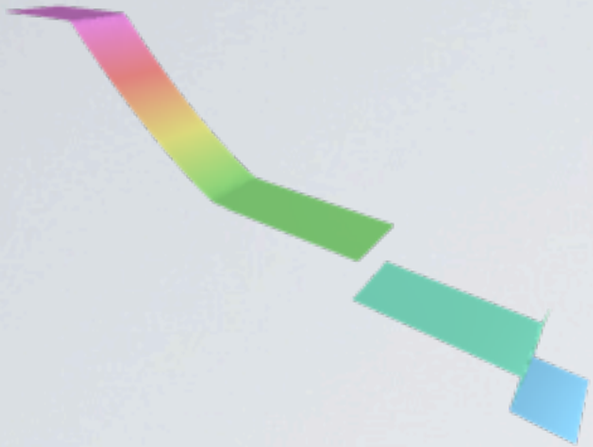

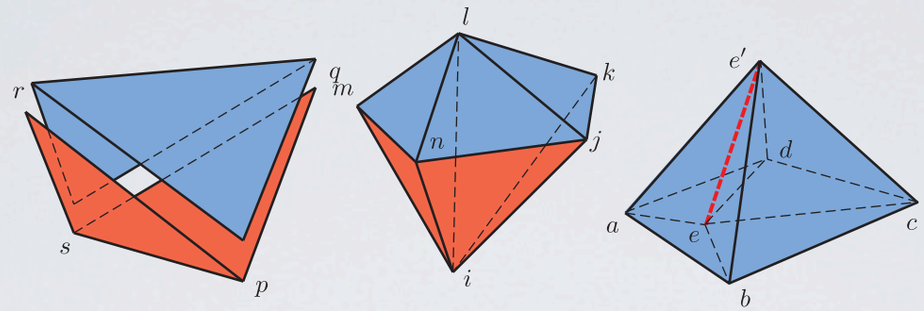

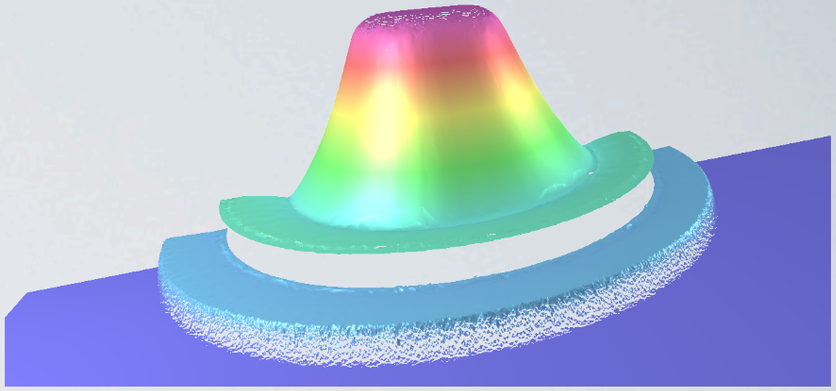

Riemann Solutions for Spacetime Discontinuous Galerkin Methods

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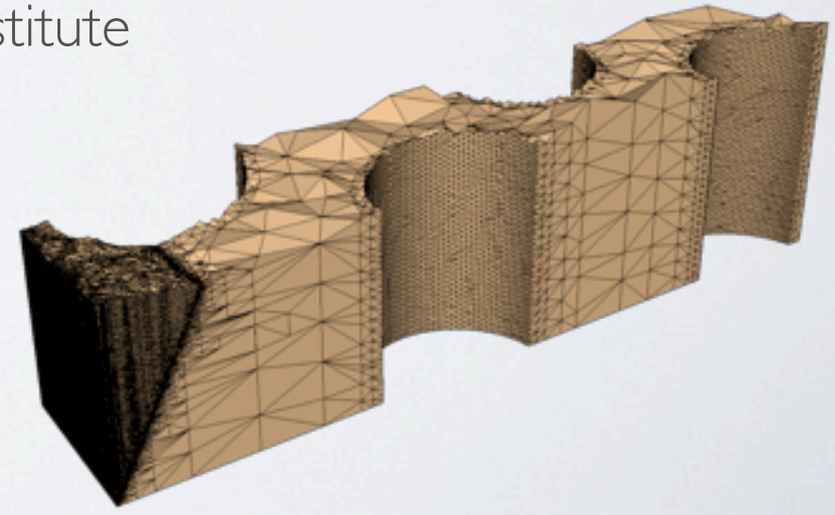
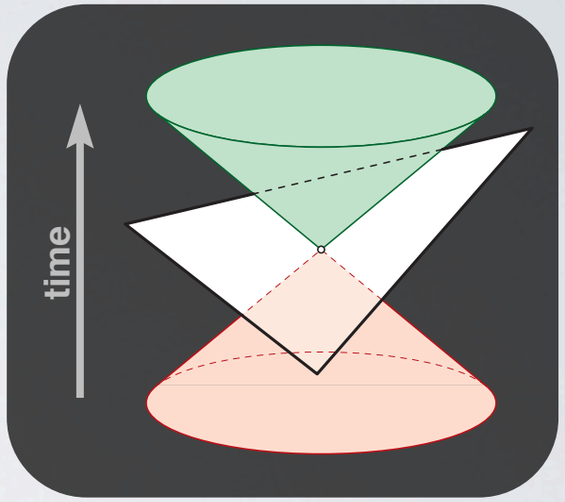
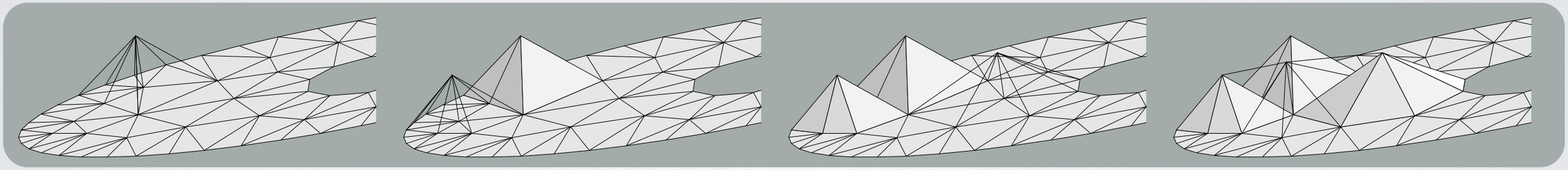


SDG directly discretizes spacetime with causal meshes.

