Comparison of Finite Element Methods for Elastodynamics Problem

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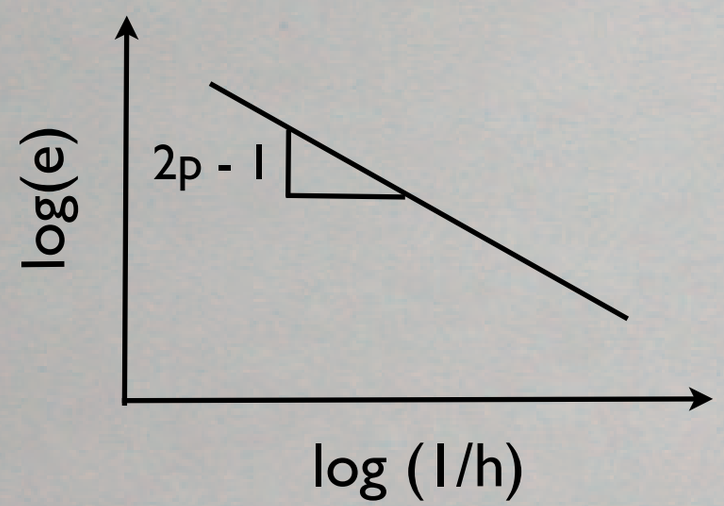
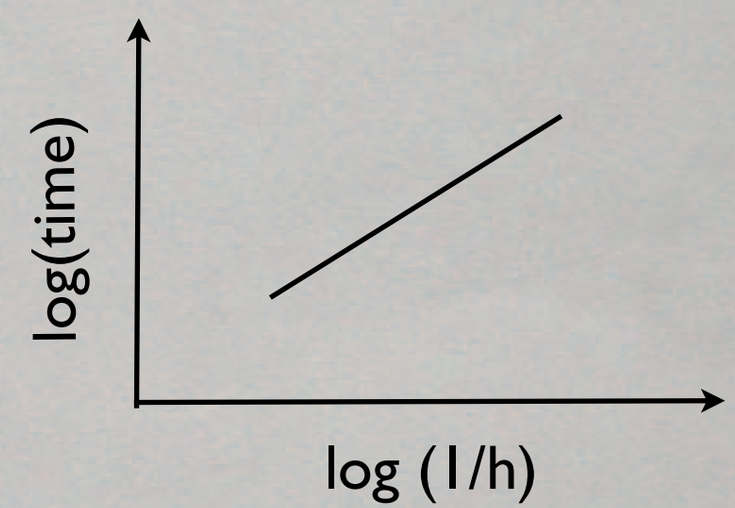
Comparison Criteria

- Convergence Rates: error vs. element size (h)
- Solution Scaling: cost vs. number of elements (N)





- **Efficiency: cost vs. error**
- Memory usage
- Software Engineering aspects: e.g. ease of implementation and extensibility

Numerical method and problem types

- Underlying mathematical model:
  - linear vs nonlinear
  - type of PDE: Elliptic, Parabolic, and Hyperbolic
- Solution features: shocks and singularities
- Specifics of the numerical method:
  - Formulation type (single-field and multi-field formulations)
  - Element type
  - Integration scheme
- Adaptive Implementations: *h*- and *hp*-adaptive
- Parallel computing

Sample numerical methods: A. Time marching schemes

- Displacement and velocity are both primitive fields.

$$\mathbf{U} = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}$$

- Solve linear momentum balance and kinematic compatibility.
- Consistent mass matrices are used

$$\mathbf{M} = \begin{bmatrix} \hat{u}_i \hat{u}_j & 0 \\ 0 & \hat{v}_i \hat{v}_j \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} 0 & -\hat{u}_i \hat{v}_j \\ 0 & \nabla \hat{v}_i \cdot \mathbb{C}[\nabla \hat{v}_j] \end{bmatrix}$$

$$\mathbf{M}\,\dot{\mathbf{U}} + \mathbf{K}\,\mathbf{U} = 0 \Longrightarrow \dot{\mathbf{U}} = \mathbf{M}^{-1}\mathbf{K}\,\mathbf{U}$$

- deal.ii is used to implement time marching schemes.

A. Time marching schemes
Standard ODE: $\dot{u} = f(t, u).$

- Forward Euler: explicit, first order accurate

$$\dot{u}_{n+1} = u_n + (\Delta t)\, f(t_n, u_n)$$

- Explicit Runge-Kutta (Strong stability Preserving (SSP) variant):

$$u_0 = u(t^n)$$

$$u_i = k_i(\Delta t) f(t_{i-1}, u_{i-1}) + \sum_{j=1}^{i} c_{ij} u_j \quad \text{for } i = 1 - 5$$

$$u(t^{n+1}) = u_5$$

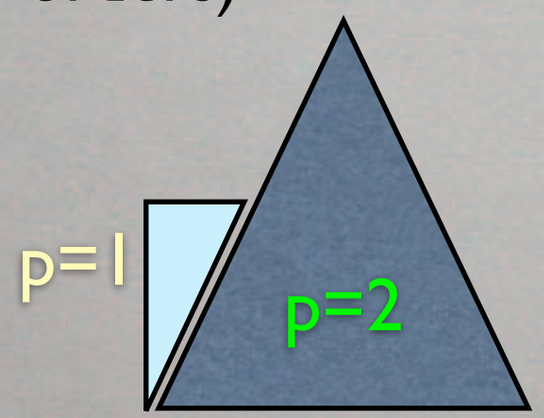
- Bathe's method (2007)
  - Implicit and 2nd order accurate
  - Combines a trapezoidal rule step with a backward difference formula

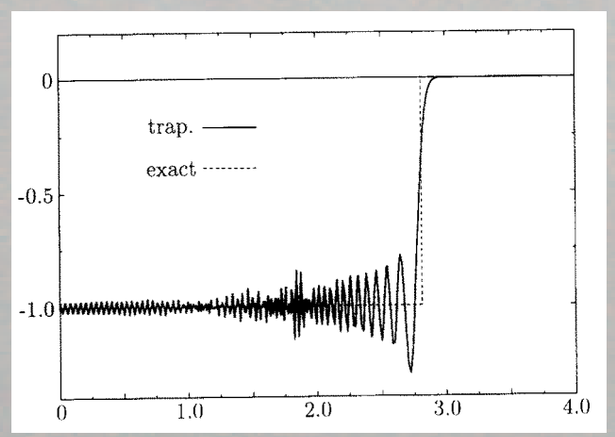
$$u_{n+1/2} = u_n + \frac{\Delta t}{4}\left(\dot{u}_n + f(t_{n+1/2}, u_{n+1/2})\right)$$

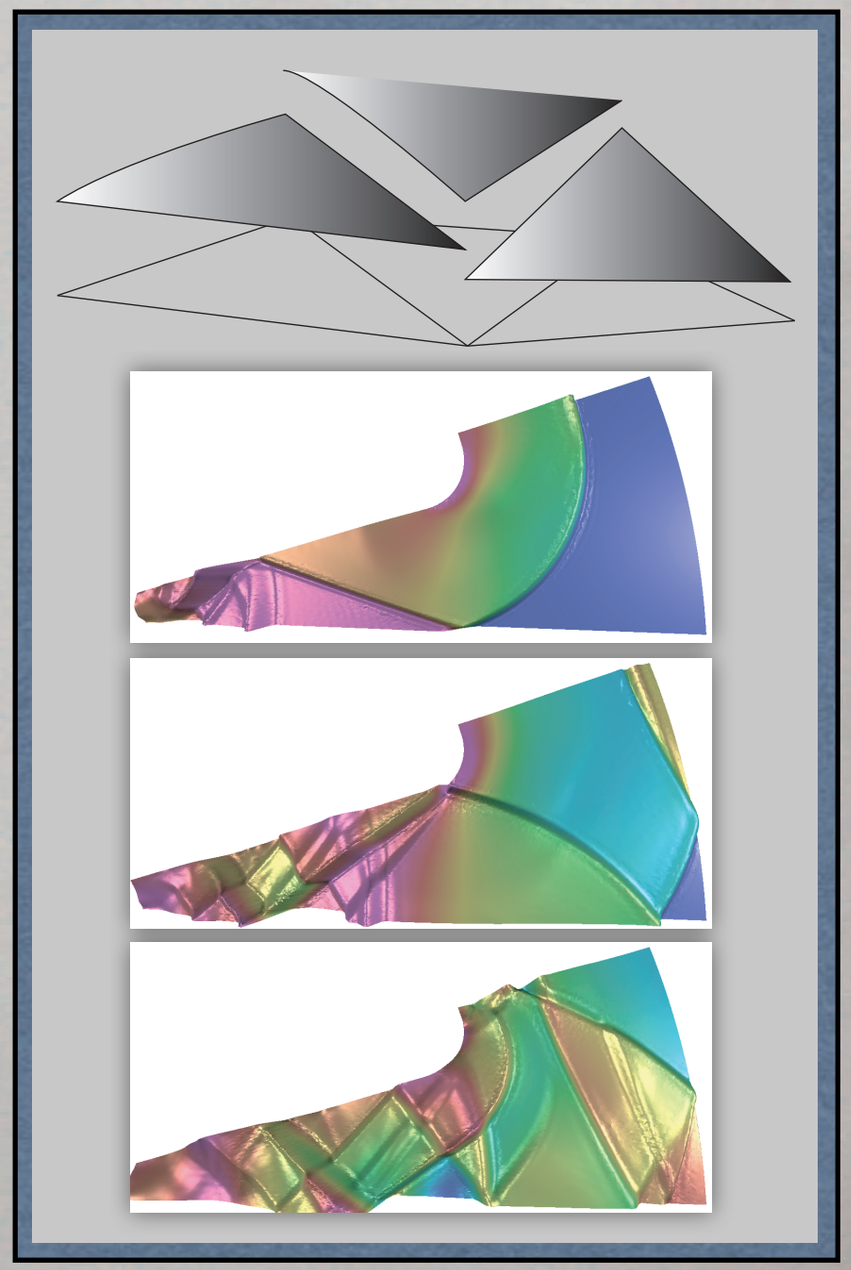
$$u_{n+1} = \frac{1}{3}\left(4u_{n+1/2} - u_n + (\Delta t) f(t_{n+1}, u_{n+1})\right)$$

B. Spacetime Discontinuous Galerkin(SDG) finite element method

- Discontinuous basis functions
  - Element-wise balance property
  - Superior performance for resolving discontinuous features without spurious oscillation or excessive dissipation
  - Support for nonconforming meshes
  - Flexibility in shape functions (polynomial orders)

