Riemann Solutions for Spacetime
Discontinuous Galerkin Methods

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SDG directly discretizes spacetime with causal meshes.

- Developed at the Univ. of Illinois Urbana-Champaign
  - R. Haber, J. Erickson, et al.
- Uses the characteristic structure of hyperbolic PDEs to construct patches of elements that decouple from the domain.
- Given an initial triangulation, a local patch of elements is constructed and solved immediately
- *Linear computational complexity* in the number of patches/elements
SDG tent-pitching/solution procedure enables highly adaptive spacetime meshes.

- Linearized elastodynamics: shock reflection off of a stationary crack

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SDG with adaptive meshing sharply resolves waves and discontinuities.

- Discontinuous elements permit sharp discontinuity resolution

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Differential forms notation provides a clear & concise statement of balance laws.

- No natural metric for spacetime vectors, so tensor calculus is inadequate
  - No inner product, no normal vectors
  - No Stokes' theorem, no weak formulations!
- Forms: normal free, direct expressions of spacetime balance laws
- Basic mathematical constructs and identities:

  Spacetime manifold: \( \mathcal{D} \subset \mathbb{E}^d \times \mathbb{R} \)

  One form: \( \mathbf{dx} := e_i dx^i = e_1 dx^1 + e_2 dx^2 + e_3 dx^3 \)

  \(d\)-form: \( \star \mathbf{dx} := e^i \star dx^i = e^1 dx^2 \wedge dx^3 - e^2 dx^1 \wedge dx^3 + e^3 dx^1 \wedge dx^2 \)

  Stokes’ Theorem: \( \int_Q d\mathbf{\omega} = \int_{\partial Q} \mathbf{\omega} \)

  \(d(w \star \mathbf{dx}) = (\nabla w) \Omega \quad d(a \wedge \star \mathbf{dx}) = (\nabla \cdot a) \Omega \)

  \(d(w \star dt) = \dot{w} \Omega \quad d(a \wedge \star dt) = \dot{a} \Omega \)
Differential forms notation provides a clear & concise statement of balance laws.

- Governing equations on a spacetime control volume $Q$:
  \[
  \int_{\partial Q} F(u) - \int_Q S(u) = 0
  \]

- Localized governing equations:
  \[
  dF - S = 0, \\
  \]

- Flux derivative has a familiar form:
  \[
  F(u) := u \diamond dt + f(u) \diamond dx \\
  dF = (\dot{u} + \nabla \cdot f(u)) \, dx^1 \, dx^2 \, dx^3 \, dt
  \]

- Jump terms arise due to the distributional nature or derivatives:
  \[
  [F] |_{\Gamma_J} := (F^* - F) |_{\Gamma_J} = 0
  \]

- Standard Bubnov-Galerkin weak formulation:
  Find $u \in U$ such that for every $Q \in \mathcal{P}$
  \[
  -\int_Q [d\hat{w} \wedge F(u) + \hat{w} \wedge S(u)] + \int_{\partial Q} \hat{w} \wedge F^*(u) = 0 \quad \forall \, \hat{w} \in \mathcal{U}^Q
  \]
The inter-elemental flux \( F^* \) is determined by solving a local, one dimensional Riemann problem.

- **Riemann problem**: Given a hyperbolic PDE and an interface with discontinuous data across it, determine the intermediate solution states (and satisfy Rankine-Hugoniot).
  - Information travels between cells only in the “normal” direction
  - The Riemann problem becomes one directional
  - Necessary to introduce a local coordinate system
  - *Source terms do not influence the spacetime Riemann flux*
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**Question**: How do we transform our differential forms notation into a local coordinate system so that the Riemann problem is the familiar one (e.g. from CFD)?
Coordinate transformations with forms are analogous to the “typical” transformations.

- Transform global to local coordinates: \( \{x^i, t\}_{i=1}^d \rightarrow \{\tilde{x}^i, \tilde{t}\}_{i=1}^d \)
- The spatial coordinates are rotated, and time is shifted.
- Spacetime fluxes give both spatial and temporal fluxes through an arbitrarily oriented manifold!
- Basis vectors transform via a rotation tensor: \( \mathbf{e}_I := Q_I^i \mathbf{e}_i, \quad \mathbf{e}_t := \mathbf{e}_t. \)

- Vector-valued differential forms are objective: \( \omega = \omega, \quad \star \omega = \star \omega \)
- We are left to properly transform the “fields” that are coefficients of the vector valued forms, e.g. \( u \) in \( u \wedge \star \omega \)
Standard Riemann solvers can be used once the formulation is in the local coordinate system.