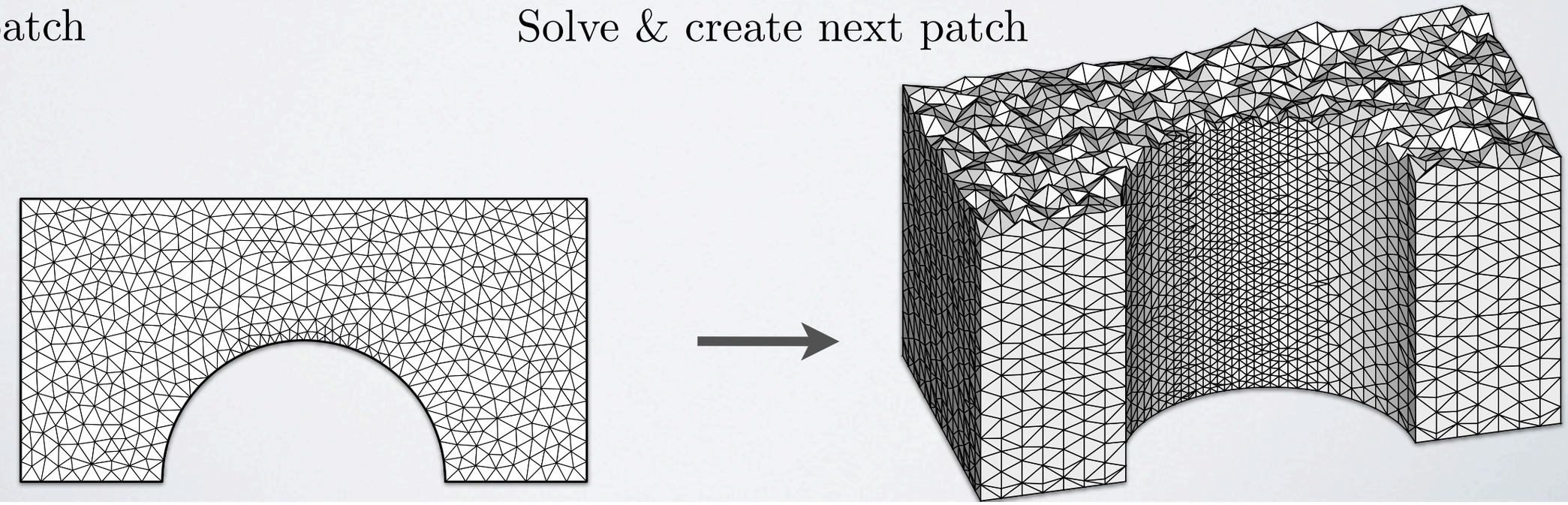
- Developed at the Univ. of Illinois Urbana-Champaign
  - R. Haber, J. Erickson, et al.
- Uses the characteristic structure of hyperbolic PDEs to construct patches of elements that decouple from the domain.
- Given an initial triangulation, a local patch of elements is constructed and solved immediately
- *Linear computational complexity* in the number of patches/elements





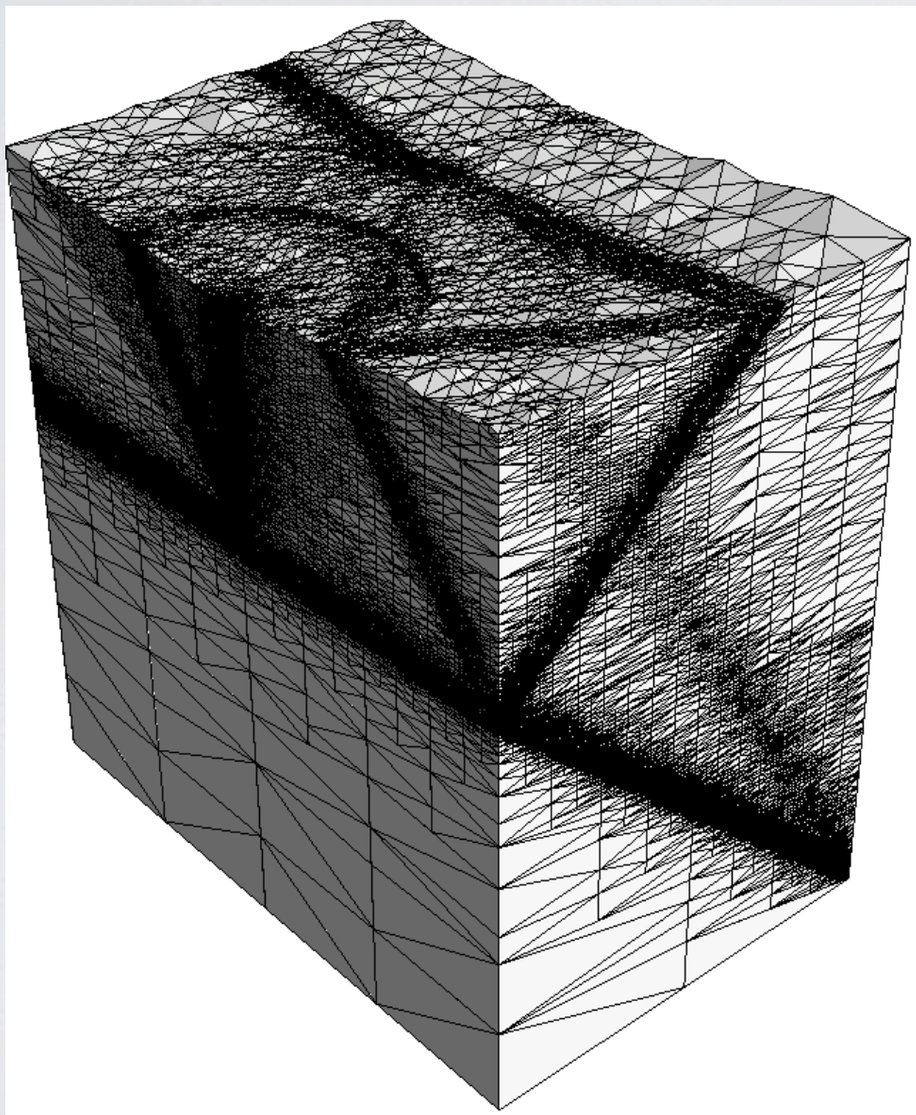
Create patch

Solve & create next patch



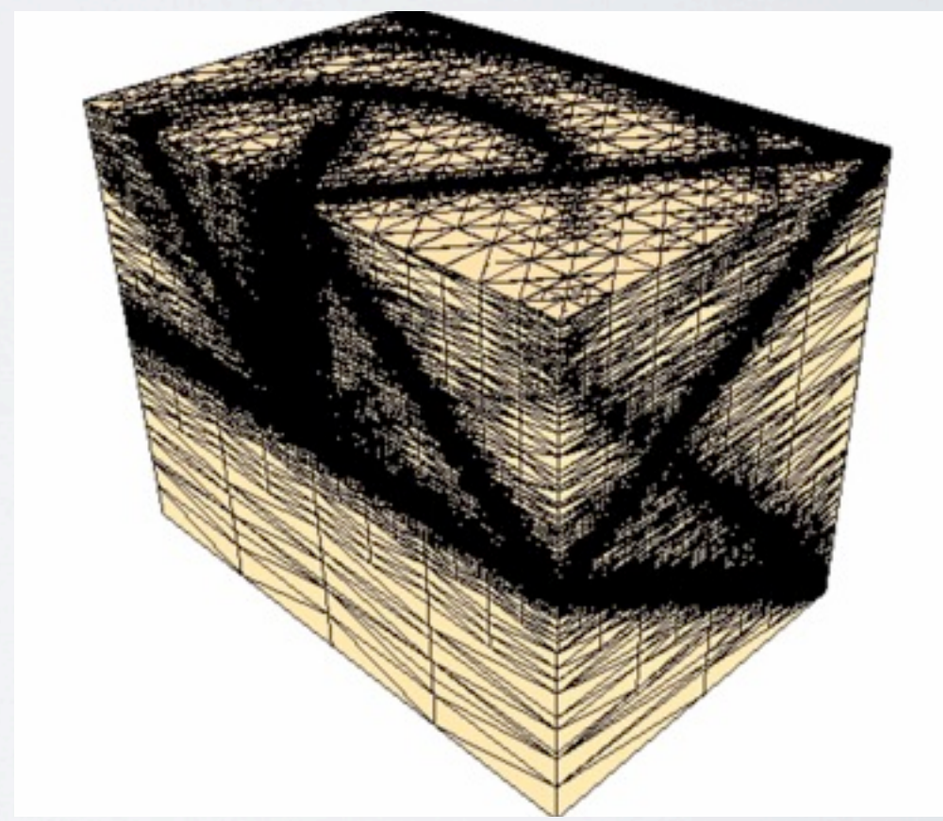
SDG tent-pitching/solution procedure enables highly adaptive spacetime meshes.

