such conditions, the other strengths can be approximated as constant factors of uniaxial tensile strength field, without introducing much error. This greatly simplifies several aspects of the proposed multiscale approach in this manuscript, as we consider a single inhomogeneous strength field for $S_N$ in Sec. 3.4.

In this section, we studied how fracture strengths vary by changing the spatial location, angle of loading, and type of strength. Covariance functions can be used by the Karhunen–Loève [64,65] or other similar methods to generate a large number of random field realizations that are consistent with fields homogenized with a given $L_{SVE}$. This approach in general is faster than generating domains with an actual microstructure (i.e., the microcracks here) and using the SVEs to homogenize them. Thus, understanding the form of covariance function is of great importance. For the two microcrack distributions considered and for $L_{SVE} = 1$, the corresponding random fields are stationary in space. In addition, the spatial covariance function is spherical and tends to zero at a length scale below $L_{SVE}$. The angular covariance function is stationary in angle for the isotropic domain and tends to zero for loading angle differences greater than about 20 deg.

### 3.4 Dynamic Fracture Analysis

For fragmentation and other problems that lack macroscopic stress concentration points, maintaining material inhomogeneity is of great importance. We study the fragmentation of a $32 \times 32$ mm² square rock domain (same VE size used in Ref. [33]) under bi-axial tensile loading condition. We use the inhomogeneous fracture strength fields for $L_{SVE} = 1$ for both isotropic and anisotropic domains, to maintain the highest level of material inhomogeneity with the SVE sizes considered. Since $n = 5$, cf. Sec. 2.1, any of the fracture strength fields for $S_N(\theta)$ is represented by a $160 \times 160$ resolution material property mesh. The value at each grid point is equal to $S_N(\theta)$ obtained for the SVE centered at that point. Using a step size of 2 deg for $\theta$, 90 strength fields $S_N(\theta)$ are computed and stored for macroscopic fracture analysis for each of the two rock domains.

![Fig. 13 The measure of anisotropy, $A_{AN}$ for the anisotropic domain for different SVE sizes: (a) $L_{SVE} = 1$ mm, (b) $L_{SVE} = 2$ mm, (c) $L_{SVE} = 4$ mm, and (d) $L_{SVE} = 8$ mm](Downloaded from http://asmedigitalcollection.asme.org/risk/article-pdf/6/2/021008/6520233/risk_006_02_021008.pdf by University of Tennessee user on 10 September 2020)
Four of such meshes for isotropic and anisotropic domains are shown in Figs. 6 and 11, respectively.

Initial and boundary conditions were applied consistent with a uniform, isotropic tensile field that ramps linearly in time. The components of the displacement field \( (U, V) \) are

\[
U(X, Y, t) = \dot{e}_X t, \quad V(X, Y, t) = \dot{e}_Y t
\]

where \((X, Y)\) and \(t\) are space and time coordinates, and \(\dot{e}\) is the specified normal strain rates for both \(X\) and \(Y\) directions. Dirichlet boundary conditions, i.e., a prescribed velocity field, is applied on the entire boundary of the domain at all times. This resembles a boundary condition, i.e., a prescribed velocity field, is applied on Ref. [33]; that is, Young’s Modulus conditions were specified with material properties taken from Sec. 2.7.

Both rock domains are simulated up to time \(t = 12 \mu s\). A strain rate of \(\dot{e} = 20/s\) is used to gradually load them. A spatially uniform and temporally increasing stress response persists until the stress value reaches the minimum tensile strength of \(S_N(X, Y, \theta)\) over all spatial points and angles of loading, at location \((X_m, Y_m)\) and for the angle of loading \(\theta_m\). At this instant, a crack is nucleated at \((X_m, Y_m)\) and propagated along the direction \(\theta_m\). The uniform stress field no longer persists past this instant, and more cracks are nucleated and/or propagated until the square domain completely fails.

We use homogenization boundary integrals [66] to obtain macroscopic strain \(E\) and stress \(\Sigma\) tensors for each time \(t \in [0, 12 \mu s]\). Figure 18 compares the response of the two rock domains in terms of their macroscopic strain versus stress response. The maximum attainable stresses of the isotropic domain are max \((\Sigma_{XX}) = 19.16 \text{ MPa}\) and max \((\Sigma_{YY}) = 18.39 \text{ MPa}\). The corresponding values for the anisotropic rock domain are max \((\Sigma_{XX}) = 21.82 \text{ MPa}\) and max \((\Sigma_{YY}) = 21.94 \text{ MPa}\). Thus, the anisotropic rock domain has a higher macroscopic strength under the multi-axial tensile loading condition. This is due to the fact that over the majority of angles, the fracture strength \(S_N\) is much higher than the isotropic domain other than a small angular range about the angle of weakest strength. At time \(t = 12 \mu s\), the isotropic domain has reached failure, but the anisotropic domain has not yet reached failure.