One-dimensional Riemann problem for hyperbolic systems

\[ u_t + A u_x = 0, \quad u = \{u_1, u_2, \ldots, u_n\}^T, \]
\[ u = \begin{cases} 
  u_\alpha, & x_1 < 0 \\
  u_\beta, & x_1 > 0,
\end{cases} \]

Remarks:

› \( A \) is the (linearized) flux Jacobian matrix
› Source terms do not affect the Riemann solution in the SDG context
› Eigen-decomposition/diagonalisation:

\[ A = \Gamma \Lambda \Gamma^{-1} \]
\[ \Gamma = [\gamma_1 | \cdots | \gamma_n] \]
\[ \Lambda = \begin{pmatrix} 
  c_1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & c_n
\end{pmatrix} \]
Riemann solutions in characteristic variables are then ‘textbook’.

Change of variables: \[ \boldsymbol{w}_\alpha = \Gamma^{-1}\boldsymbol{u}_\alpha, \quad \boldsymbol{w}_\beta = \Gamma^{-1}\boldsymbol{u}_\beta \]

Riemann solution: \[ \boldsymbol{w}^* = \left\{ w_{\beta 1}, \ldots, w_{\beta p}, w_{\alpha p+1}, \ldots, w_{\alpha m} \right\}^T \]

Riemann solution in primitive variables: \[ \boldsymbol{u}^* = \Gamma \boldsymbol{w}^* \]

\[ u_{i}^* = \left( \sum_{j=1}^{p} \Gamma_{ij}(\Gamma^{-1})_{jk}u_{\beta k} \right) + \left( \sum_{j=p+1}^{m} \Gamma_{ij}(\Gamma^{-1})_{jk}u_{\alpha k} \right) \]

Spatial flux: \[ F_{i}^*(\boldsymbol{u}) = A_{ij}u_{j}^* = \left( \sum_{j=1}^{p} \Gamma_{ij} \lambda^j(\Gamma^{-1})_{jk}u_{\beta k} \right) + \left( \sum_{j=p+1}^{m} \Gamma_{ij} \lambda^j(\Gamma^{-1})_{jk}u_{\alpha k} \right) \]

Note: The linearized eigenstructure is often much easier to compute than the entire Riemann solution.
Hyperbolic thermal model (non-Fourier): semi-analytic Riemann solver is not necessary.

Constitutive MCV equation: \[ \frac{1}{\kappa} (\tau \dot{q} + q) + \nabla T = 0 \]

Primitive variables: \[ \mathbf{u} = \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\} = \left\{ \begin{array}{c} CT \\ (\tau / \kappa) q \end{array} \right\}, \quad \{ u_1, u_2, u_3 \} = \left\{ \begin{array}{c} CT, \frac{\tau}{\kappa} q_n, \frac{\tau}{\kappa} q_t \end{array} \right\} \]

\[ \mathbf{F}(\mathbf{u}) = \left\{ \begin{array}{c} F_1 \\ F_2 \end{array} \right\} = \left\{ \begin{array}{c} CT \dot{t} + q \wedge \star \mathbf{d}x \\ (\tau / \kappa) q \wedge \mathbf{d}t + T \wedge \mathbf{d}x \end{array} \right\}, \quad \mathbf{S}(\mathbf{u}) = \left\{ \begin{array}{c} S_1 \\ S_2 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ (1 / \kappa) q \Omega \end{array} \right\} \]

\[ \{ c \}^3_{i=1} = \{-c_T, 0, c_T\}, \quad c_T = \sqrt{\frac{\kappa}{C_T}} \]

\[ \gamma_1 = \left\{ -\sqrt{\frac{C \kappa}{\tau}}, 1, 0 \right\}, \quad \gamma_2 = \{0, 0, 1\}, \quad \gamma_3 = \left\{ \sqrt{\frac{C \kappa}{\tau}}, 1, 0 \right\} \]
Hyperbolic thermal model (non-Fourier): semi-analytic Riemann solver is not necessary.