- Linearized elastodynamics: shock reflection off of a stationary crack



Time

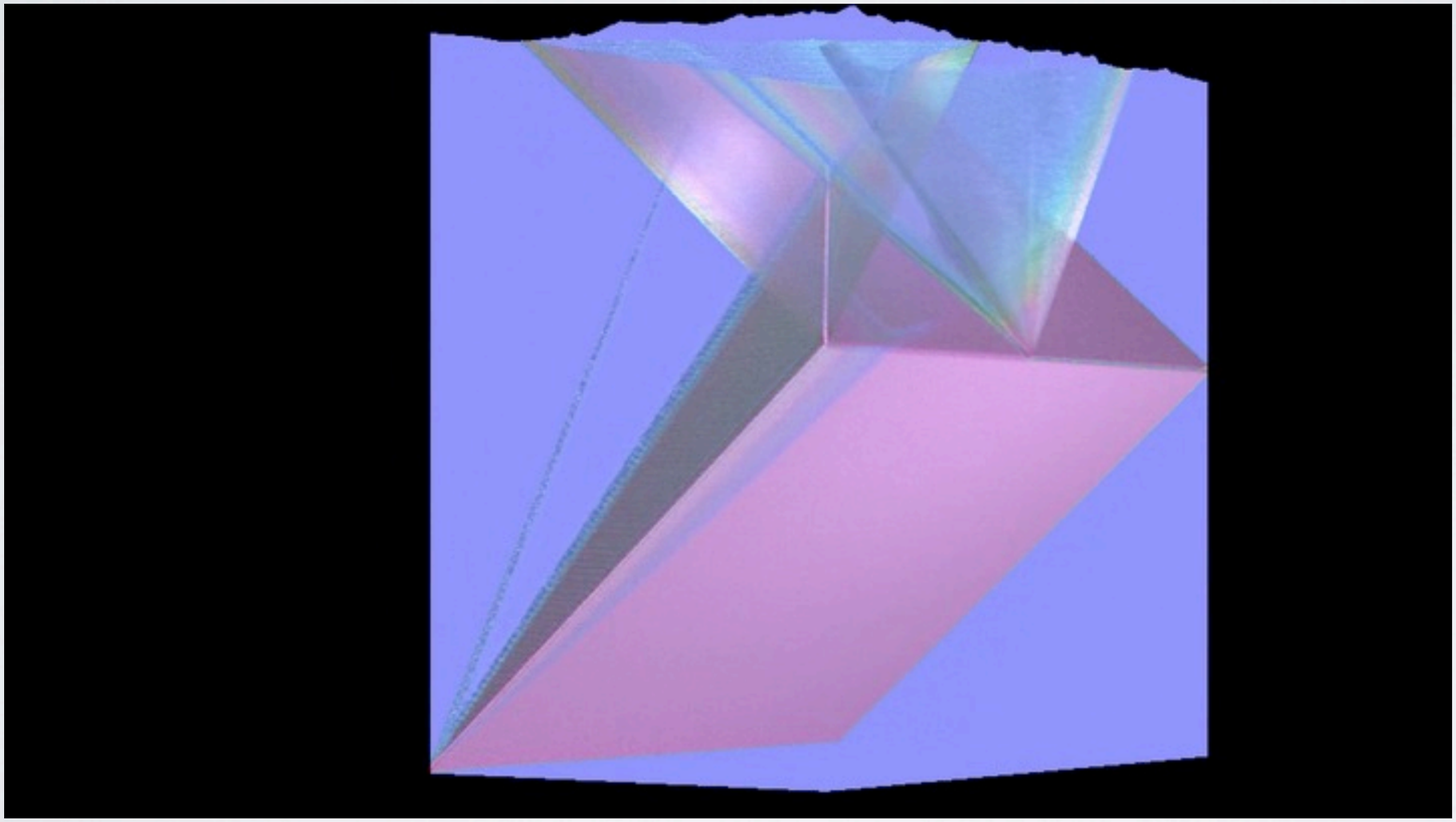
click to play movie



SDG with adaptive meshing sharply resolves waves and discontinuities.

- Discontinuous elements permit sharp discontinuity resolution

click to play movie



Differential forms notation provides a clear & concise statement of balance laws.

- No natural metric for spacetime vectors, so tensor calculus is inadequate
  - No inner product, no normal vectors
  - No Stokes' theorem, no weak formulations!
- Forms: normal free, direct expressions of spacetime balance laws
- Basic mathematical constructs and identities:

Spacetime manifold: $\mathcal{D} \subset \mathbb{E}^d \times \mathbb{R}$

One form: $\mathbf{dx} := \mathbf{e}_i \mathrm{dx}^i = \mathbf{e}_1 \mathrm{dx}^1 + \mathbf{e}_2 \mathrm{dx}^2 + \mathbf{e}_3 \mathrm{dx}^3$

d-form: $\star\mathbf{dx} := \mathbf{e}^i \star\mathrm{dx}^i = \mathbf{e}^1 \mathrm{dx}^2 \wedge \mathrm{dx}^3 - \mathbf{e}^2 \mathrm{dx}^1 \wedge \mathrm{dx}^3 + \mathbf{e}^3 \mathrm{dx}^1 \wedge \mathrm{dx}^2$

Stokes' Theorem: $\displaystyle\int_Q \mathbf{d}\boldsymbol{\omega} = \int_{\partial Q} \boldsymbol{\omega}$

$$\mathbf{d}(w\star\mathbf{dx}) = (\nabla w)\boldsymbol{\Omega} \qquad \mathbf{d}(\mathbf{a} \wedge \star\mathbf{dx}) = (\boldsymbol{\nabla} \cdot \mathbf{a})\boldsymbol{\Omega}$$

$$\mathbf{d}(w\star\mathrm{dt}) = \dot{w}\boldsymbol{\Omega} \qquad \mathbf{d}(\mathbf{a} \wedge \star\mathrm{dt}) = \dot{\mathbf{a}}\boldsymbol{\Omega}$$

Differential forms notation provides a clear & concise statement of balance laws.

- Governing equations on a spacetime control volume Q:

$$\int_{\partial\mathcal{Q}} \mathbf{F}(\mathbf{u}) - \int_{\mathcal{Q}} \mathbf{S}(\mathbf{u}) = \mathbf{0}$$

- Localized governing equations:

$$\mathbf{dF} - \mathbf{S} = \mathbf{0},$$

- Flux derivative has a familiar form:

$$\mathbf{F}(\mathbf{u}) := \mathbf{u}\star\mathrm{dt} + \mathbf{f}(\mathbf{u})\star\mathbf{dx}$$

$$\mathbf{dF} = (\dot{\mathbf{u}} + \nabla\cdot\mathbf{f}(\mathbf{u}))\,\mathrm{dx}^1\mathrm{dx}^2\mathrm{dx}^3\mathrm{dt}$$

- Jump terms arise due to the distributional nature or derivatives:

$$[\![\mathbf{F}]\!]\,\big|_{\Gamma_J} := (\mathbf{F}^* - \mathbf{F})\,\big|_{\Gamma_J} = \mathbf{0}$$

- Standard Bubnov-Galerkin weak formulation:

$$\text{Find } \mathbf{u} \in \mathcal{U} \text{ such that for every } \mathcal{Q} \in \mathcal{P}$$

$$-\int_{\mathcal{Q}} [\mathbf{d}\hat{\mathbf{w}} \wedge \mathbf{F}(\mathbf{u}) + \hat{\mathbf{w}} \wedge \mathbf{S}(\mathbf{u})] + \int_{\partial\mathcal{Q}} \hat{\mathbf{w}} \wedge \mathbf{F}^*(\mathbf{u}) = 0 \quad \forall\ \hat{\mathbf{w}} \in \mathcal{U}^{\mathcal{Q}}$$

The inter-elemental flux F* is determined by solving a local, one dimensional Riemann problem.

