ADAPTIVE DISCONTINUOUS GALERKIN METHOD FOR ELASTODYNAMICS ON UNSTRUCTURED SPACETIME GRIDS

Reza Abedi*, Shuo-Heng Chung**, Yong Fan*, Shripad Thite**, Jeff Erickson**, Robert B. Haber*
*Department of Theoretical & Applied Mechanics, University of Illinois at Urbana-Champaign, 104 S. Wright St., Urbana, IL 61801, USA
**Department of Computer Science, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA

Summary We present an adaptive spacetime discontinuous Galerkin (SDG) method for linearized elastodynamics. The SDG method uses a simple Bubnov-Galerkin projection that delivers stable and oscillation–free solutions, with $\mathcal{O}(N)$ complexity and exact momentum balance on every spacetime element. An extended version of the Tent Pitcher algorithm generates unstructured spacetime grids that support simultaneous grading in space and time. We present results in 1D and 2D $\times$ time, emphasizing problems with shocks. [1]

INTRODUCTION

Effective numerical methods for elastodynamics are needed in diverse engineering applications, ranging from seismic exploration to automotive design for crashworthiness. Accuracy requires highly refined spatial and temporal discretizations when shocks are present, so the computational expense can be unacceptably large. We present an adaptive procedure for elastodynamic analysis based on a spacetime discontinuous Galerkin (SDG) finite element method [1] that delivers low-dissipation and oscillation-free solutions. The discontinuous SDG basis functions facilitate our adaptive implementation, since nonconforming grids and jumps in polynomial order are automatically supported. We use unstructured spacetime meshes and a direct patch–by–patch solution procedure with O(N) complexity. Extreme refinement is only required along the spacetime trajectory of shocks, and we are not forced to impose global constraints on the time step size. The patch–by–patch solution technique supports local adaptive operations. The following sections outline the SDG formulation, describe the adaptive spacetime meshing procedures, and present some numerical results.

SPACETIME DISCONTINUOUS GALERKIN FORMULATION FOR ELASTODYNAMICS

Consider an open spacetime analysis domain, $D \subset E^d \times R$, where d is the spatial dimension. Let $\mathcal{V}$ denote the space of admissible SDG displacement fields on D which are piecewise H^1, but which may suffer discontinuities across a collection of d–dimensional jump manifolds. The SDG formulation derives from the following Bubnov-Galerkin weighted residual statement. Find a spacetime displacement solution $\mathbf{u} \in \mathcal{V}$ such that

$$\int_Q \mathbf{w}\cdot\mathcal{L}\left(\mathbf{dM}+\rho\mathbf{b}\right) - \int_{\partial Q}\left\{\mathbf{w}\cdot\mathbf{i}\left(\mathbf{dM}+\rho\mathbf{b}\right)+\dot{\mathbf{w}}\cdot\left(\mathbf{M}^*-\mathbf{M}\right)+\left(\boldsymbol{\varepsilon}^*-\boldsymbol{\varepsilon}\right)\wedge\mathbf{i}\hat{\mathbf{M}}\right\} + \int_{\partial Q^-}\mathbf{w}_0\cdot\kappa\left(\mathbf{u}_0^*-\mathbf{u}_0\right)\mathbf{i}\boldsymbol{\Omega} = 0 \tag{1}$$