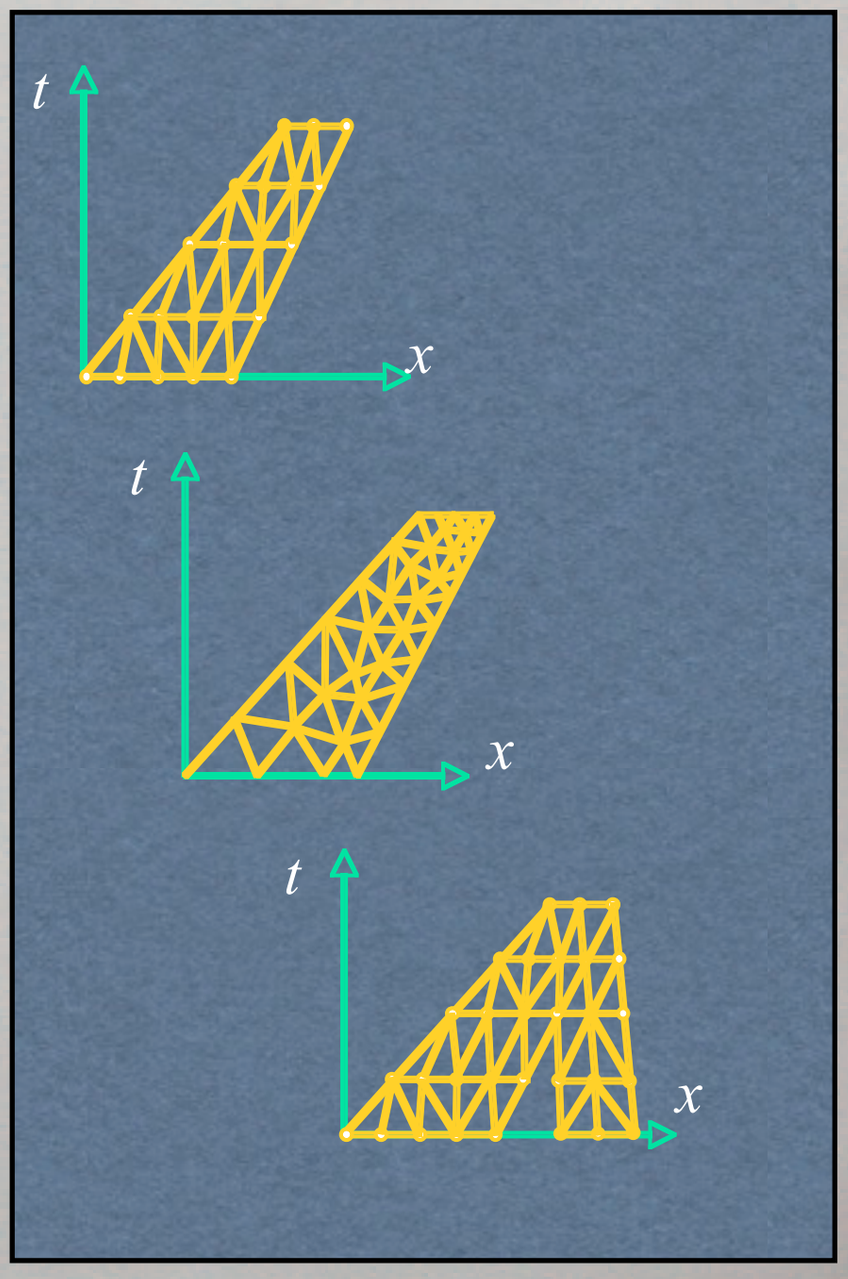






B. Spacetime Discontinuous Galerkin(SDG) finite element method

- Direct discretization of spacetime
  - Replaces separate temporal integration
  - Unified treatment of initial and boundary conditions
- Unstructured spacetime mesh
  - Simultaneous mesh gradation in space and time
  - Eliminates mesh tangling in moving–grid methods



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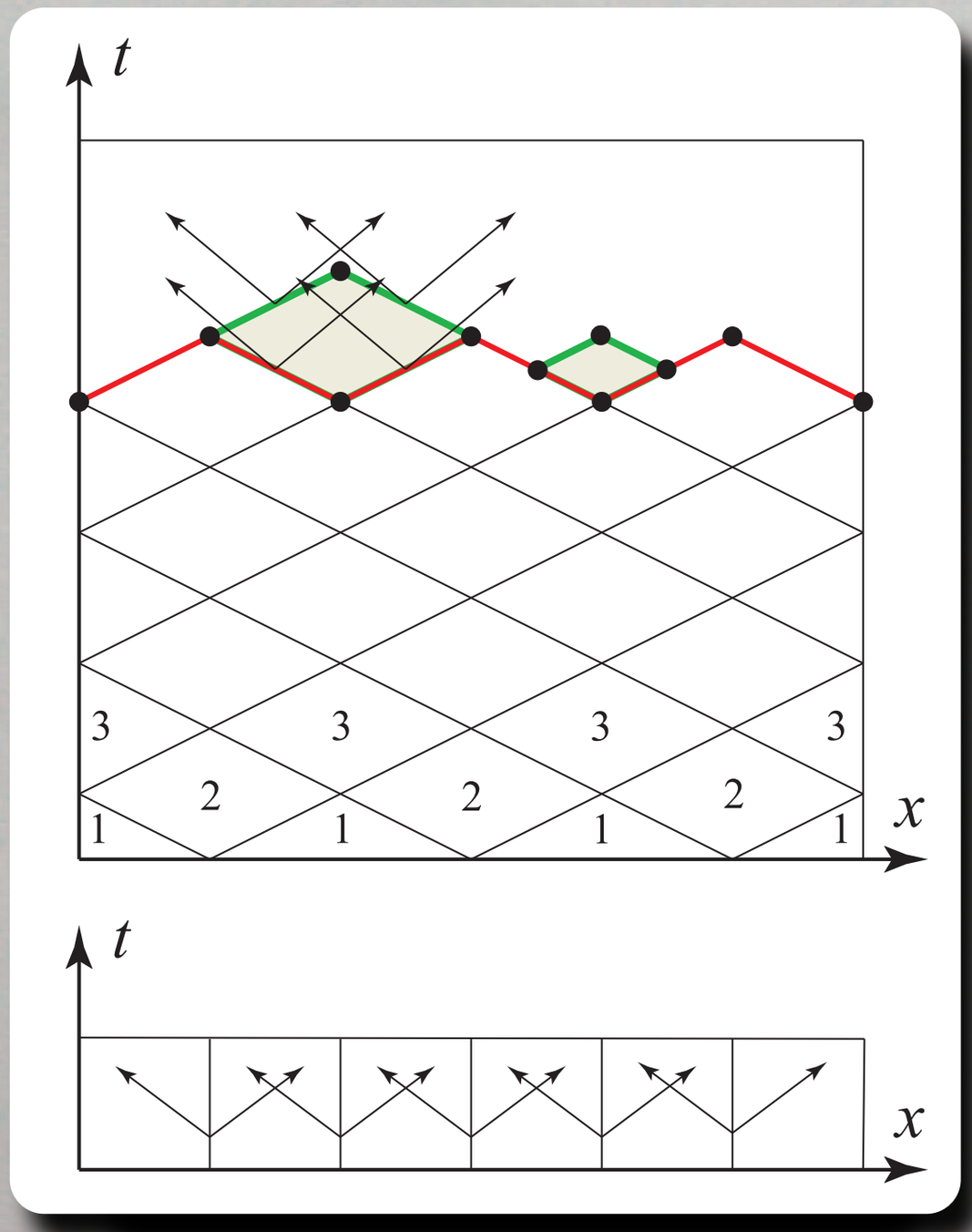
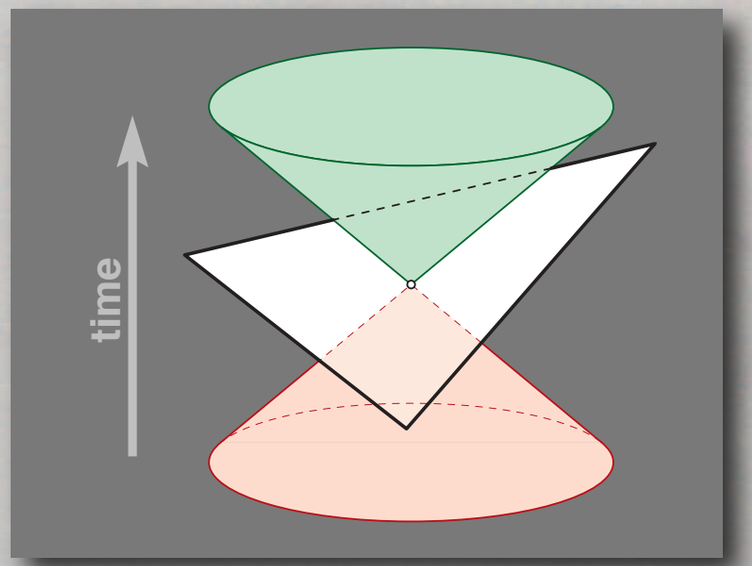
Causal spacetime mesh and scalable Advancing-Front Solution Strategy

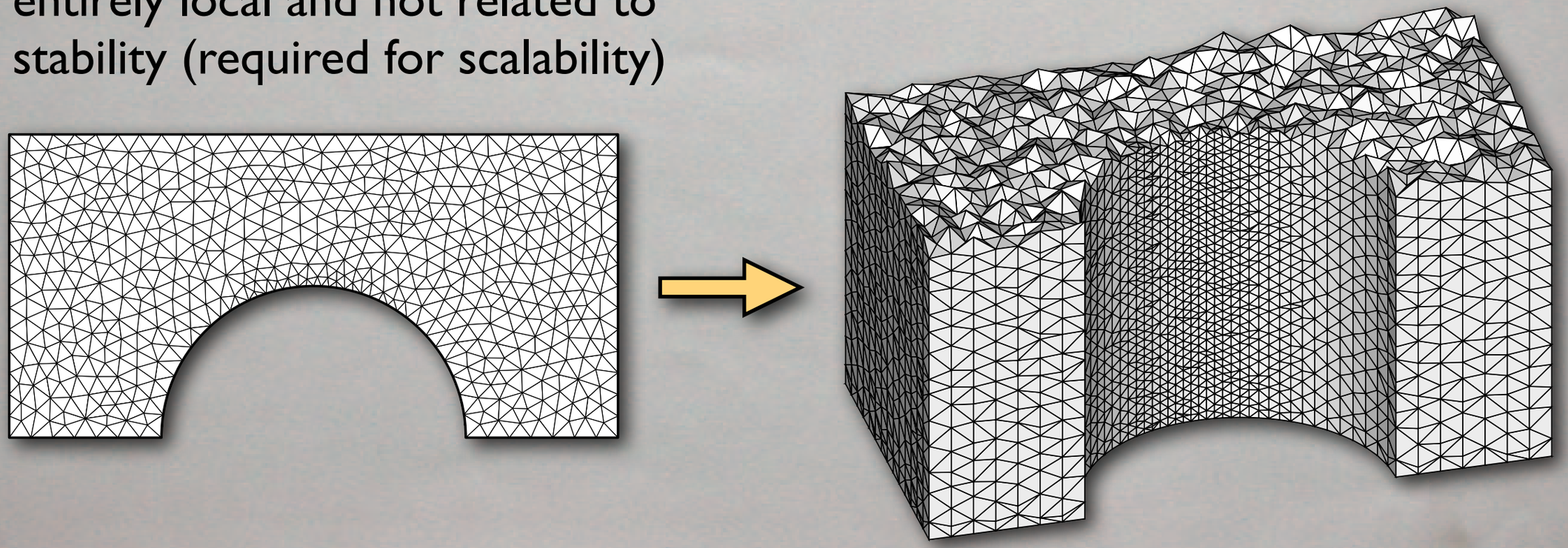


Tent Pitcher: causal spacetime meshing

- Given a space mesh, Tent Pitcher constructs a spacetime mesh such that every facet on sequence of advancing fronts is spacelike (patch height bounded by causality constraint)
- Similar to CFL condition, except entirely local and not related to stability (required for scalability)