- **Riemann problem**: Given a hyperbolic PDE and an interface with discontinuous data across it, determine the intermediate solution states (and satisfy Rankine-Hugoniot).
- Information travels between cells only in the "normal" direction
- The Riemann problem becomes one directional
- Necessary to introduce a local coordinate system
- *Source terms do not influence the spacetime Riemann flux*

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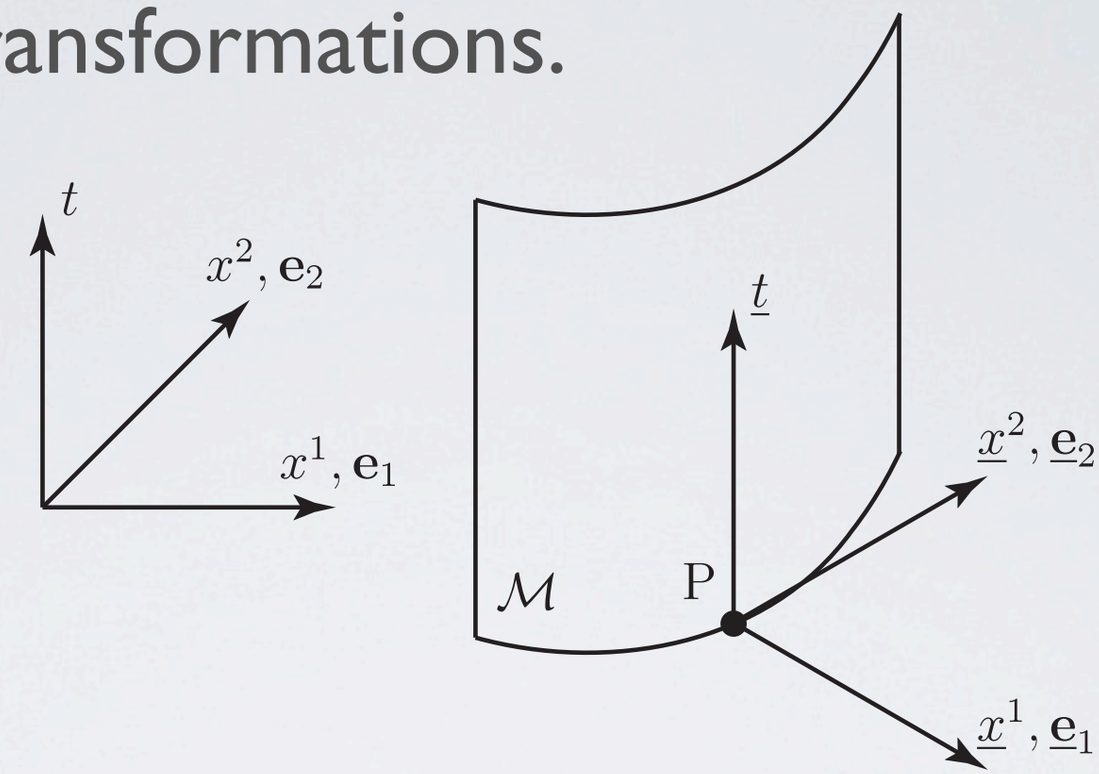
Question: How do we transform our differential forms notation into a local coordinate system so that the Riemann problem is the familiar one (e.g. from CFD)?

Coordinate transformations with forms are analogous to the "typical" transformations.



- Transform global to local coordinates: $\{x^i, t\}_{i=1}^d \to \{\underline{x}^i, \underline{t}\}_{i=1}^d$
- The *spatial* coordinates are rotated, and time is *shifted*
- Spacetime fluxes give *both* spatial and temporal fluxes through an arbitrarily oriented manifold!
- Basis vectors transform via a rotation tensor: $\underline{\mathbf{e}}_I := Q_I{}^i \mathbf{e}_i, \quad \underline{\mathbf{e}}_t := \mathbf{e}_t.$

- *Vector*-valued differential forms are objective: $\mathbf{d}\underline{\mathbf{x}} = \mathbf{dx}, \quad \star\mathbf{d}\underline{\mathbf{x}} = \star\mathbf{dx}$
- We are left to properly transform the "fields" that are coefficients of the vector valued forms, e.g. $\mathbf{u}$ in $\mathbf{u} \wedge \star\mathbf{dx}$

Standard Riemann solvers can be used once the formulation is in the local coordinate system.

One-dimensional Riemann problem for hyperbolic systems

$$\boldsymbol{u}_{,t} + \mathbf{A}\boldsymbol{u}_{,n} = \mathbf{0}, \quad \boldsymbol{u} = \{u_1, u_2, \ldots, u_n\}^T,$$

$$\mathbf{u} = \begin{cases} \mathbf{u}_\alpha, & \underline{x}_1 < 0 \\ \mathbf{u}_\beta, & \underline{x}_1 > 0, \end{cases}$$

Remarks:

- $\mathbf{A}$ is the (linearized) flux Jacobian matrix
- Source terms do not affect the Riemann solution in the SDG context
- Eigen-decomposition/diagonalisation:

$$\mathbf{A} = \boldsymbol{\Gamma}\boldsymbol{\Lambda}\boldsymbol{\Gamma}^{-1}$$

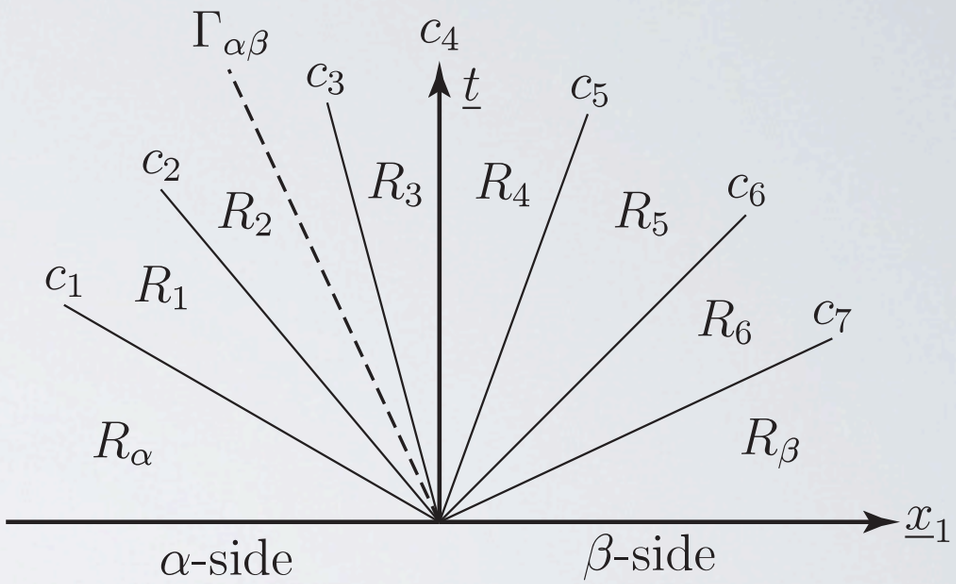
$$\boldsymbol{\Gamma} = [\gamma_1 | \cdots | \gamma_n]$$

$$\boldsymbol{\Lambda} = \begin{pmatrix} c_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & c_n \end{pmatrix}$$

Riemann solutions in characteristic variables are then 'textbook'.