for all $\mathbf{w} \in \mathcal{V}$, and for all open subdomains $Q \subset D$ such that $\mathbf{u}|_Q \in H^1(Q)$ but where $\mathbf{u}$ may jump across ∂Q. The variables in (1) are differential forms with vector coefficients: $\mathbf{M}$ is the spacetime momentum flux, a d–form comprised of the stress and momentum density; the body force $\mathbf{b}$ is a $(d+1)$–form (ρ is mass density); the strain-velocity $\boldsymbol{\varepsilon}$ is a 1–form; and Ω is the standard basis for $(d+1)$–forms. The notations $\mathcal{L}$ and $\mathbf{i}$ denote the Lie derivative and insertion operators, both defined with respect to the time direction. A superscript '$*$' indicates a Godunov boundary value from a local Riemann problem, and $\hat{\mathbf{M}}$ is the momentum flux associated with the weighting function $\mathbf{w}$. A subscript '0' indicates a mapping into the zero–energy subspace of steady, infinitesimal–rigid displacement fields. Equation (1) enforces momentum balance via the equation of motion, $\mathbf{dM}+\rho\mathbf{b} = \mathbf{0}$, and the momentum flux jump condition, $\mathbf{M}^*-\mathbf{M} = \mathbf{0}$ on ∂Q. Kinematic compatibility is weakly enforced via the jump conditions, $\boldsymbol{\varepsilon}^*-\boldsymbol{\varepsilon} = \mathbf{0}$ on ∂Q and $\mathbf{u}_0^*-\mathbf{u}_0 = \mathbf{0}$ on the time–inflow boundary, ∂Q^-. Integration by parts using the Cartan identity yields the SDG weak form on Q,

$$\int_Q\left(\mathbf{d}\dot{\mathbf{w}}\wedge\mathbf{M}-\dot{\mathbf{w}}\cdot\rho\mathbf{b}\right) - \int_{\partial Q}\left\{\dot{\mathbf{w}}\cdot\mathbf{M}^*+\left(\boldsymbol{\varepsilon}^*-\boldsymbol{\varepsilon}\right)\wedge\mathbf{i}\hat{\mathbf{M}}\right\} + \int_{\partial Q^-}\mathbf{w}_0\cdot\kappa\left(\mathbf{u}_0^*-\mathbf{u}_0\right)\mathbf{i}\boldsymbol{\Omega} = 0\ \forall\mathbf{w}\in\mathcal{V} \tag{2}$$

The SDG finite element method is obtained from (2) by associating Q with each element in a spacetime meshing of D, and by equipping each element with an independent set of discrete basis functions. The SDG method has low dissipation and it delivers exact momentum balance on every element in terms of the physically meaningful Godunov fluxes $\mathbf{M}^*$. It requires no stabilization and generates oscillation–free solutions, even when shocks are present.

CAUSALITY AND THE ADAPTIVE TENT-PITCHER ALGORITHM

A mesh is called patch–wise causal if the elements can be grouped into patches such that the wave characteristics at every point on every patch boundary are either all inward or all outward. The causal property establishes a partial ordering of

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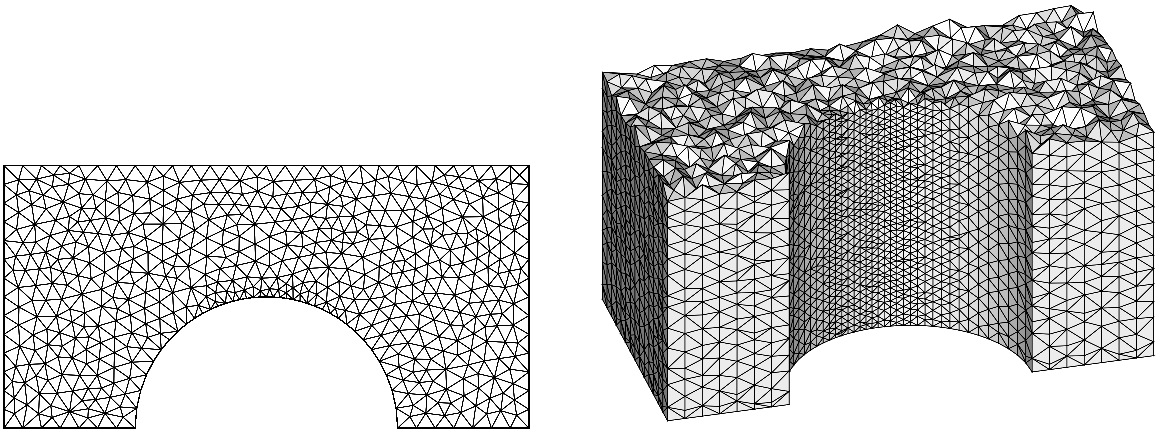

Figure 1. An input space mesh and the resulting spacetime mesh computed by Tent Pitcher [2]