Another interesting feature is that for the anisotropic rock, aside from a small region early in macroscopic failure initiation stages, \(\Sigma_{YY}\) is lower than \(\Sigma_{XX}\) over a zone where significant macroscopic softening and unloading occurs in Fig. 18(b). Eventually, however, both stresses take almost the same values once they tend to zero past this zone. Given that for the anisotropic domain, the lowest strengths in the homogenized \(S_N(X, Y, \theta)\) field are for angles \(\theta \in (25, 35) \text{ deg}\); overall, the domain is expected to have a lower strength in \(Y\), compared to \(X\), direction. For the single realization of the microcracks with anisotropic angle statistics in Fig. 4(b), this happens for the range of strains discussed previously. We have repeated the fracture analysis of the same two rock domains under different loading rates \(e\). While the results are not presented for brevity, the same difference between \(\Sigma_{XX}\) and \(\Sigma_{YY}\) for the anisotropic domain is observed for other loading rates. In addition, maximum macroscopic stresses of the anisotropic
domain consistency remain higher than those of the isotropic domain.

The strain and kinetic energy densities of the two solutions are shown in Fig. 19 for $t = 12 \mu s$. The strain energy density is mapped to the color field, where light to dark color indicates a range of zero to 1000 J/m$^3$. The kinetic energy is mapped to the height field such that regions of high kinetic energy appear closer. Because the anisotropic domain has not reached failure, higher strain levels are shown in the domain. The isotropic domain shows cracks growing in several directions; however, the anisotropic domain shows the angularly biased crack field where, as expected, the majority of cracks are aligned close to 25–35 deg. The majority of cracks with other orientations connect three to four main cracks that are aligned with the weakest plane of the rock.

Figure 20 depicts the front meshes at the time of failure $t = 12 \mu s$ for the two rock domains. These meshes show the adaptivity of the aSDG method for propagating cracks and controlling solution errors. In addition, the level of interfacial damage parameter is indicated by blue (zero damage) to red (full damage) colors on crack surfaces. For both cases, the solution starts with a spatial triangulation containing no cracks. The effect of initial microcracks in the domain is modeled by anisotropic and inhomogeneous mesoscopic fields of $S_N$ as those shown in Figs. 6 and 11. As the cracks are nucleated with specified propagation directions, the front mesh is continuously adapted to align element boundaries with the specified directions. In addition, adaptive operations ensure that the numerical errors in the bulk and on fracture surfaces are below the user-specified tolerances. This is reflected in finer elements closer to the crack tips in the figure. As a result of these adaptive operations, the front meshes in Figs. 20(a) and 20(b) contain 23,891 and 26,615 triangles, respectively, compared to only 803 triangles in the initial front mesh at $t = 0$.

### 4 Conclusion

An upscaling approach was proposed in which an angle-dependent inhomogeneous fracture strength field was
homogenized from the distribution of cracks at the microscale. The mesoscopic strength field is in turn used for fracture simulations at the macroscale. The computational cost is deemed to be lower compared to direct numerical simulation of the material with explicit representation of its microstructure. The microcrack statistics of a real material, Yuen-Long marble, as described in Ref. [33], was used to generate cracks at the microscale. The proposed multiscale approach can be used to systematically propagate the statistics and randomness of a material at the microscale to uncertainties in its macroscopic failure response.

We use a mixed mode crack propagation criterion, while taking into account the frictional effect for mode II fracture, to derive angle-dependent uniaxial tensile, uniaxial compressive, and shear strengths for arbitrary orientations of the loading and a single crack. These strengths are used to derive corresponding angle-dependent strengths for circular SVEs, containing potentially multiple cracks. A 2D macroscopic domain is then traversed with overlapping SVEs in X and Y directions. The homogenized fracture strengths of each SVE are assigned to its center point. Two rock domains with isotropic and anisotropic distribution of microcracks were homogenized with different SVE sizes. Macroscopic fracture responses were consistent with the underlying statistics of microcracks. For example, for the anisotropic domain, the majority of macroscopic cracks were aligned with the weakest directions of the homogenized mesoscopic strength field.

The pointwise and two-point statistics of the random fields were analyzed. As the SVE size increased, the homogenized field became smoother and higher frequency local variations were lost. This corresponded to lower mean and variation of the homogenized strengths for higher SVE sizes. The spatial covariance function of the strength field was shown to only depend on the distance of two points. The angular covariance function of the isotropic rock should be a function of relative angular distance of two points. In contrast, this function was shown to be much more complex for the anisotropic rock domain.

The effectiveness of this method for homogenization and fracture simulation was demonstrated; however, there are several areas of improvement for future research. First, in this study only fracture properties were assumed to be anisotropic. For more realistic modeling, the elasticity tensor should also be considered inhomogeneous and anisotropic [67], and be homogenized by SVEs. Second, we only presented a single realization of a microcracked domain and its corresponding mesoscopic fracture strength field (for a given SVE size). Many realizations of the microcracked domain (or mesoscopic fracture strength fields) should be generated and analyzed to provide a statistical representation of macroscopic fracture response. Third, the one-point and
two-point statistics of the mesoscopic random fields, studied here, can be used to circumvent the microscale domain realization and analysis, and directly generate statistically consistent random fields at the mesoscale. Fourth, finite element analysis can be used to circumvent the microscale domain realization and analysis, and directly generate statistically consistent random fields, studied here, to two-point statistics of the mesoscopic random fields, studied here.

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References