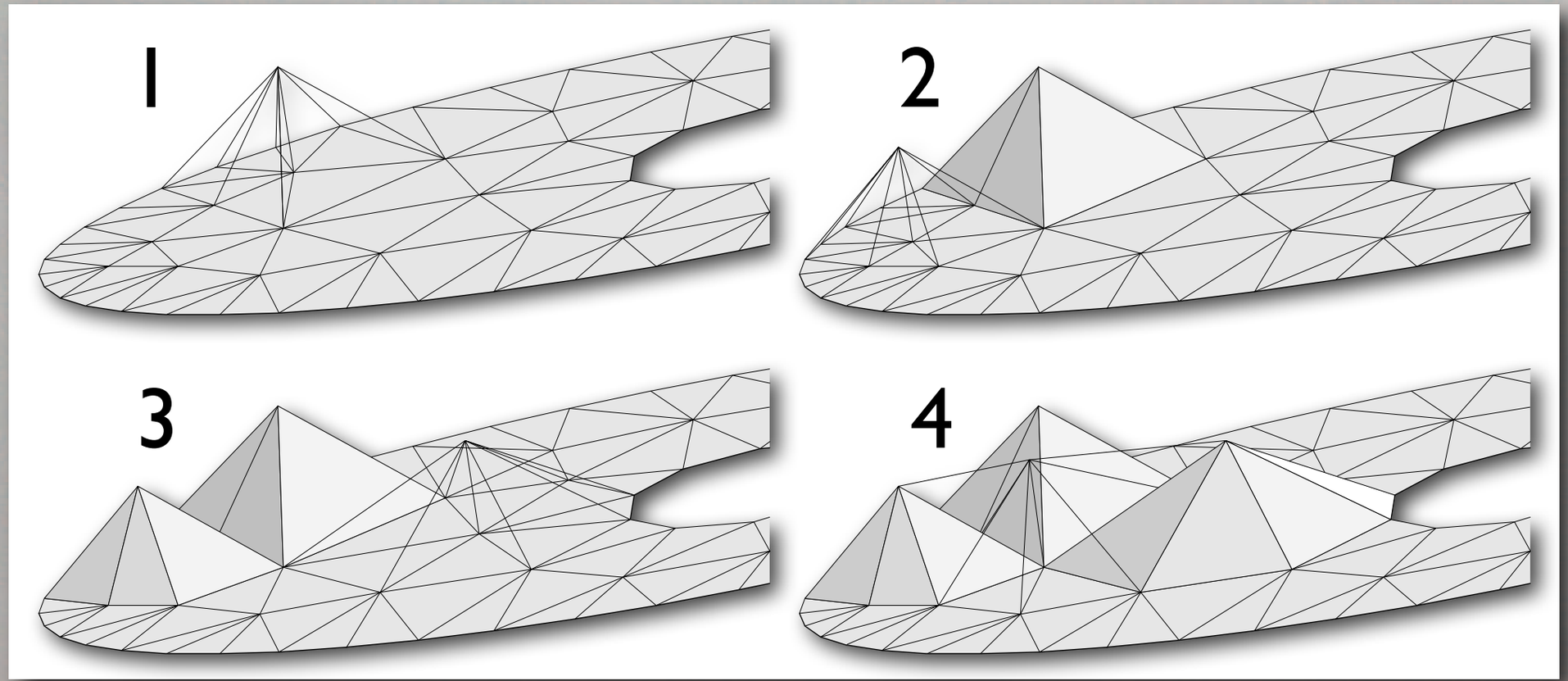


causality constraint



Tent Pitcher: patch–by–patch meshing & solution

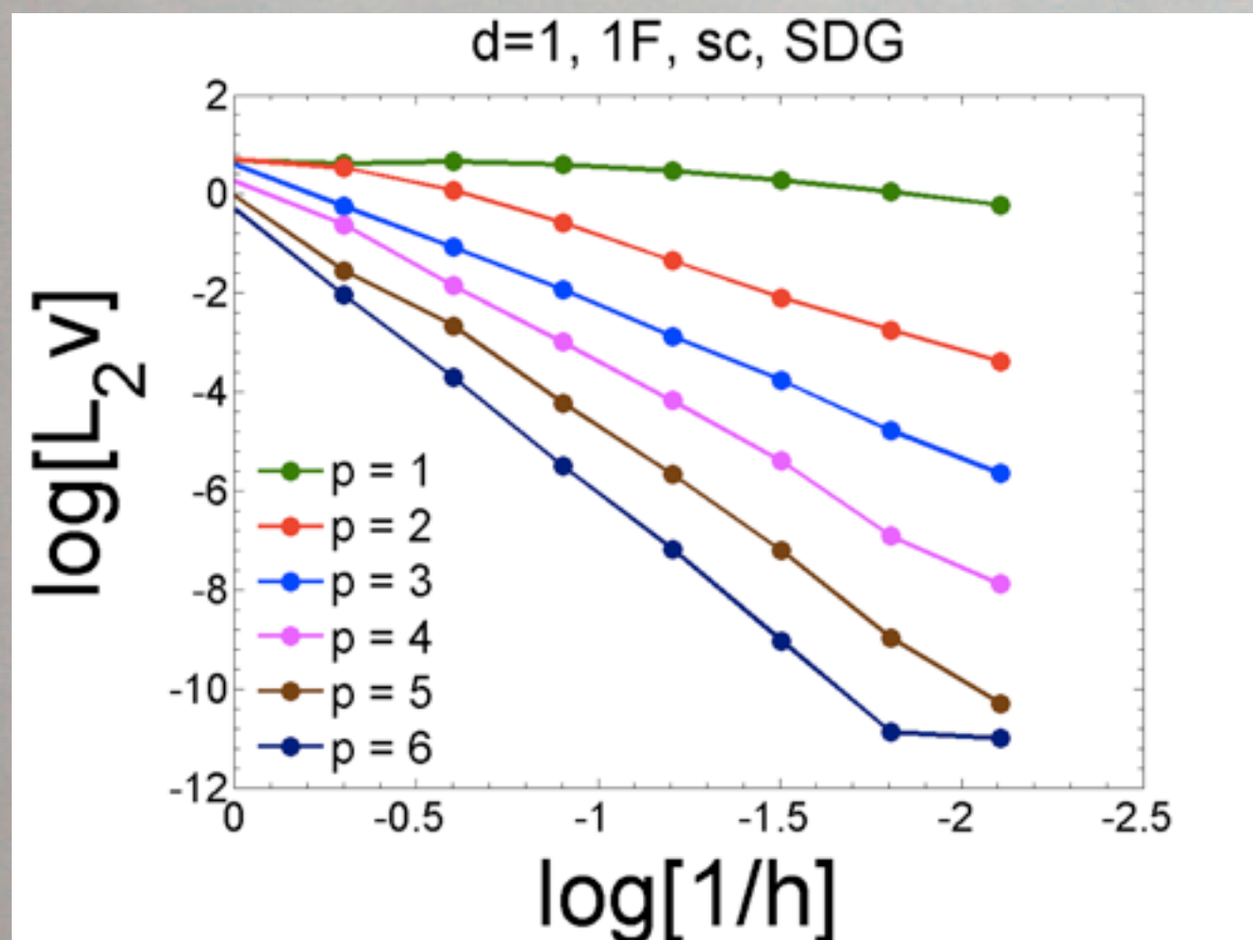
- Meshing and solutions are interleaved
  - Patches ('tents') of tetrahedra are solve immediately
- patches are created and solved in parallel