Constitutive MCV equation: 
\[
\frac{1}{\kappa} (\tau \dot{q} + q) + \nabla T = 0
\]

Primitive variables: 
\[
\mathbf{u} = \begin{cases}
\mathbf{u}_1 \\
\mathbf{u}_2
\end{cases} = \begin{cases}
CT \\
(\tau/\kappa)q
\end{cases}, \quad \{u_1, u_2, u_3\} = \begin{cases}
CT, \frac{\tau}{\kappa} q_n, \frac{\tau}{\kappa} q_t
\end{cases}
\]

\[
\mathbf{F}(\mathbf{u}) = \begin{cases}
\mathbf{F}_1 \\
\mathbf{F}_2
\end{cases} = \begin{cases}
CT \ast dt + q \wedge \ast dx \\
(\tau/\kappa)q \ast dt + T \ast dx
\end{cases}, \quad \mathbf{S}(\mathbf{u}) = \begin{cases}
\mathbf{S}_1 \\
\mathbf{S}_2
\end{cases} = \begin{cases}
0 \\
(1/\kappa)q \Omega
\end{cases}
\]

Exact Riemann solution

\[
\mathbf{u}_1^* = \langle \mathbf{u}_1 \rangle - \frac{1}{2} \sqrt{\frac{C\kappa}{\tau}} [\mathbf{u}_2] \quad \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \text{ or } R_2
\]

\[
\mathbf{u}_2^* = \langle \mathbf{u}_2 \rangle - \frac{1}{2} \sqrt{\frac{\tau}{C\kappa}} [\mathbf{u}_1] \quad \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \text{ or } R_2
\]

\[
\mathbf{u}_3^* = \begin{cases}
\mathbf{u}_3^\alpha & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \\
\mathbf{u}_3^\beta & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_2
\end{cases}
\]
Hyperbolic heat conduction permits ‘thermal shocks’.

Thermal shock loading of a matrix with circular voids.
Hyperbolic heat conduction permits ‘thermal shocks’.

Thermal shock loading of a matrix with inclusions.
Linearized elastodynamics: weak shocks and solution dependent crack propagation

Enforce linear momentum and kinematic compatibilities

\[
\mathbf{u} = \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \\ \mathbf{E} \end{pmatrix}, \quad \{u_i\}_{i=1}^7 = \{u_n, u_t, \rho v_n, \rho v_t, E_{nn}, E_{nt}, E_{tt}\}
\]

Eigensystem

\[
\{c\}_{i=1}^7 = \{-c_D, -c_S, 0, 0, 0, c_S, c_D\}, \quad c_D = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_S = \sqrt{\frac{\mu}{\rho}}
\]

\[
\gamma_1 = \{0, 0, \rho c_D, 0, 1, 0, 0\},
\gamma_2 = \{0, 0, 0, 2\rho c_S, 0, 1, 0\},
\gamma_3 = \{1, 0, 0, 0, 0, 0, 0\}
\gamma_4 = \{0, 1, 0, 0, 0, 0, 0\}
\gamma_5 = \{0, 0, 0, 0, -\lambda, 0, \lambda + 2\mu\}
\gamma_6 = \{0, 0, 0, -2\rho c_S, 0, 1, 0\},
\gamma_7 = \{0, 0, -\rho c_D, 0, 1, 0, 0\}
\]
Linearized elastodynamics: Exact Riemann solution

\( u_1^* = \begin{cases} f_1^\alpha & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \text{ or } R_2 \\ f_1^\beta & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_3 \text{ or } R_4 \end{cases} \)

\( u_2^* = \begin{cases} f_2^\alpha & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \text{ or } R_2 \\ f_2^\beta & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_3 \text{ or } R_4 \end{cases} \)

\( u_3^* = \left< f_3 \right> + \frac{\rho c_D}{2} \left( [f_5] + \frac{\lambda}{\lambda + 2\mu} [f_7] \right), \text{ for all regions} \)

\( u_4^* = \begin{cases} f_4^\alpha & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \\ \left< f_4 \right> + \rho c_s [f_6] & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_2 \text{ or } R_3 \\ f_4^\beta & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_4 \end{cases} \)

\( u_5^* = \frac{1}{2\rho c_D} [f_3] + \left< f_5 \right> + \frac{\lambda}{2(\lambda + 2\mu)} [f_7], \text{ for all regions} \)

\( u_6^* = \begin{cases} f_6^\alpha & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \\ \frac{1}{4\rho c_s} [f_4] + \left< f_6 \right> & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_2 \text{ or } R_3 \\ f_6^\beta & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_4 \end{cases} \)

\( u_7^* = \begin{cases} f_7^\alpha & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \text{ or } R_2 \\ f_7^\beta & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_3 \text{ or } R_4 \end{cases} \)
Linearized elastodynamics: weak shocks are resolved over multiple scales with spacetime adaptive meshing

click to play movie
Linearized elastodynamics: weak shocks and solution dependent crack propagation

Contact and cohesive zone models are viewed as special Riemann conditions on crack faces.

