Spacetime Interfacial Damage Model for Elastodynamic Fracture with Riemann Contact Conditions

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16th U.S. National Congress of Theoretical and Applied Mechanics
June 27 - July 2  2010
Features of spacetime discontinuous Galerkin finite element methods

- Inter-element discontinuous basis functions
- Weak enforcement of balance/conservation jump conditions in spacetime (e.g., Rankine–Hugoniot conditions for conservation laws)
- Enables exact conservation per element and $O(N)$ complexity for hyperbolic problems
Features of spacetime discontinuous Galerkin finite element methods

- Direct discretization of spacetime
- Replaces separate temporal integration
- Unstructured spacetime mesh eliminates tangling in moving–boundary problems
- Unambiguous numerical framework for initial/boundary conditions (vs. finite volume, finite difference)
Causal Spacetime Mesh and $O(N)$ Advancing-Front Solution Strategy
Tent Pitcher: causal spacetime meshing

- Given a space mesh, Tent Pitcher constructs a spacetime mesh such that every facet on sequence of advancing fronts is spacelike (patch height bounded by *causality constraint*)

- Similar to CFL condition, except entirely *local* and not related to stability (required for O(N) solution)
Tent Pitcher: patch–by–patch meshing & solution

- Patches (‘tents’) of tetrahedra; solve immediately for $O(N)$ method with rich parallel structure.
- Maintain “space mesh” as advancing, space-like front with non-uniform time coordinates.
Force-like fields
(d,d+1 -forms with vector coefficients)

- Body force ((d + 1)-form): \( b = b\Omega \)
- Spacetime momentum flux (d-form): \( M = p - \sigma \)

\( M|_\Gamma = \text{linear momentum flux across } d\text{–manifold } \Gamma \)

- Linear momentum density: \( p = p \star dt \)
- Stress/traction: \( \sigma = \sigma \wedge \star dx \)
- Stokes Theorem: \( \int_{\partial Q} M = \int_Q dM \)
Momentum Balance

- Integral form of linear momentum balance:

\[
\int_{\partial \Omega} M = \int_{\Omega} \rho b \quad \forall \Omega \subset D
\]

\[
\int_{\Omega} (dM - \rho b) = 0 \quad \forall \Omega \subset D \quad \text{(Stokes Thm.)}
\]

- Local form with jump part:

\[
(dM - \rho b)|_{D \setminus \Gamma_J} = 0
\]

\[
[M]|_{D \cap \Gamma_J} = 0 \rightarrow (M^* - M)|_{Q \cap \Gamma_J} = 0
\]

\[
M^* = \text{Riemann or prescribed value}
\]
Adaptive refinement

Refine space mesh using *newest vertex method* to maintain element quality.
More adaptive meshing

New spacetime adaptive meshing operations:

- Edge flip; vertex deletion; inclined tent poles (ALE) for smoothing, tracking and repositioning

- Spacetime format eliminates projection error
Energy balance and dissipation error indicator for adaptive meshing

- On every spacetime element $Q$, SDG solution satisfies:

$$
\varphi^Q = \int_Q \ddot{u} \wedge \rho \bm{b} + \frac{1}{2} \int_{\partial Q} (\dot{\bm{u}}^* \wedge \bm{M}^* + \varepsilon^* \wedge i\bm{M}^*)
$$

$$
= \frac{1}{2} \int_{\partial Q} \left\{ (\ddot{\bm{u}}^* - \dot{\bm{u}}) \wedge (\bm{M}^* - \bm{M}) + (\varepsilon^* - \varepsilon) \wedge i(\bm{M}^* - \bm{M}) \right\}
$$

Right hand side is non-negative numerical dissipation for $Q$.

- To control adaptively the numerical dissipation, we use the element-wise error indicator:

$$
\varphi^Q \approx \text{tol}_\varphi
$$
Cohesive error measure and adaptive error indicator

- Cohesive energy error on cohesive surface trajectory, $\Gamma^C$:

$$
\epsilon_C = \|(t - t_{coh}) \cdot v\|_{L_1(\Gamma^C)}
$$

- Cohesive error indicator for element $Q$:

$$
\epsilon_Q^C = \|(t - t_{coh}) \cdot v\|_{L_1(\partial Q_{coh})} \approx tol_c
$$
Crack-tip Wave Scattering
Crack-tip Wave Scattering

click to play movie
Crack-tip Wave Scattering
Crack-tip Wave Scattering
Delay Damage Evolution
Cohesive Model

\[ D = \text{Area fraction of debonded surface} \]
Mesoscopic interface subdivisions