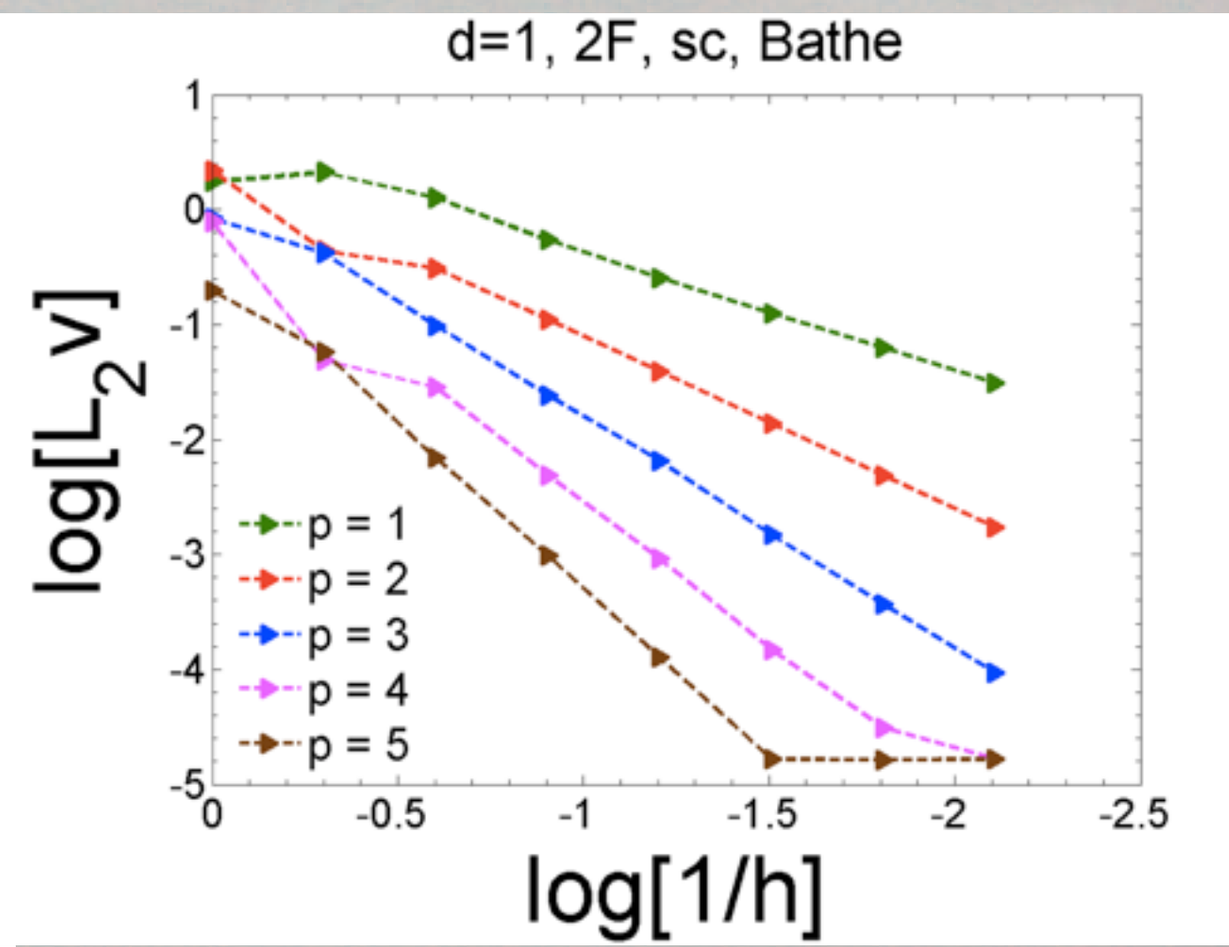
FEM comparisons

Convergence rate

SDG



Bathe



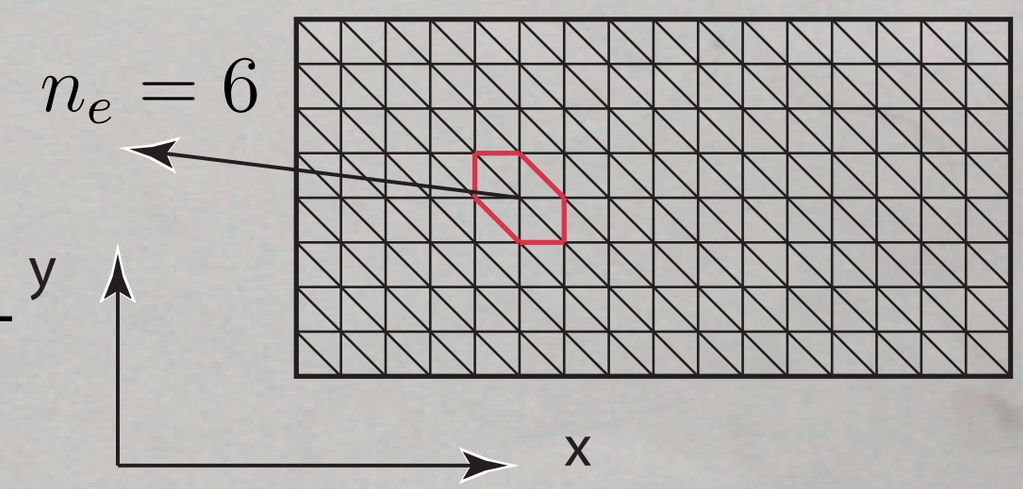
$$e = Ch^{ap+b} \quad \Rightarrow \log(e) = log(C) - (ap+b)\log(1/h)$$

- a and b depend on both spatial and temporal discretization and integration schemes

Solution Scaling

Various parts of the solution scale differently against the number of elements:

n_e: number of elements in a patch
n_p: number of patches
n: total number of elements $= n_p \langle n_e \rangle$



c_p: computational cost for a patch
c: total cost $\propto n_p c_p$
$c_p \propto n_e f_{EA} + n_e^{\beta} f_{LU}$

- SDG Method:

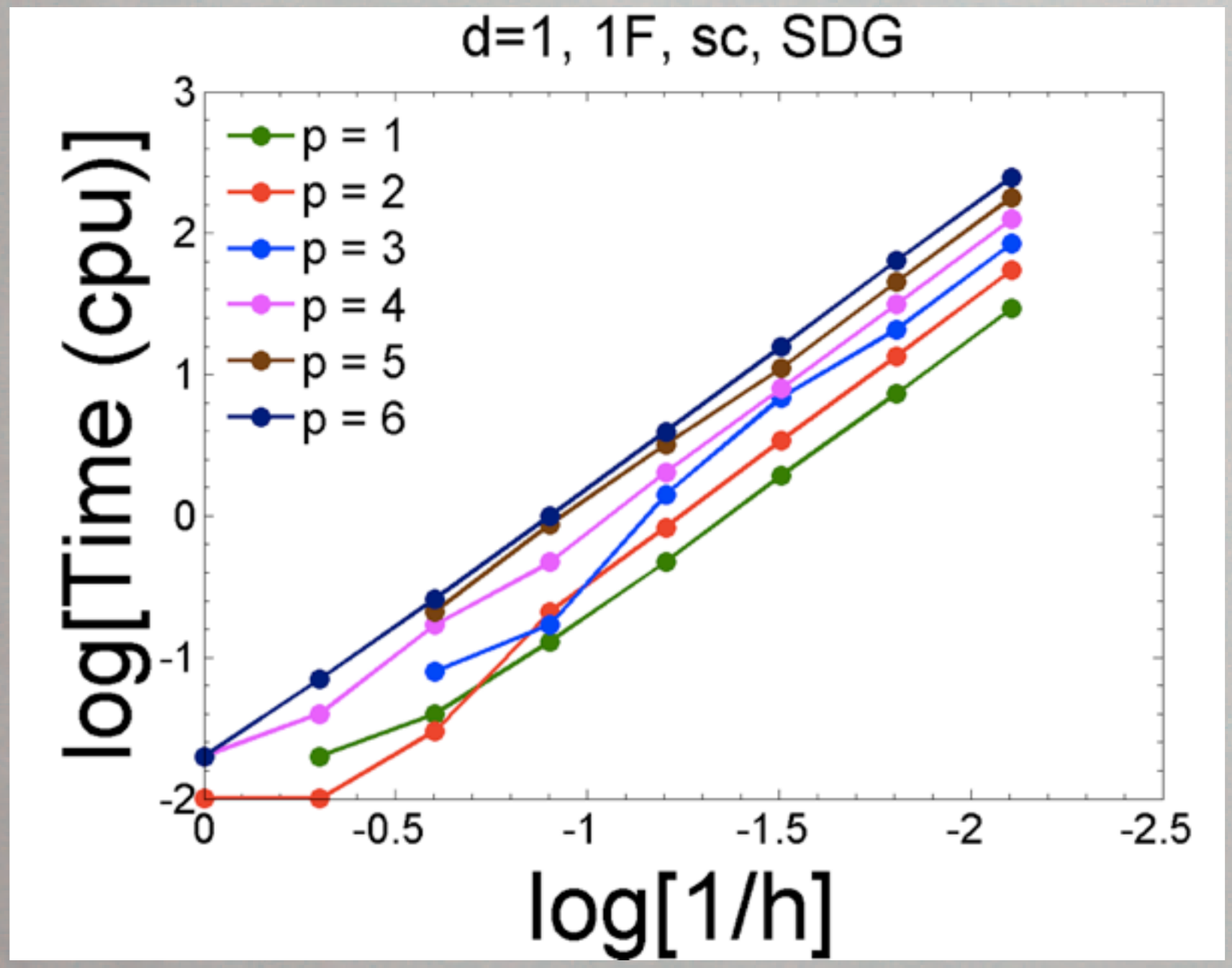
$$c \propto n_p c_p \propto n c_p \quad \Rightarrow \quad c \propto n \; c_p \propto h^{-(d+1)} \; c_p$$

- Globally coupled time marching schemes: $n_p = 1 (n_e = n) \Rightarrow$

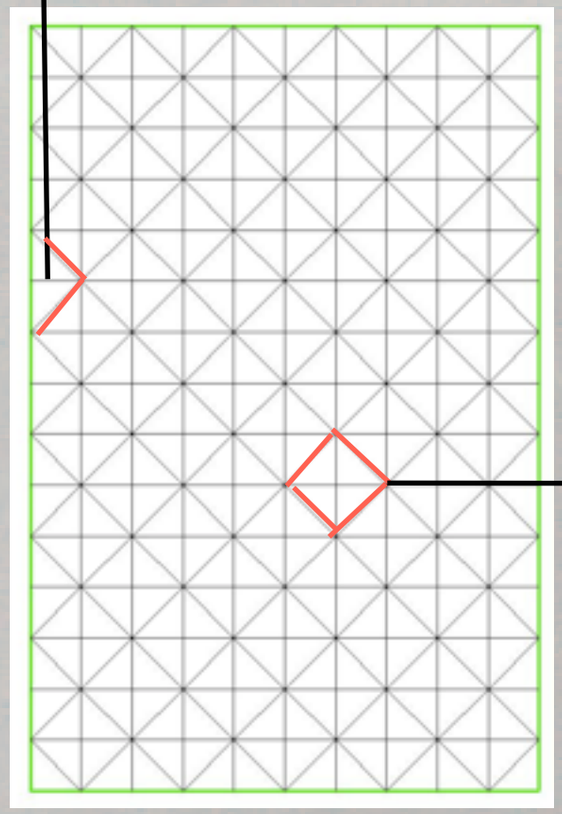
$$c \propto n \; f_{EA} + n^{\beta} \; f_{LU}$$