[Click to play movie]
Linearized elastodynamics: weak shocks and solution dependent crack propagation

Mesh refinement enables prediction of crack paths with high fidelity.
Generalized thermoelasticity: coupled multiphysics benefit from using a semi-analytic exact Riemann solver

- Governing equations combine elastodynamics and thermal model
- Stress is temperature dependent
- Energy balance now has a velocity term
- Primitive variables:
  \[ \{u_i\}_{i=1}^{10} = \{u_n, u_t, \rho v_n, \rho v_t, E_{nn}, E_{nt}, E_{tt}, T, q_n, q_t\} \]

- Coupled wavespeeds are complicated:
  \[
c_d = \left[ \frac{1}{2} \left( c_D^2 + c_T^2 + \beta - \sqrt{-4c_D^2 c_T^2 + (c_D^2 + c_T^2 + \beta)^2} \right) \right]^{1/2}
  \]
  \[
c_t = \left[ \frac{1}{2} \left( c_D^2 + c_T^2 + \beta + \sqrt{-4c_D^2 c_T^2 + (c_D^2 + c_T^2 + \beta)^2} \right) \right]^{1/2}
  \]

- Exact Riemann solutions get ‘nasty’; e.g. the simplest one is
  \[
  (u_8)^* = CT^* = \langle u_8 \rangle - \frac{1}{2\rho(c_d + c_t)} \left( \rho C(c_d c_t + c_T^2) [u_9] + kT_0 [u_3] \right)
  \]
Generalized thermoelasticity: coupled multiphysics benefit from using a semi-analytic exact Riemann solver

- Eigensystem is much easier to obtain and implement
- Symbolic computer algebra can be used
- Only need to replace eigenvalues/vectors for each subproblem

\[
\begin{align*}
\gamma_1 &= \{0, 0, \rho c_D^2 (c_t - c_T)(c_t + c_T), 0, c_t (c_D^2 - c_d^2), 0, 0, -c_D^2 c_t C \theta, c_D^2 \theta, 0\}, \\
\gamma_2 &= \{0, 0, \rho c_D^2 (c_d - c_T)(c_d + c_T), c_d (c_D - c_t)(c_D + c_t), 0, 0, -c_D^2 c_d C \theta, c_D^2 \theta, 0\}, \\
\gamma_3 &= \{0, 0, 0, 2 \rho c_S, 0, 1, 0, 0, 0, 0\}, \\
\gamma_4 &= \{0, 1, 0, 0, 0, 0, 0, 0, 0, 0\}, \\
\gamma_5 &= \{0, 0, 0, 0, 0, 0, 0, 0, 0, 1\}, \\
\gamma_6 &= \{0, 0, 0, 0, -\lambda, 0, \rho c_D^2, 0, 0, 0\}, \\
\gamma_7 &= \{1, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \\
\gamma_8 &= \{0, 0, 0, -2 \rho c_S, 0, 1, 0, 0, 0, 0\}, \\
\gamma_9 &= \{0, 0, \rho c_D^2 (c_d - c_T)(c_d + c_T), c_d (c_t^2 - c_D^2), 0, 0, C c_d c_D^2 \theta, c_D^2 \theta, 0\}, \\
\gamma_{10} &= \{0, 0, \rho c_D^2 (c_t - c_T)(c_t + c_T), c_t (c_d - c_D)(c_d + c_D), 0, 0, C c_D^2 c_t \theta, c_D^2 \theta, 0\}
\end{align*}
\]
Riemann solutions for unstructured spacetime DG methods are as straightforward as any other method.

- Don’t let the differential forms notation put you off!
- The forms are objective with respect to coordinate systems
- Rotations into local coordinates are familiar
- Source terms in equations *do not matter* with SDG methods!
- Complicated systems of PDEs from multiphysics problems can benefit substantially from a semi-analytic exact Riemann solver
- PDEs do not need to be linear, just linearized at the flux location
- Extensions to exact non-linear Riemann solvers should be straightforward, but with a much higher computational cost