Undamaged \((D = 0)\)  

Macroscopic

Damaged \((D > 0)\)

Debonded

Bonded

Mesoscopic

\(N(x) \subset \tilde{\Gamma}\)

Contact

Stick

Slip

Regularization

\(x\)

\(1 - D\)

\(D\)

\(1 - \eta\)

\(\eta\)

\(1 - \gamma\)

\(\gamma\)

Averaging

\(\lambda_1\)

\(\lambda_2\)

\(1\)

\(D\)
Contact conditions

• Common numerical idealizations of contact
  ✦ Discrete models - goal is to prevent node penetration
  ✦ Elastostatic contact conditions
  ✦ Penalty methods and other inexact formulations
    - \( s = f(\Delta u) = K \Delta u(\Delta u < 0) \) but \( \Delta u = 0! \)
      - Approach exact solution for large \( K \Rightarrow \) divergence
  ✦ Variational Inequality/Lagrange multiplier methods
    - Lagrange variables increase the problem size
    - typically require an implicit solution scheme
Riemann Contact/Separation Solutions

- Direct solution from momentum balance and compatibility jump conditions
- Matching conditions at material interface
- Preserves characteristic structure of solution
- Weak enforcement of the continuum, dynamic jump conditions converges to correct result
Interface Matching condition
Bonded and contact-stick

- **Matching conditions**
  - Momentum: \([\tilde{s}] = 0\)
  - Compatibility: \([\tilde{v}] = 0\)

- **Riemann Solution**
  - \(s^I_j = \frac{[w(j)/\rho c(j)]}{[(\rho c(j))^{-1}]}\)
  - \(v^I_j = \frac{[w(j)]}{[(\rho c(j))^{-1}]}\)

- Contact-stick and bonded solutions are identical

- Contact-stick and separation solutions are physically distinct:
  - Contact-stick
    \[\Delta u = 0\]
    \[s = ?\]
  - Separation
    \[\Delta u = ?\]
    \[s = 0 \text{ or } s = f(\Delta u)(TSL)\]
Interface Matching condition

**Contact-slip**

- **Matching conditions**
  - Momentum: \( \mathbb{[\dot{s}]} = 0 \)
  - Compatibility: \( \mathbb{[\dot{\nu}_1]} = 0 \)
  - Coulomb’s law: \( s_{ji}^{II} = k\langle -s_{jI}^1 \rangle + (e_\nu)^j \) if \( j \neq 1 \)

- **Riemann Solution**

\[
\begin{align*}
\dot{s}_{ji}^{II} &= \begin{cases} 
   \dot{s}_{1I}^1 & j = 1 \\
   k\langle -s_{jI}^1 \rangle + (e_\tau)^j & j \neq 1 
\end{cases} \\
\dot{v}_{j}^{II} &= \begin{cases} 
   \dot{v}_1^I \\
   u^{(j)}/\rho c^{(j)} - s_{ji}^{II}/\rho c^{(j)} = \ddot{v}_j + \left( \dot{s}_{j}^j - s_{ji}^{II} \right)/\rho c^{(j)} & j \neq 1 
\end{cases}
\end{align*}
\]

- **Slip velocity** \( \mathbb{[\ddot{v}]} \to 0 \) at stick-slip transition \( \Rightarrow \) Discontinuous response from \( (e_\nu)^j \)

- many regularizations proposed to solve the problem (Karnopp, Mostaghel, Quinn)

- \( \tau_I > 0 \) at stick-slip transition \( \Rightarrow \) Continuous response from \( \tau_I > 0 \)
Interface Matching condition
Separation

• Matching conditions
  – Momentum and constitutive equation: \( \dot{s} = 0 \)
  – No compatibility condition

• Riemann Solution
  – \( s^{III} = 0 \)
  – \( \mathcal{U}_j^{III} = \frac{w(j)}{\rho_c(j)} \)
Mesoscopic subdivision
Damaged area (D)

• General Properties of Damage parameter
  ✦ Nondecreasing (irreversible process)
  ✦ Stress Induced (Ravi-Chandar, Yang 1997)
  ✦ Exact Bonded solution is recovered for \( D = 0 \)
  ✦ Stress free (crack) surface condition at \( D = 1 \)

• Rate Dependent damage model
  ✦ Hardening effects at high strain rates (experimental results from Fineberg and Marder 1999)
  ✦ Introduces a fracture related length scale
  ✦ Mesh dependency of static damage models (strain softening failure, Bazant and Belytschko 1984)
Damage Evolution