Solution Scaling

SDG



2 elements

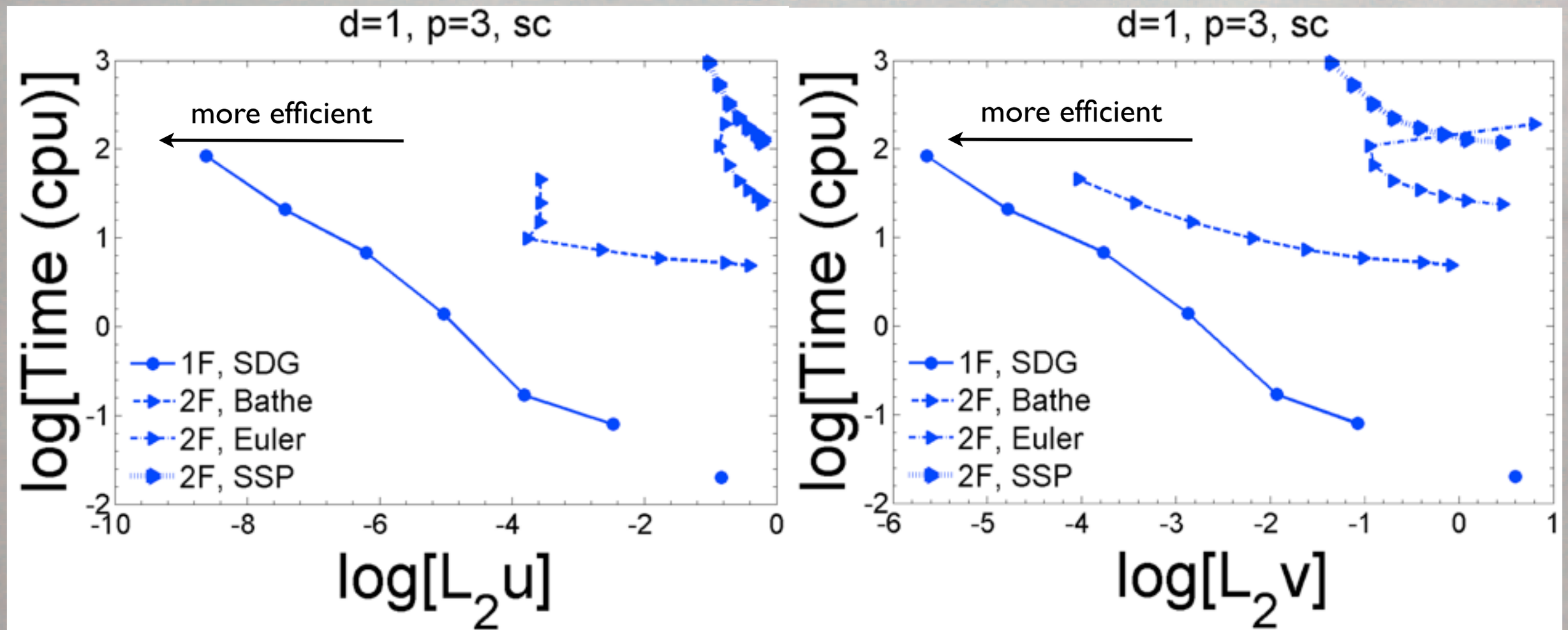


y

x

4 elements

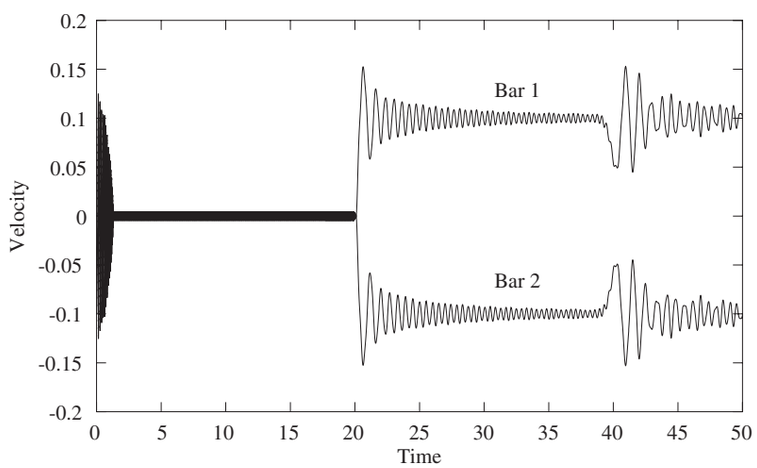
Efficiency

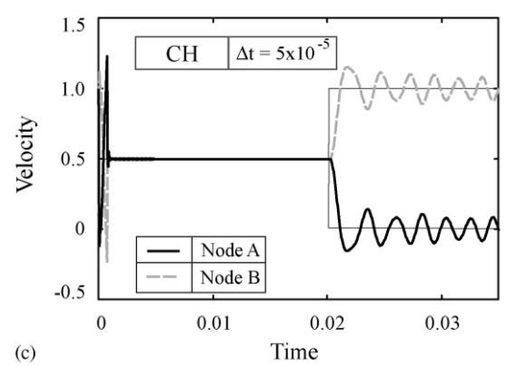


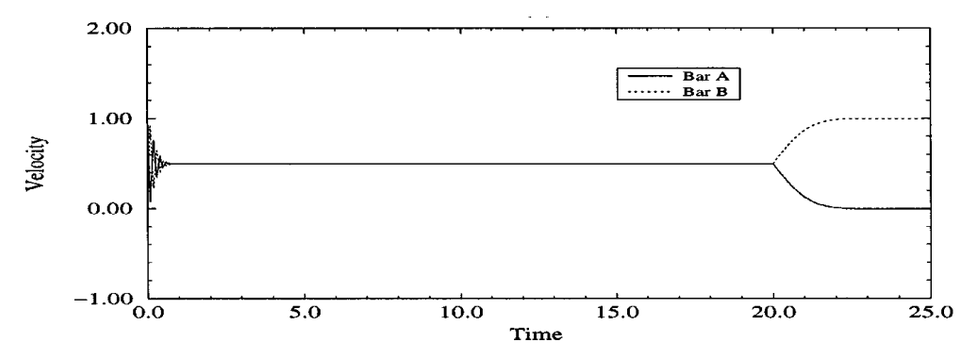
- SDG solutions are between 1 to 2 orders of magnitude more efficient than Bathe's method.
- RK1 method (forward Euler) exhibit numerical instabilities at high resolutions

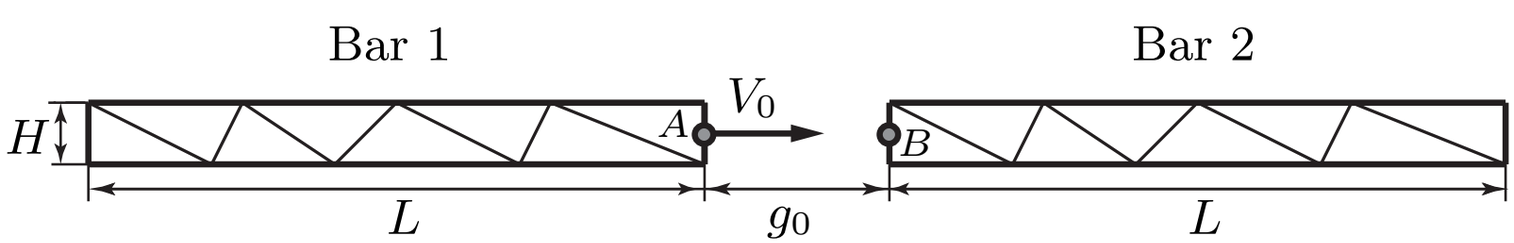
Lack of regularity: Numerical examples

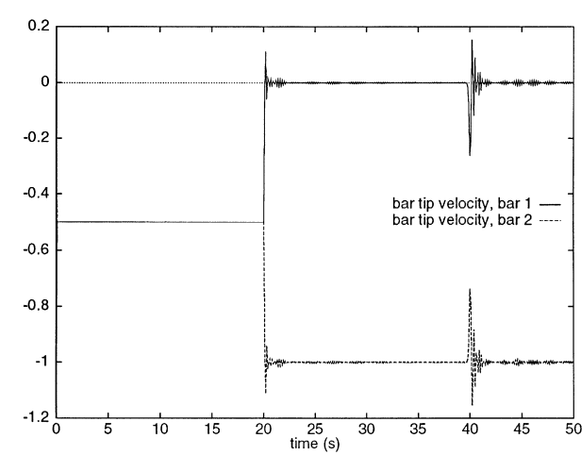
- Benchmark contact problem studied by T. J.R. Hughes *et. al.* (1976); T. A. Laursen, V. Chawla (1997); A. Czekanski, S. A. Meguid (2001); F. Cirak, M. West (2005), etc.