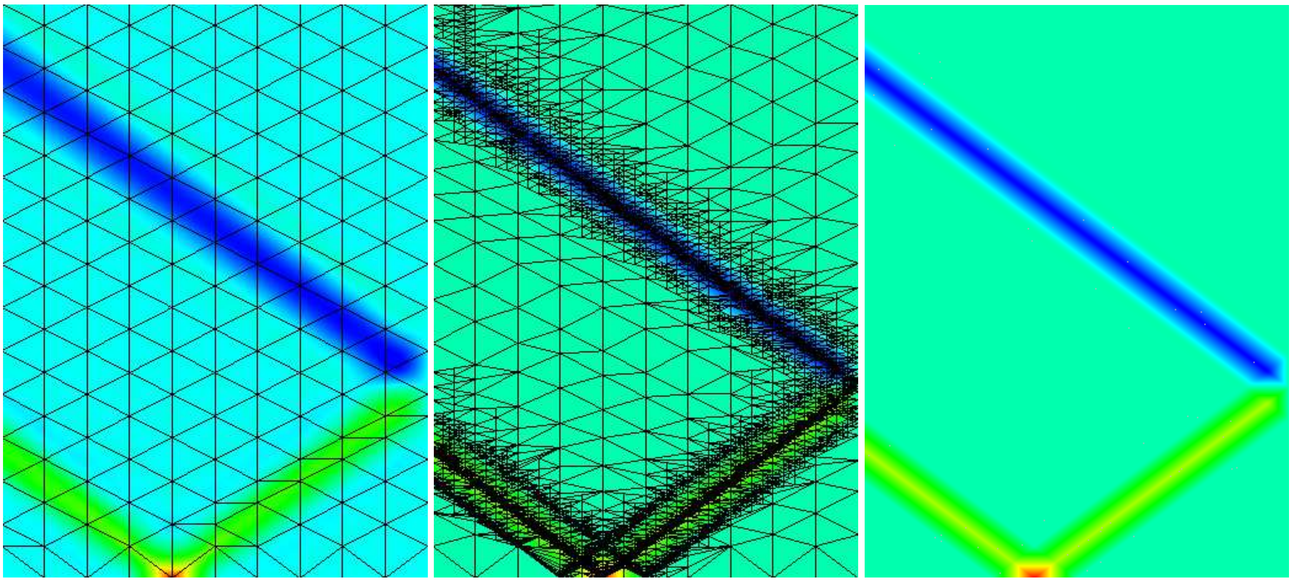

Figure 2. SDG displacement solutions for shock propagation in an elastic rod without adaptivity (left) and with adaptive refinement and coarsening (center and right). The space domain is along the horizontal axis, and time is increasing upwards.

patches where the solution on a given patch is independent of the solutions on all subsequent patches; this enables patch–by–patch solution procedures with $\mathcal{O}(N)$ complexity. Tent Pitcher [2] is an advancing–front meshing procedure that generates patchwise–causal spacetime meshes over an arbitrary spatial triangulation, as shown in Fig. 1. We immediately compute the SDG solution on each new patch generated by Tent Pitcher. Then, based on an appropriate error indicator, we decide whether to accept the patch (with or without a request for coarsening on subsequent patches) or to discard the patch and demand a local h–refinement operation. In the latter case, or when coarsening is requested, an adaptive implementation of Tent-Pitcher [3] exploits the ability of the SDG formulation to accommodate nonconforming meshes in carrying out the adaptation while maintaining the patch–wise causal property.

The adaptive SDG method has been implemented for linear elastodynamics in one and two spatial dimensions. The example depicted in Fig. 2 illustrates the benefits of spacetime adaptivity in a shock propagation problem. Shock loading is applied to an elastic rod, with a transmitting boundary on the left and a fixed boundary on the right. The smearing of the shock bands in the nonadaptive solution indicates numerical error. The strongly graded adaptive mesh resolves the fine details of the shocks with no visible smearing. Restricting the refinement to the shock trajectories reduces the computational expense relative to adaptive procedures constrained by a global time–step size.

References

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