Follow delay damage evolution model of Allix, Feissel and Thévenet:

\[
\tilde{\sigma} = \frac{\sqrt{\sigma_N^2 + (\beta \sigma_T)^2}}{\sigma_c}
\]

\[
\tilde{D} = f(\tilde{\sigma})
\]

\[
\dot{D} = \begin{cases} 
\frac{1}{\tau} [1 - H(\langle f(\tilde{\sigma}) - D \rangle_+)] & \text{if } D < 1 \\
0 & \text{if } D = 1
\end{cases}
\]

where \( \tau \) is a characteristic time scale for debonding, and \( H(0) = 1 \) and \( \lim_{x \to \infty} H(x) = 0 \). We use

\[
H(x) = e^{-ax}
\]

This model implies

\[
D = \tilde{D} \Rightarrow \dot{D} = 0
\]
Mesoscopic subdivision
Contact area ($\eta$)

- Contact/Separation transition may introduce shocks in tractions and velocities
- Regularization is required for shock capturing schemes
- Regularization is based on normal contact traction and macroscopic traction
- A minimum separation value of $r_c$ is ensured in the regularized model
- Surface toughness may provide physical regularization
- No regularization is required for stick-slip areas

![Diagram showing contact scheme prior to regularization and regularized contact scheme.](image)
Crack closure example: brittle fracture under cyclic, dynamic loading conditions

\[ \bar{\sigma}(t), \bar{\pi}(t) \]

- Free surface
- Crack free surface
- Cohesive surface
- Fixed boundary
- Transmitting boundary
- Initial crack tip
- Crack closure
- Distance \( a_0 \)
- Length \( L \)
Crack closure example: brittle fracture under cyclic, dynamic loading conditions

[link to play movie]
Solution-dependent crack paths

• Inclined tent poles, edge bisection and smoothing extend cohesive zone in any desired direction

• Effective stress criterion for extending cohesive surface
  • maximum value determines direction for extension

• Nucleation + coalescence = branching
  • Probabilistic nucleation criterion models random defects
  • Introducing cohesive surface does not affect behavior until damage accumulates
Solution-dependent crack propagation

Geometry and Loading

PMMA:
- $E = 3.24 GPa$.
- $\rho = 1190 \frac{Kg}{m^3}$.
- $w = 0.020 MPa$.
- $\tau = 10^{-2} \mu s$.

based on Sharon and Fineberg experiment (1996)
Refinement Details: 1 x zoom
Refinement Details: 25 x zoom
Refinement Details: 100 x zoom
Refinement Details: 200 x zoom
Refinement Details: 2000 x zoom
Same problem with crack closure contact model, random defects and nucleation
Effect of contact conditions on crack path
Surface roughening at high crack velocities

- Almost uniform hoop stress distribution at $\dot{a} > 2/3 c_R$ (Yoffe, 1951)

- Experimentally observed roughness on crack surface (Smekal, 1953; Kerkhof, 1973; Sharon and Fineberg, 1996)

- Surface roughening and branching instabilities in dynamic fracture; Phenomenological wavy crack path (Gao, 1992)

- Dynamical stability of a propagating crack: Obrezanova (2002) at $\dot{a} > 1/3 c_R$ crack may admit one or more oscillatory modes of instability

- Stability of dynamically propagating cracks in brittle materials: Uenishi et. al. (2001): Surface roughness induced by increase of crack velocity
Single crack velocity agreement with experiments

- Material degradation and energy loss due to induced heat should be contributed in the model (Sharon and Fineberg 1996)
- Bulk material dissipative and damage models to be incorporated
- Material degradation around the crack tip, the high inertia zone, and subsequent drop in wave speed (Ravi et al., 2007)
- Surface tension or hardness in computing the limiting velocity, Kerkhof (1997)
Summary

• Robust adaptive model for crack growth
  ✦ Ensures accurate rendering of cohesive models
  ✦ Eliminates all mesh-dependence effects
  ✦ Supports nucleation
  ✦ Captures multi-scale behavior
  ✦ Captures the details of crack propagation (Branching, micro branching, surface roughening)

• New dynamic, continuum contact formulation

• Open physical modeling issues
  ✦ Crack acceleration too rapid (typical of cohesive models)
  ✦ Branching pattern does not completely match experiment
  ✦ Some aspects heuristic, but have framework for developing