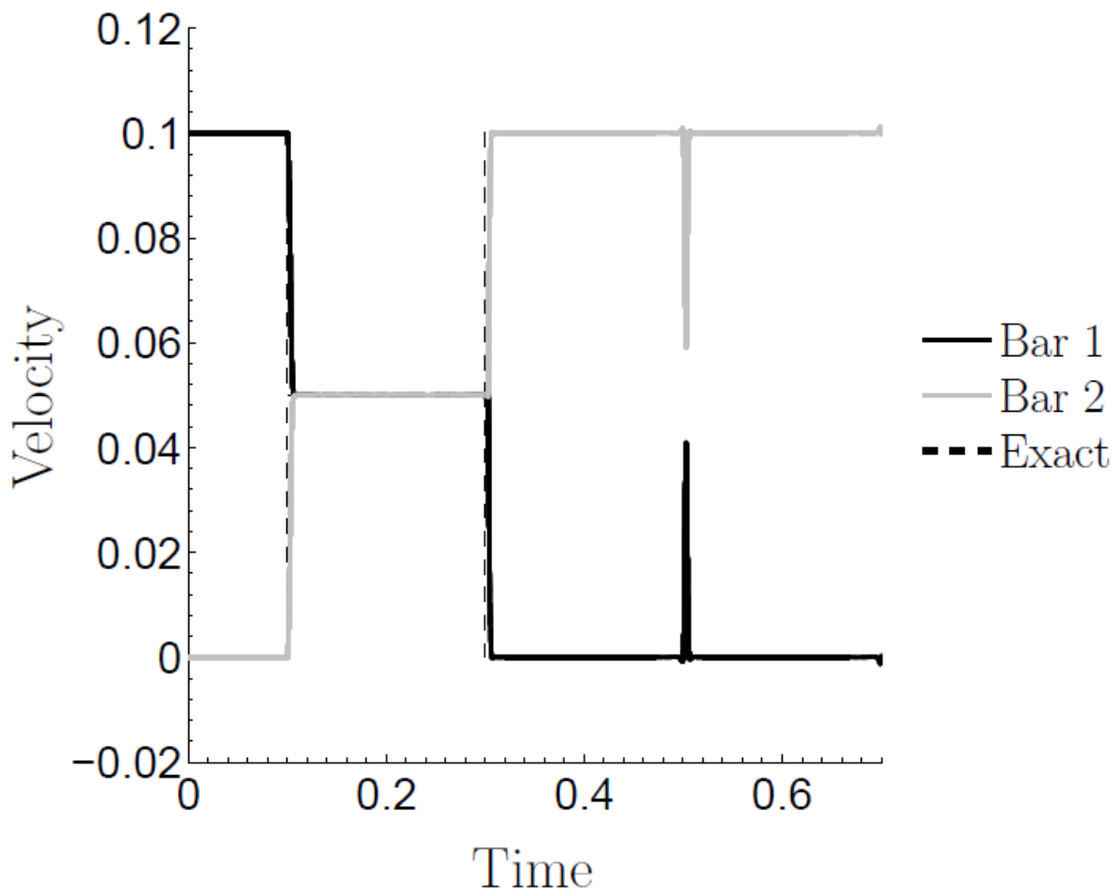


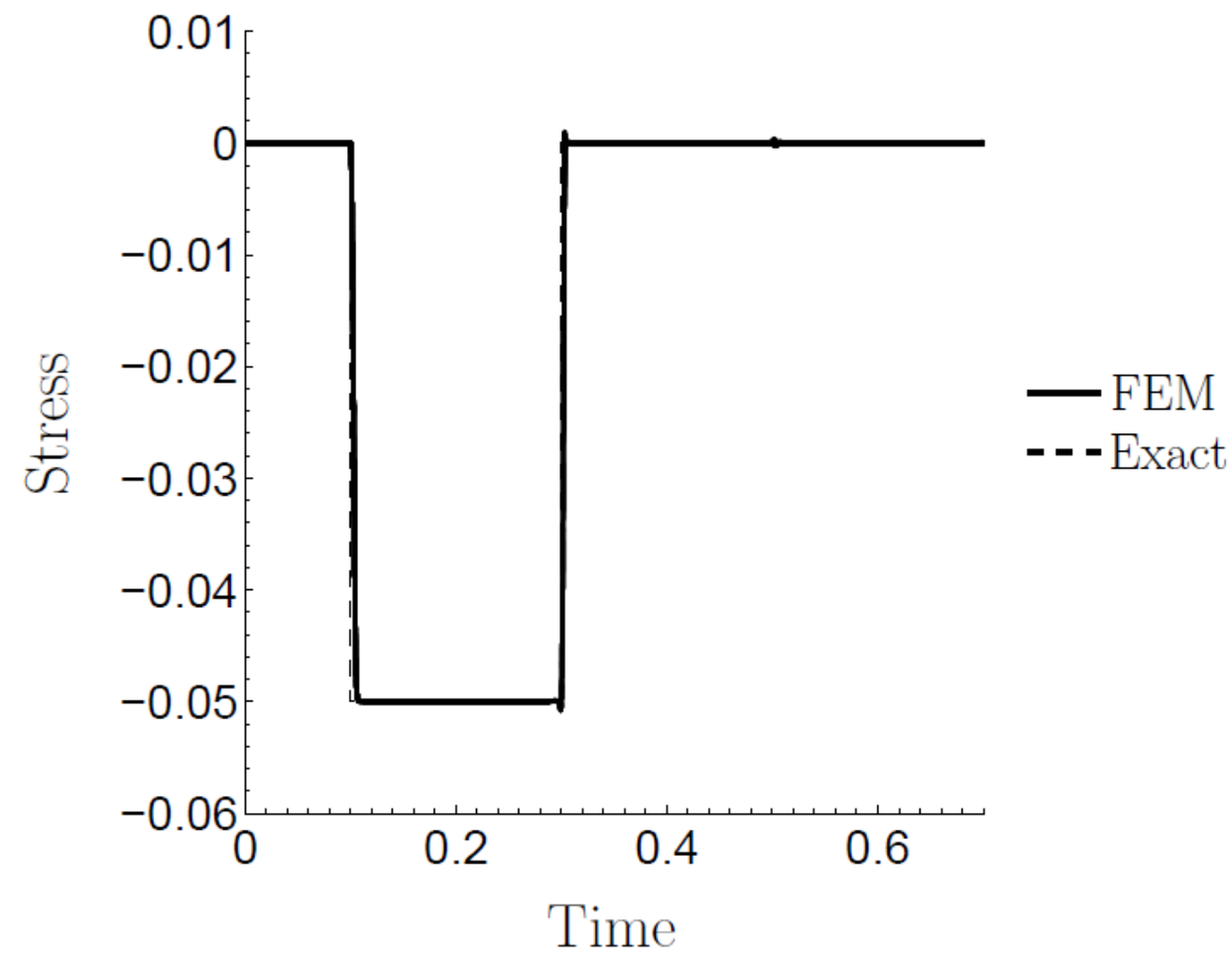




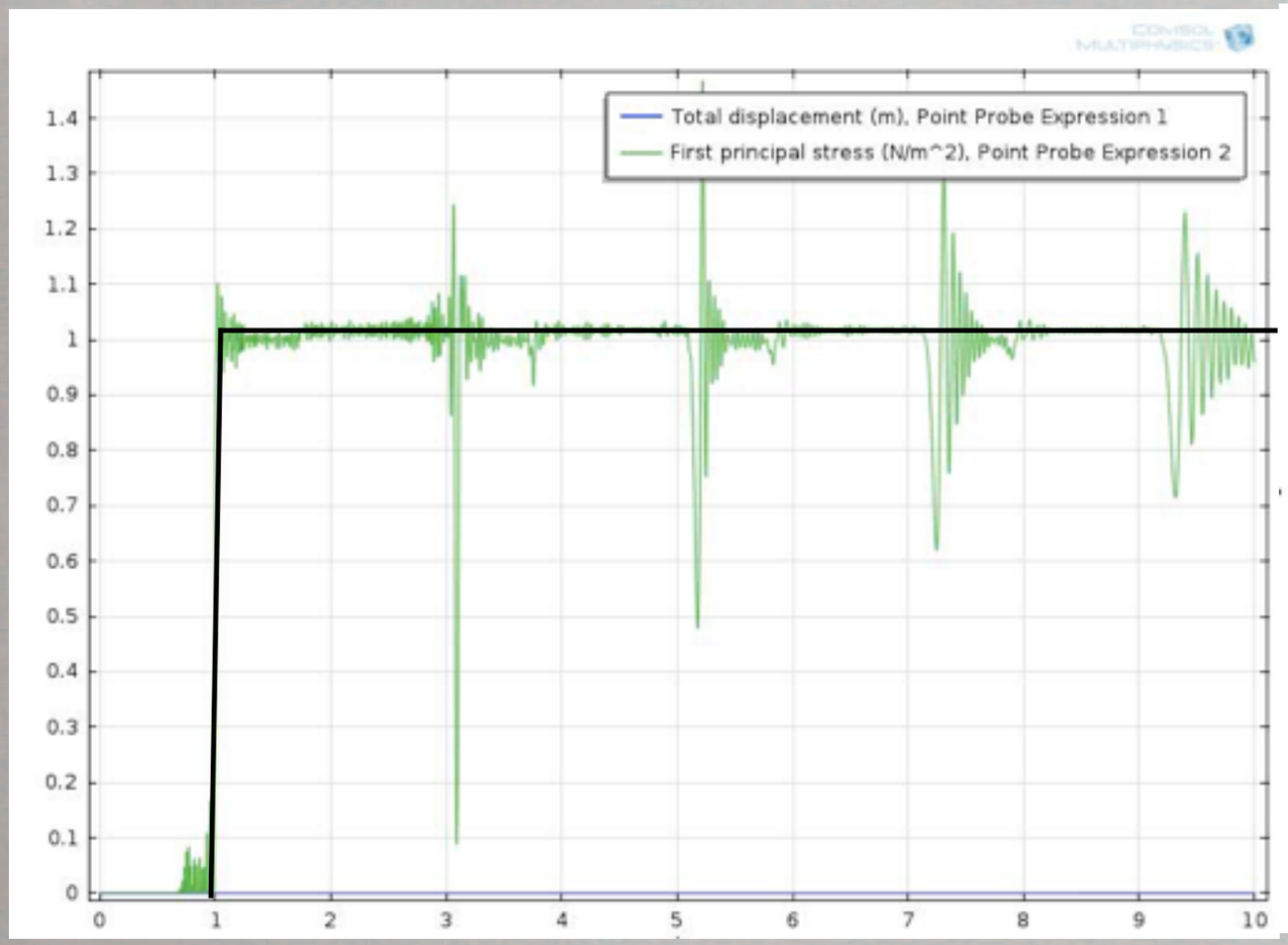


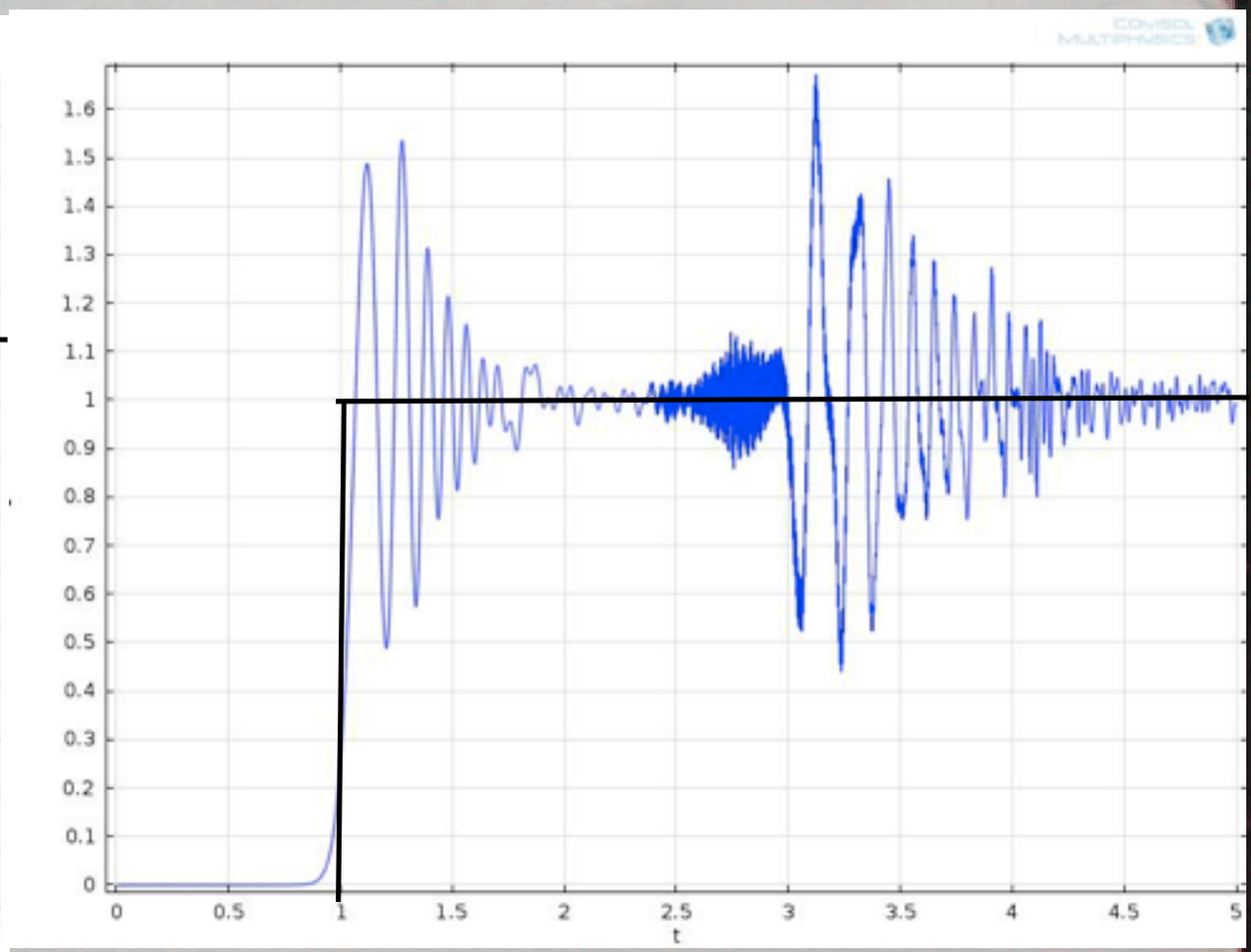
Lack of regularity
SDG solutions

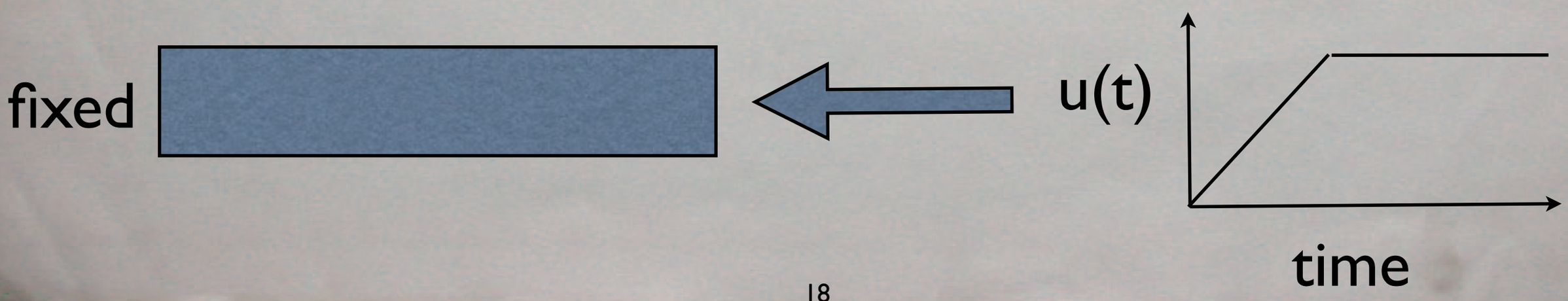




Lack of Regularity: Dynamic loading