Change of variables: $\boldsymbol{w}_\alpha = \boldsymbol{\Gamma}^{-1}\boldsymbol{u}_\alpha, \quad \boldsymbol{w}_\beta = \boldsymbol{\Gamma}^{-1}\boldsymbol{u}_\beta$

Riemann solution: $\boldsymbol{w}^* = \left\{ w_\beta^1, \ldots, w_\beta^p, w_\alpha^{p+1}, \ldots, w_\alpha^m \right\}^T$



Riemann solution in primitive variables: $\boldsymbol{u}^* = \boldsymbol{\Gamma}\boldsymbol{w}^*$

$$u_i^* = \left(\sum_{j=1}^{p} \Gamma_{ij} (\Gamma^{-1})_{jk} u_\beta^k \right) + \left(\sum_{j=p+1}^{m} \Gamma_{ij} (\Gamma^{-1})_{jk} u_\alpha^k \right)$$

Spatial flux: $F_i^*(\boldsymbol{u}) = A_{ij} u_j^* = \left(\sum_{j=1}^{p} \Gamma_{ij} \lambda^j (\Gamma^{-1})_{jk} u_\beta^k \right) + \left(\sum_{j=p+1}^{m} \Gamma_{ij} \lambda^j (\Gamma^{-1})_{jk} u_\alpha^k \right)$

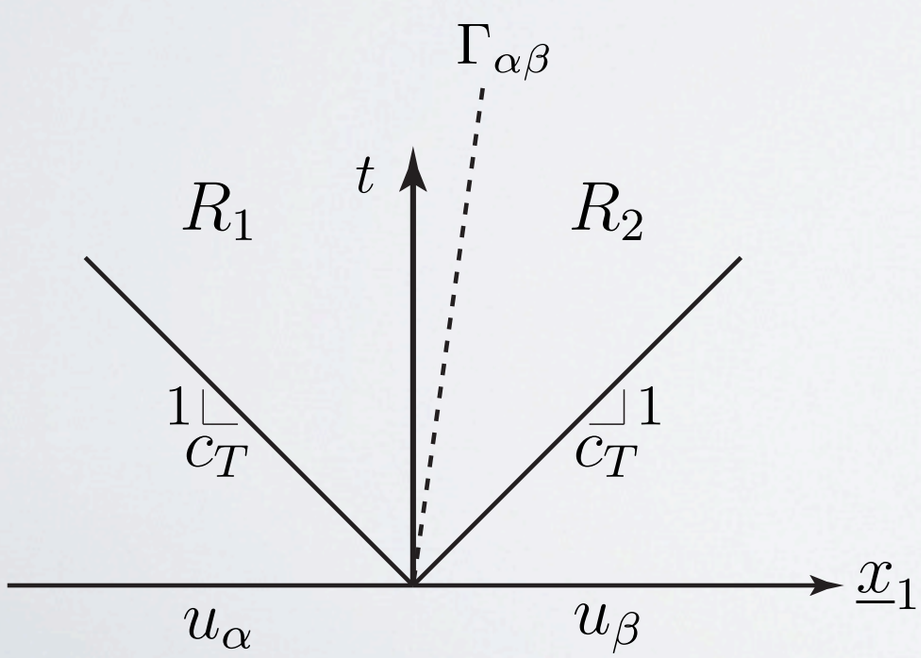
Note: The linearized eigenstructure is often much easier to compute than the entire Riemann solution.

Hyperbolic thermal model (non-Fourier): semi-analytic Riemann solver is not necessary.

Constitutive MCV equation: $\frac{1}{\kappa}\left(\tau\dot{\mathbf{q}}+\mathbf{q}\right)+\nabla T=0$

Primitive variables: $\mathbf{u}=\begin{Bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{Bmatrix}=\begin{Bmatrix}CT\\ (\tau/\kappa)\mathbf{q}\end{Bmatrix},\quad \{u_1,u_2,u_3\}=\left\{CT,\frac{\tau}{\kappa}q_n,\frac{\tau}{\kappa}q_t\right\}$

$$\mathbf{F}(\mathbf{u})=\begin{Bmatrix}\mathbf{F}_1\\ \mathbf{F}_2\end{Bmatrix}=\begin{Bmatrix}CT\star\mathrm{dt}+\mathbf{q}\wedge\star\mathbf{dx}\\ (\tau/\kappa)\mathbf{q}\star\mathrm{dt}+T\star\mathbf{dx}\end{Bmatrix},\quad \mathbf{S}(\mathbf{u})=\begin{Bmatrix}\mathbf{S}_1\\ \mathbf{S}_2\end{Bmatrix}=\begin{Bmatrix}0\\ (1/\kappa)\mathbf{q}\Omega\end{Bmatrix}$$



<u>Eigensystem</u>

$$\{c\}_{i=1}^{3}=\{-c_T,0,c_T\},\quad c_T=\sqrt{\frac{\kappa}{C\tau}}$$

$$\gamma_1=\{-\sqrt{\frac{C\kappa}{\tau}},1,0\},$$

$$\gamma_2=\{0,0,1\},$$

$$\gamma_3=\{\sqrt{\frac{C\kappa}{\tau}},1,0\}$$

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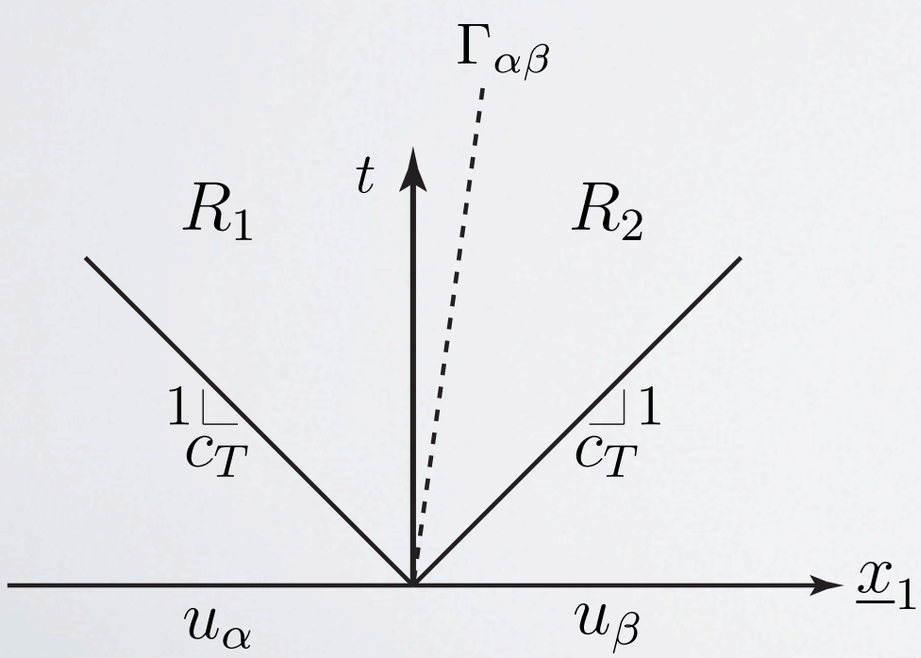