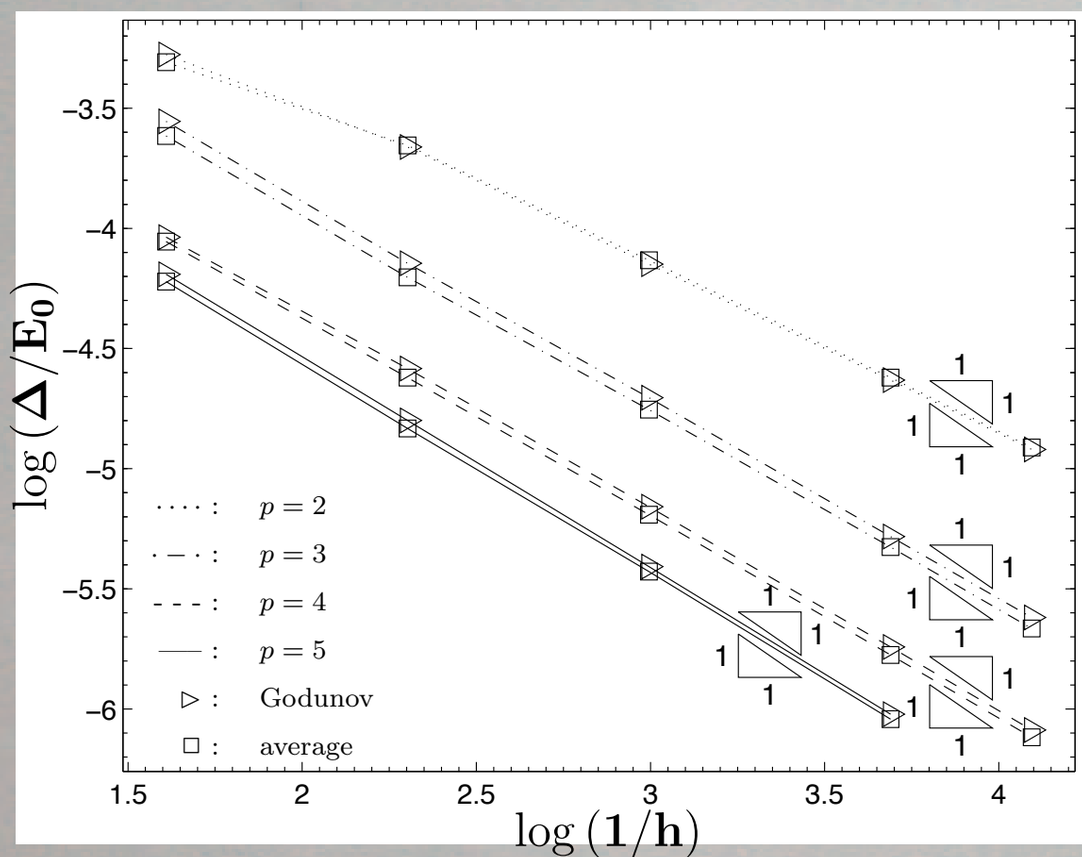


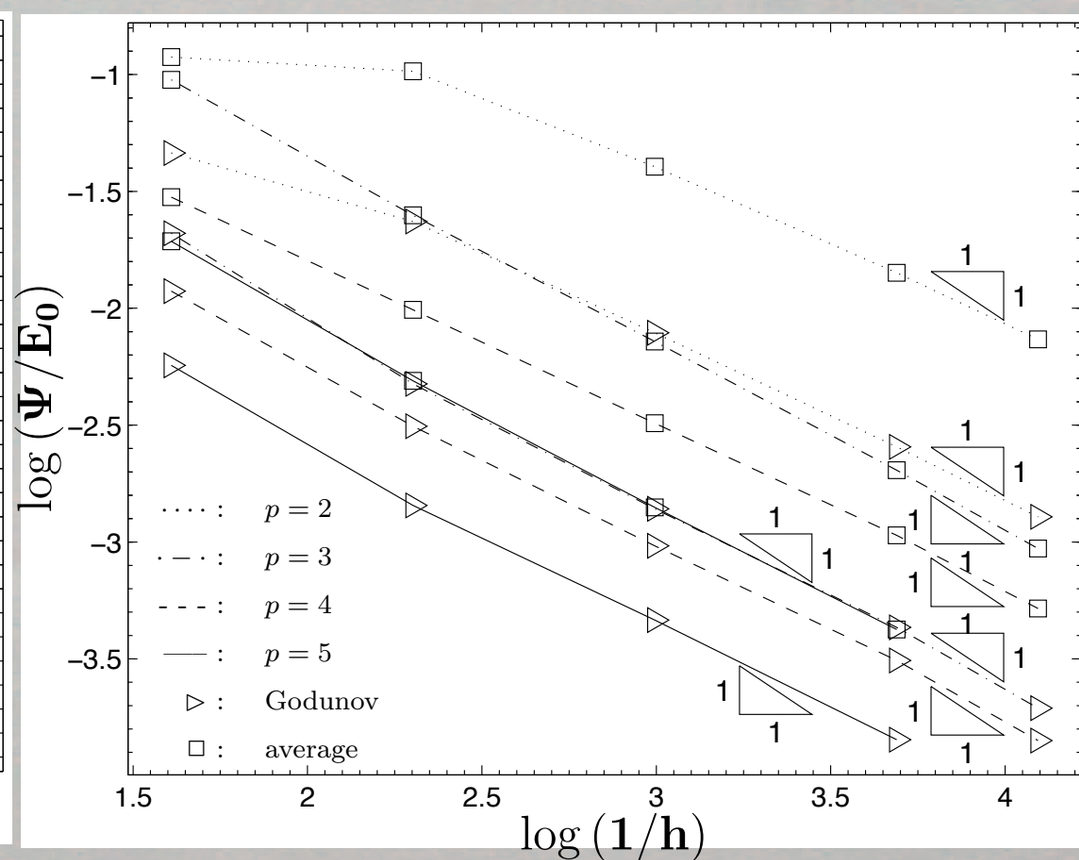




Lack of Regularity: SDG method

- SDG method a natural choice for problems with shocks and other nonsmooth features





- Choice of error measure is very important in problems with discontinuous features