<u>Exact Riemann solution</u>

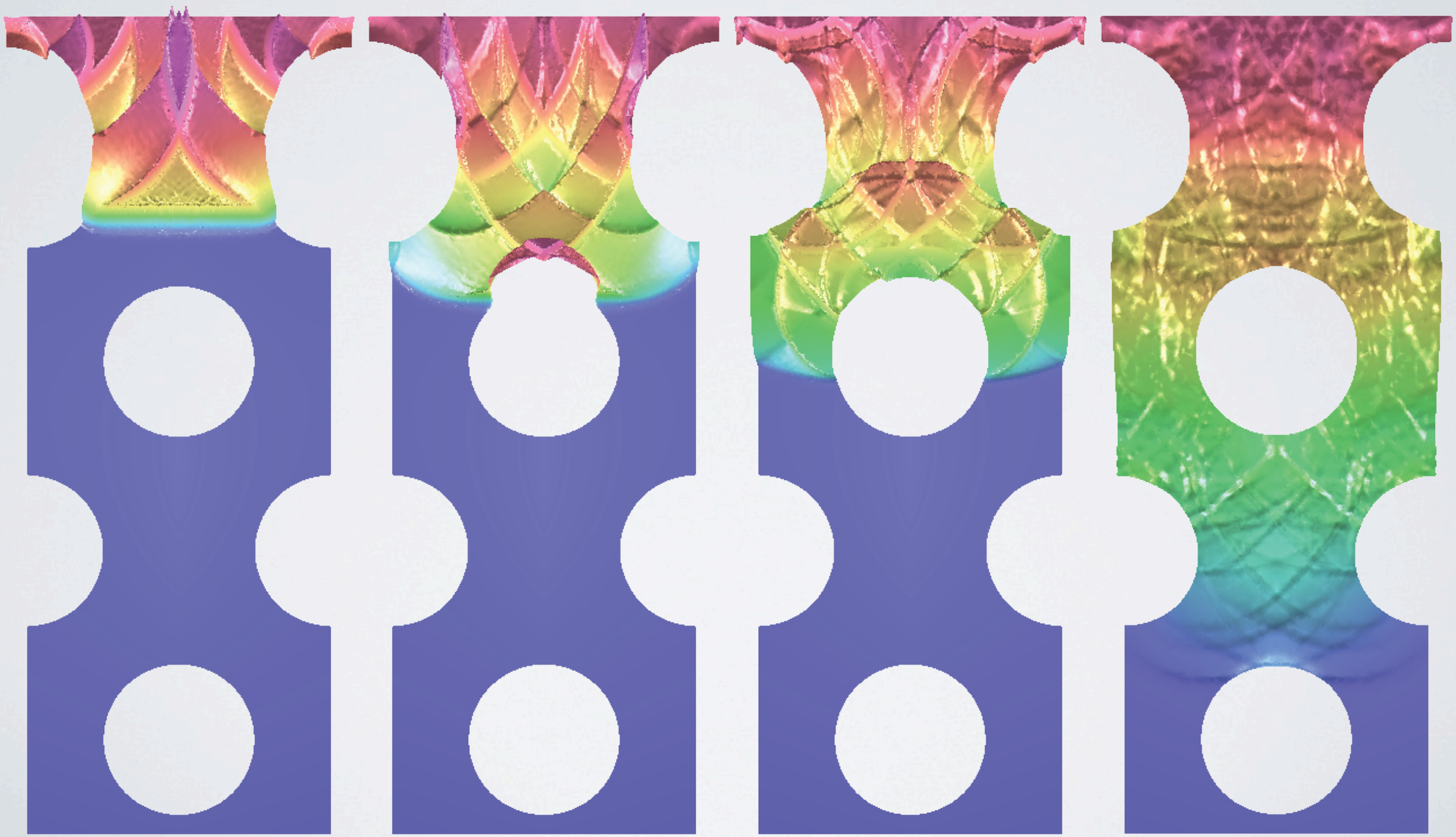
$$\underline{u}_1^*=\langle\!\langle\underline{u}_1\rangle\!\rangle-\frac{1}{2}\sqrt{\frac{C\kappa}{\tau}}\,[\![\underline{u}_2]\!] \qquad \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \text{ or } R_2$$

$$\underline{u}_2^*=\langle\!\langle\underline{u}_2\rangle\!\rangle-\frac{1}{2}\sqrt{\frac{\tau}{C\kappa}}\,[\![\underline{u}_1]\!] \qquad \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \text{ or } R_2$$

$$\underline{u}_3^*=\begin{cases}\underline{u}_3^\alpha & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1\\ \underline{u}_3^\beta & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_2\end{cases}$$

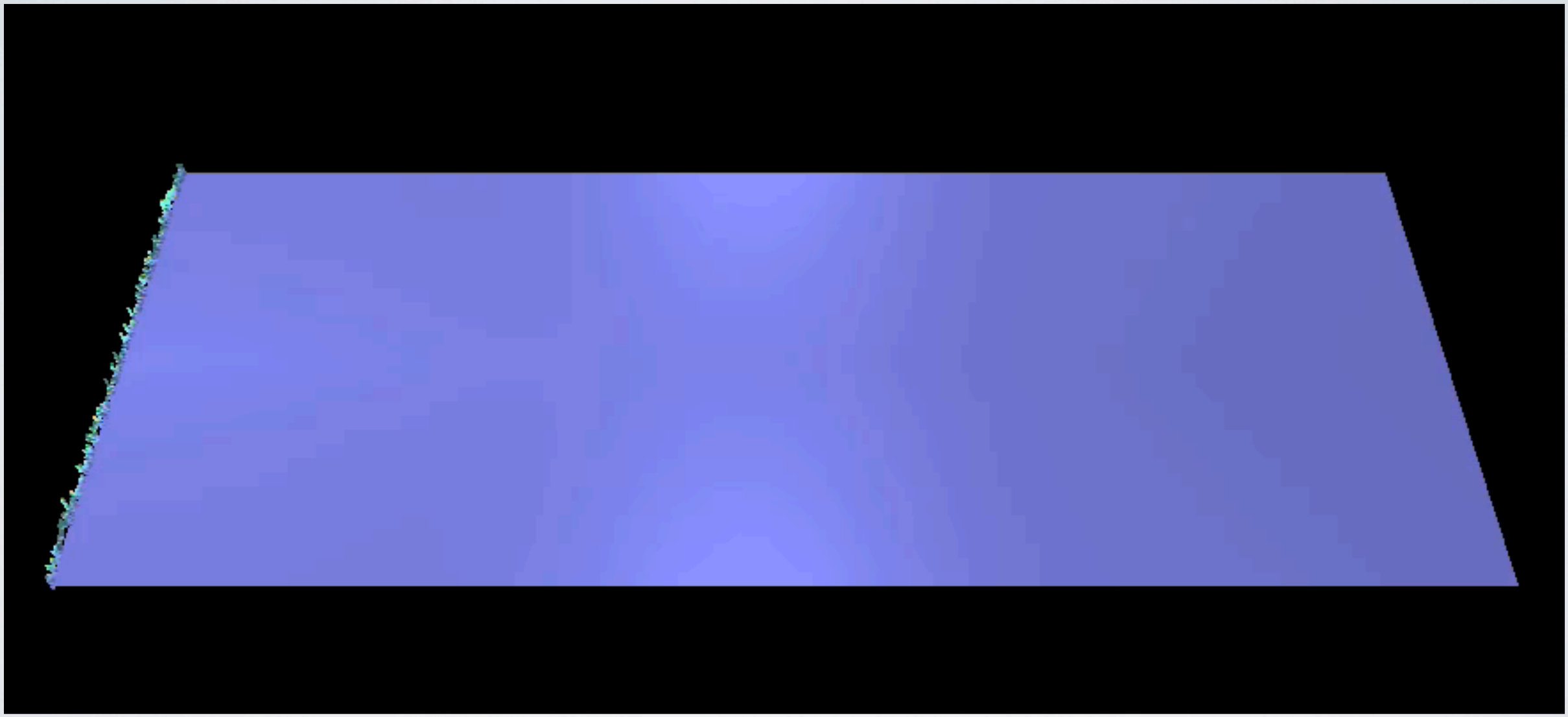
Hyperbolic heat conduction permits ‘thermal shocks’.

Thermal shock loading of a matrix with circular voids.



Hyperbolic heat conduction permits ‘thermal shocks’.

Thermal shock loading of a matrix with inclusions.



Linearized elastodynamics: weak shocks and solution dependent crack propagation

Enforce linear momentum and kinematic compatibilities

$$\boldsymbol{u} = \begin{Bmatrix} \mathbf{u} \\ \mathbf{p} \\ \mathbf{E} \end{Bmatrix}, \quad \{\underline{u}_i\}_{i=1}^{7} = \{\mathrm{u}_n, \mathrm{u}_t, \rho v_n, \rho v_t, E_{nn}, E_{nt}, E_{tt}\}$$

<u>Eigensystem</u>

$$\{c\}_{i=1}^{7} = \{-c_{\mathrm{D}}, -c_{\mathrm{S}}, 0, 0, 0, c_{\mathrm{S}}, c_{\mathrm{D}}\}, \quad c_{\mathrm{D}} = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_{\mathrm{S}} = \sqrt{\frac{\mu}{\rho}}$$

$$\begin{aligned}
\gamma_1 &= \{0, 0, \rho c_{\mathrm{D}}, 0, 1, 0, 0\}, \\
\gamma_2 &= \{0, 0, 0, 2\rho c_{\mathrm{S}}, 0, 1, 0\}, \\
\gamma_3 &= \{1, 0, 0, 0, 0, 0, 0\} \\
\gamma_4 &= \{0, 1, 0, 0, 0, 0, 0\} \\
\gamma_5 &= \{0, 0, 0, 0, -\lambda, 0, \lambda + 2\mu\} \\
\gamma_6 &= \{0, 0, 0, -2\rho c_{\mathrm{S}}, 0, 1, 0\}, \\
\gamma_7 &= \{0, 0, -\rho c_{\mathrm{D}}, 0, 1, 0, 0\}
\end{aligned}$$

Linearized elastodynamics: Exact Riemann solution

$$u_1^* = \begin{cases} f_1^\alpha & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \text{ or } R_2 \\ f_1^\beta & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_3 \text{ or } R_4 \end{cases}$$

$$u_2^* = \begin{cases} f_2^\alpha & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \text{ or } R_2 \\ f_2^\beta & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_3 \text{ or } R_4 \end{cases}$$

$$u_3^* = \langle\!\langle f_3 \rangle\!\rangle + \frac{\rho c_{\mathrm{D}}}{2} \left([\![f_5]\!] + \frac{\lambda}{\lambda + 2\mu} [\![f_7]\!] \right), \qquad \text{for all regions}$$

$$u_4^* = \begin{cases} f_4^\alpha & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \\ \langle\!\langle f_4 \rangle\!\rangle + \rho c_{\mathrm{S}} [\![f_6]\!] & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_2 \text{ or } R_3 \\ f_4^\beta & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_4 \end{cases}$$

$$u_5^* = \frac{1}{2\rho c_{\mathrm{D}}} [\![f_3]\!] + \langle\!\langle f_5 \rangle\!\rangle + \frac{\lambda}{2(\lambda + 2\mu)} [\![f_7]\!], \qquad \text{for all regions}$$

$$u_6^* = \begin{cases} f_6^\alpha & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \\ \frac{1}{4\rho c_{\mathrm{S}}} [\![f_4]\!] + \langle\!\langle f_6 \rangle\!\rangle & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_2 \text{ or } R_3 \\ f_6^\beta & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_4 \end{cases}$$

$$u_7^* = \begin{cases} f_7^\alpha & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \text{ or } R_2 \\ f_7^\beta & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_3 \text{ or } R_4 \end{cases}$$

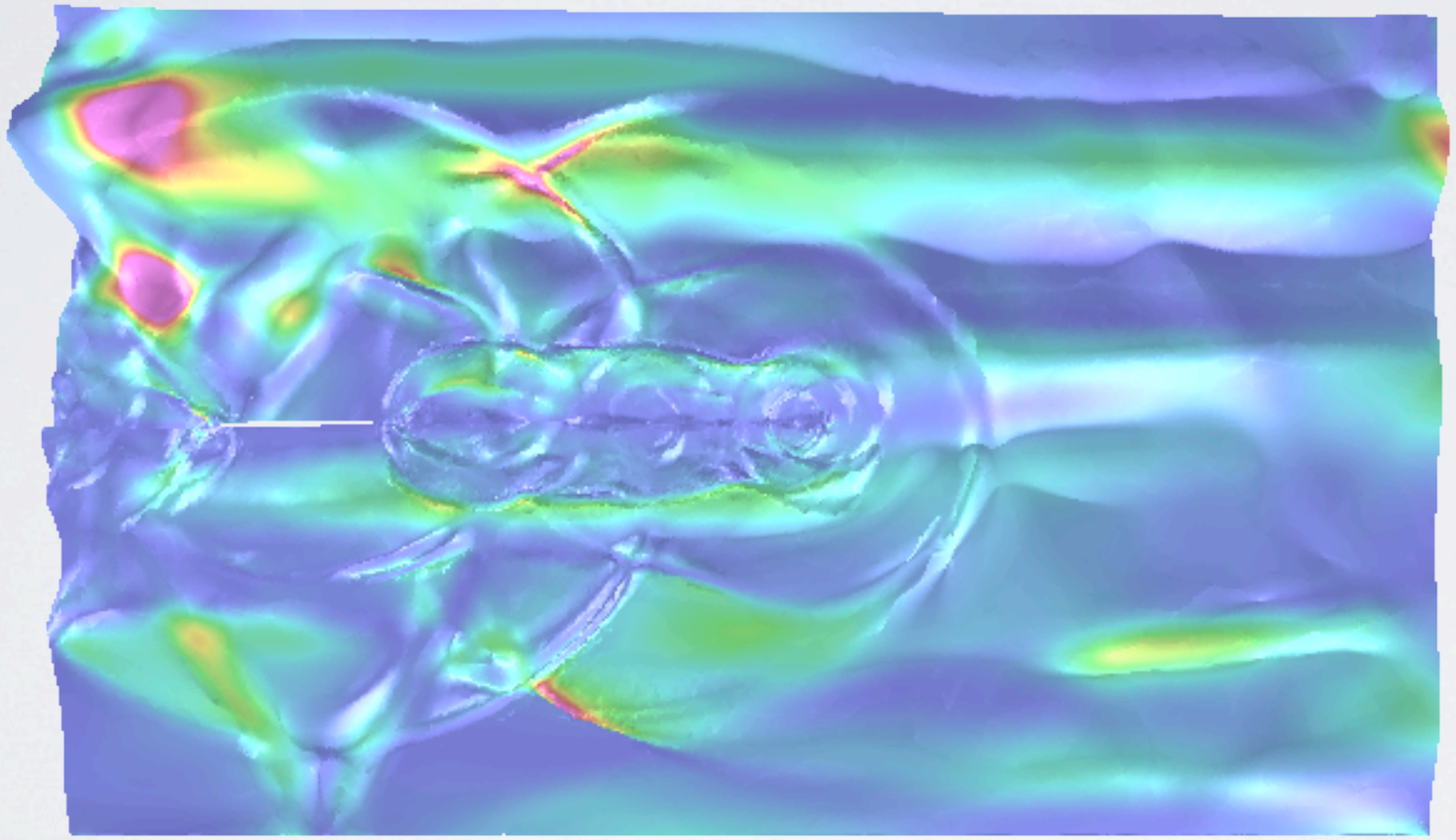
Linearized elastodynamics: weak shocks are resolved over multiple scales with spacetime adaptive meshing

click to play movie

Linearized elastodynamics: weak shocks and solution dependent crack propagation

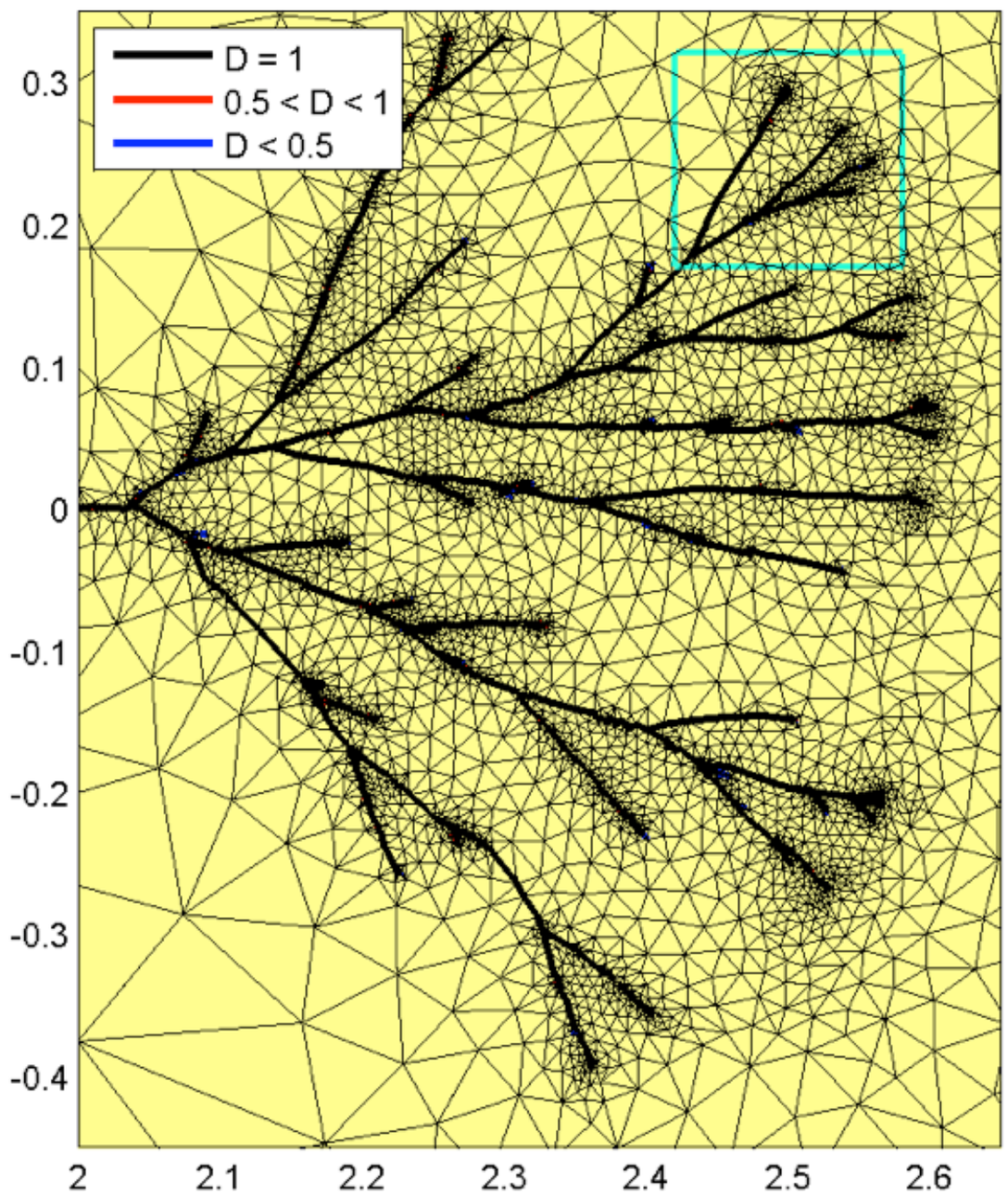
Contact and cohesive zone models are viewed as special Riemann conditions on crack faces.

click to play movie



Linearized elastodynamics: weak shocks and solution dependent crack propagation

Mesh refinement enables prediction of crack paths with high fidelity.



Generalized thermoelasticity: coupled multiphysics benefit from using a semi-analytic exact Riemann solver

- Governing equations combine elastodynamics and thermal model
- Stress is temperature dependent
- Energy balance now has a velocity term
- Primitive variables:

$$\{\underline{u}_i\}_{i=1}^{10} = \{\mathrm{u}_n, \mathrm{u}_t, \rho v_n, \rho v_t, E_{nn}, E_{nt}, E_{tt}, T, q_n, q_t\}$$

- Coupled wavespeeds are complicated:

$$c_\mathrm{d} = \left[\frac{1}{2}\left(c_\mathrm{D}^2 + c_T^2 + \beta - \sqrt{-4c_\mathrm{D}^2 c_T^2 + (c_\mathrm{D}^2 + c_T^2 + \beta)^2}\right)\right]^{\frac{1}{2}}$$

$$c_t = \left[\frac{1}{2}\left(c_\mathrm{D}^2 + c_T^2 + \beta + \sqrt{-4c_\mathrm{D}^2 c_T^2 + (c_\mathrm{D}^2 + c_T^2 + \beta)^2}\right)\right]^{\frac{1}{2}}$$

- Exact Riemann solutions get 'nasty'; e.g. the simplest one is

$$(u_8)^* = CT^* = \langle\!\langle u_8 \rangle\!\rangle - \frac{1}{2\rho(c_d + c_t)}\left(\rho C(c_d c_t + c_T^2)\,[\![u_9]\!] + kT_0\,[\![u_3]\!]\right)$$

Generalized thermoelasticity: coupled multiphysics benefit from using a semi-analytic exact Riemann solver

- Eigensystem is much easier to obtain and implement
- Symbolic computer algebra can be used
- Only need to replace eigenvalues/vectors for each subproblem

$$
\begin{aligned}
\gamma_1 &= \{0, 0, \rho c_{\mathrm{D}}^2 (c_t - c_T)(c_t + c_T), 0, c_t(c_D^2 - c_d^2), 0, 0, -c_D^2 c_t C\theta, cD^2\theta, 0\}, \\
\gamma_2 &= \{0, 0, \rho c_{\mathrm{D}}^2 (c_d - c_T)(c_d + c_T), c_d(c_D - c_t)(c_D + c_t), 0, 0, -c_D^2 c_d C\theta, c_D^2\theta, 0\}, \\
\gamma_3 &= \{0, 0, 0, 2\rho c_{\mathrm{S}}, 0, 1, 0, 0, 0, 0\}, \\
\gamma_4 &= \{0, 1, 0, 0, 0, 0, 0, 0, 0, 0\}, \\
\gamma_5 &= \{0, 0, 0, 0, 0, 0, 0, 0, 0, 1\}, \\
\gamma_6 &= \{0, 0, 0, 0, -\lambda, 0, \rho c_D^2, 0, 0, 0\}, \\
\gamma_7 &= \{1, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \\
\gamma_8 &= \{0, 0, 0, -2\rho c_{\mathrm{S}}, 0, 1, 0, 0, 0, 0\}, \\
\gamma_9 &= \{0, 0, \rho c_D^2 (c_d - c_T)(c_d + c_T), c_d(c_t^2 - c_D^2), 0, 0, C c_d c_D^2\theta, c_D^2\theta, 0\}, \\
\gamma_{10} &= \{0, 0, \rho c_D^2 (c_t - c_T)(c_t + c_T), c_t(c_d - c_D)(c_d + c_D), 0, 0, C c_D^2 c_t\theta, c_D^2\theta, 0\}
\end{aligned}
$$

Riemann solutions for unstructured spacetime DG methods are as straightforward as any other method.

- Don't let the differential forms notation put you off!
- The forms are objective with respect to coordinate systems
- Rotations into local coordinates are familiar
- Source terms in equations *do not matter* with SDG methods!
- Complicated systems of PDEs from multiphysics problems can benefit substantially from a semi-analytic exact Riemann solver
- PDEs do not need to be linear, just linearized at the flux location
- Extensions to exact non-linear Riemann solvers should be straightforward, but with a much higher computational cost