- Riemann fluxes result in small errors in such problems

Variant SDG implementations

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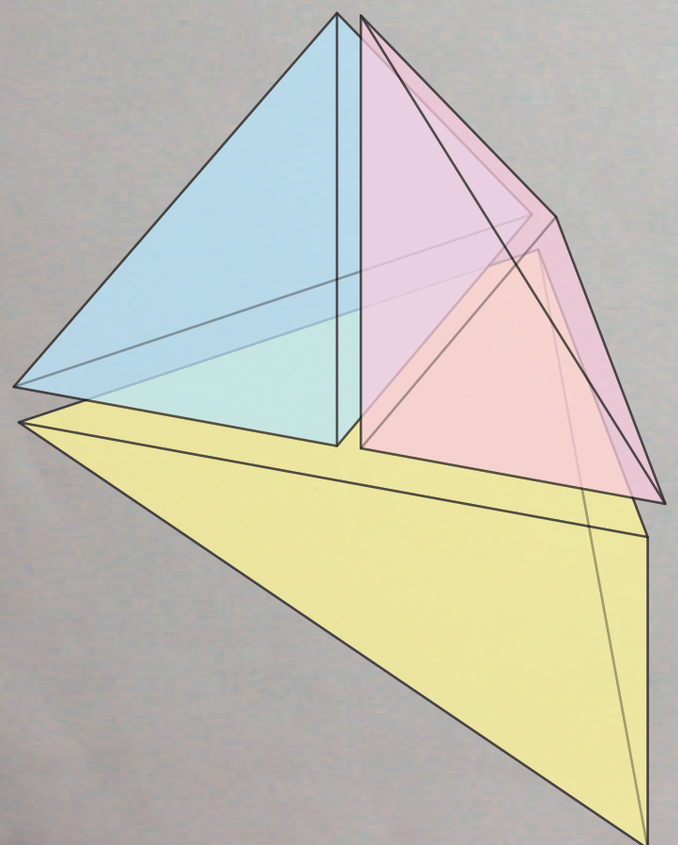
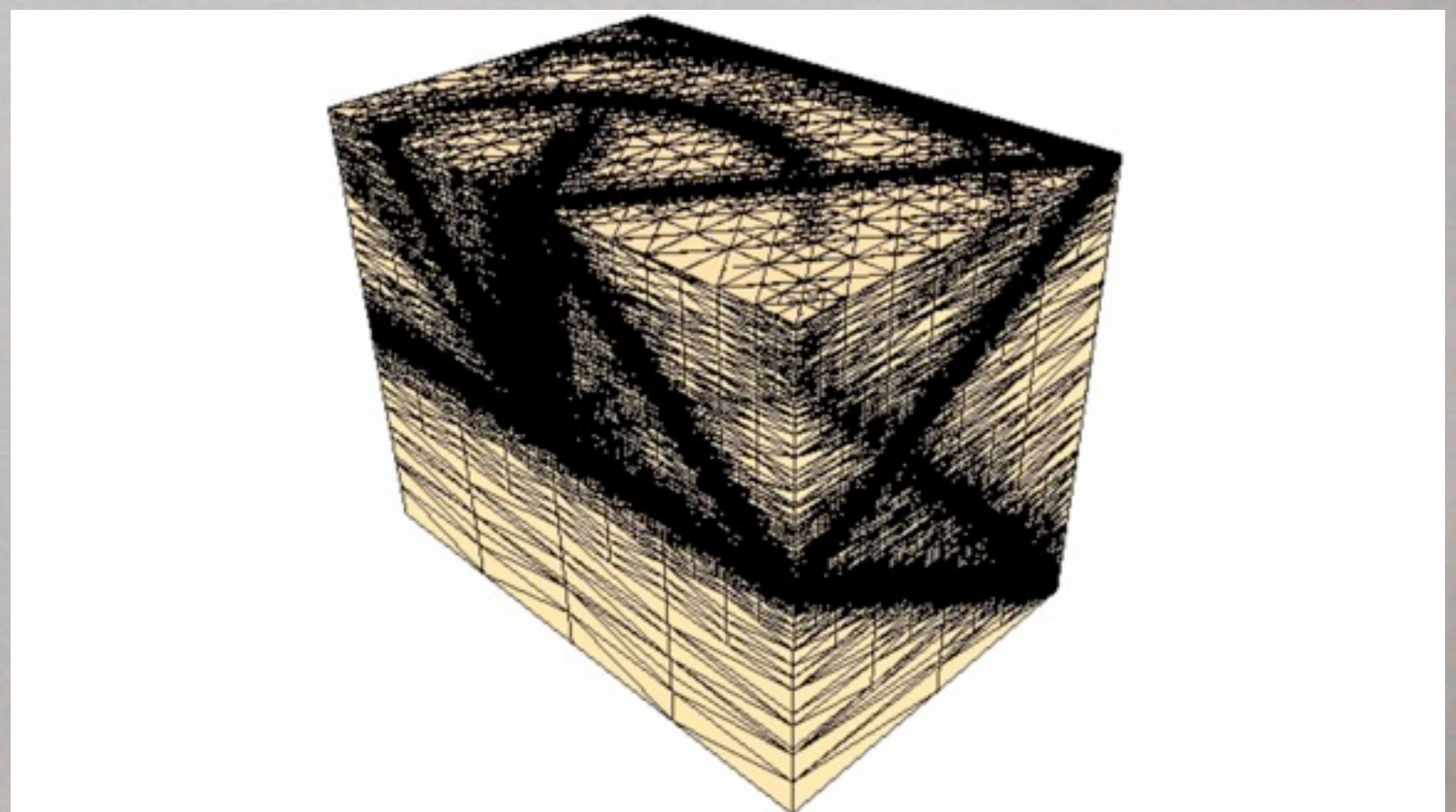
Features of adaptive SDG method

- Local adaptivity operations
- flexible h-refinement and p-enrichment
- smaller elements do not pose global time increment constraints
- no projection error



click to play movie



Crack-tip wave scattering

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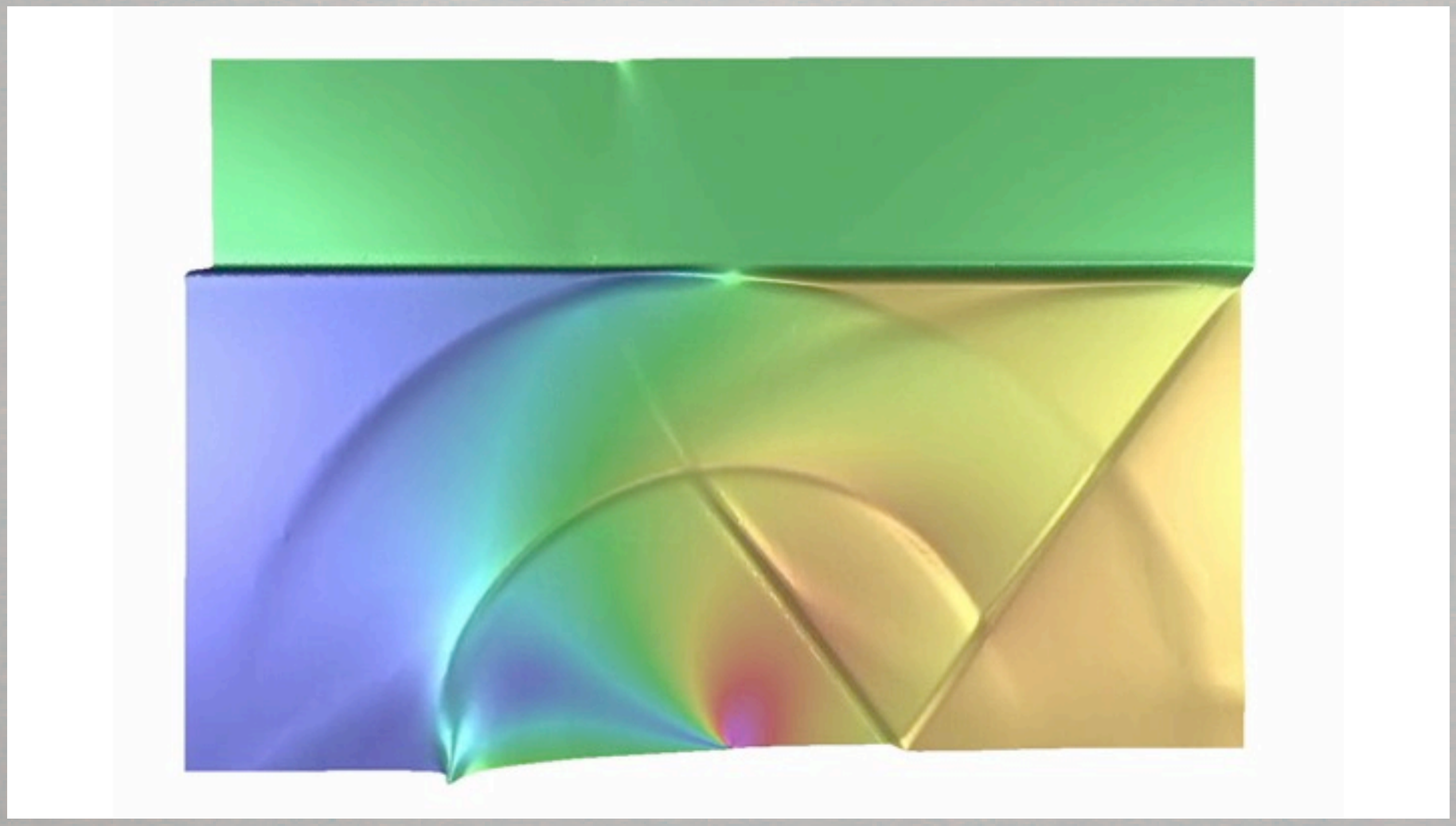

Dynamic fracture with damage-delay cohesive model, random defects and nucleation

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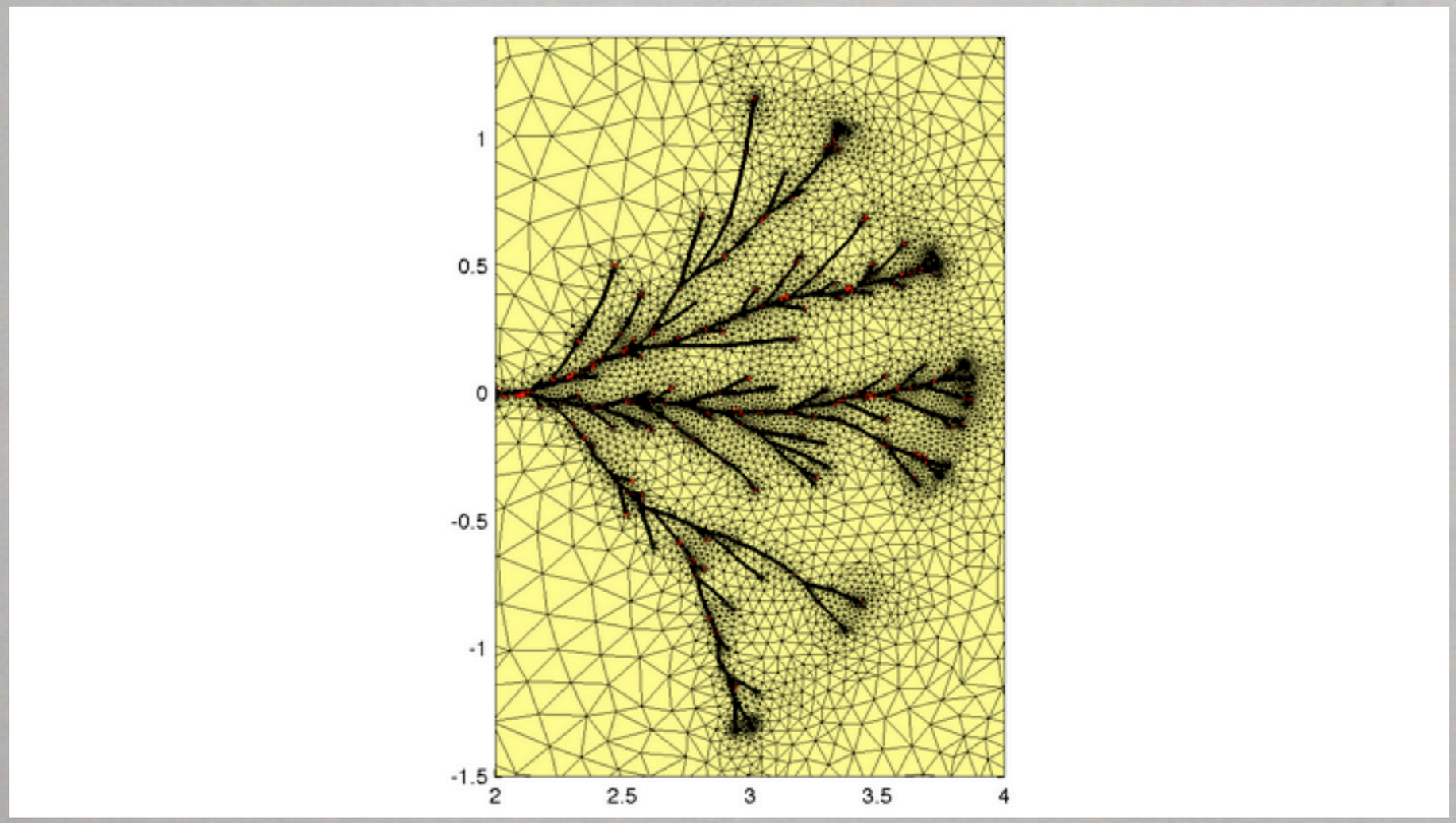
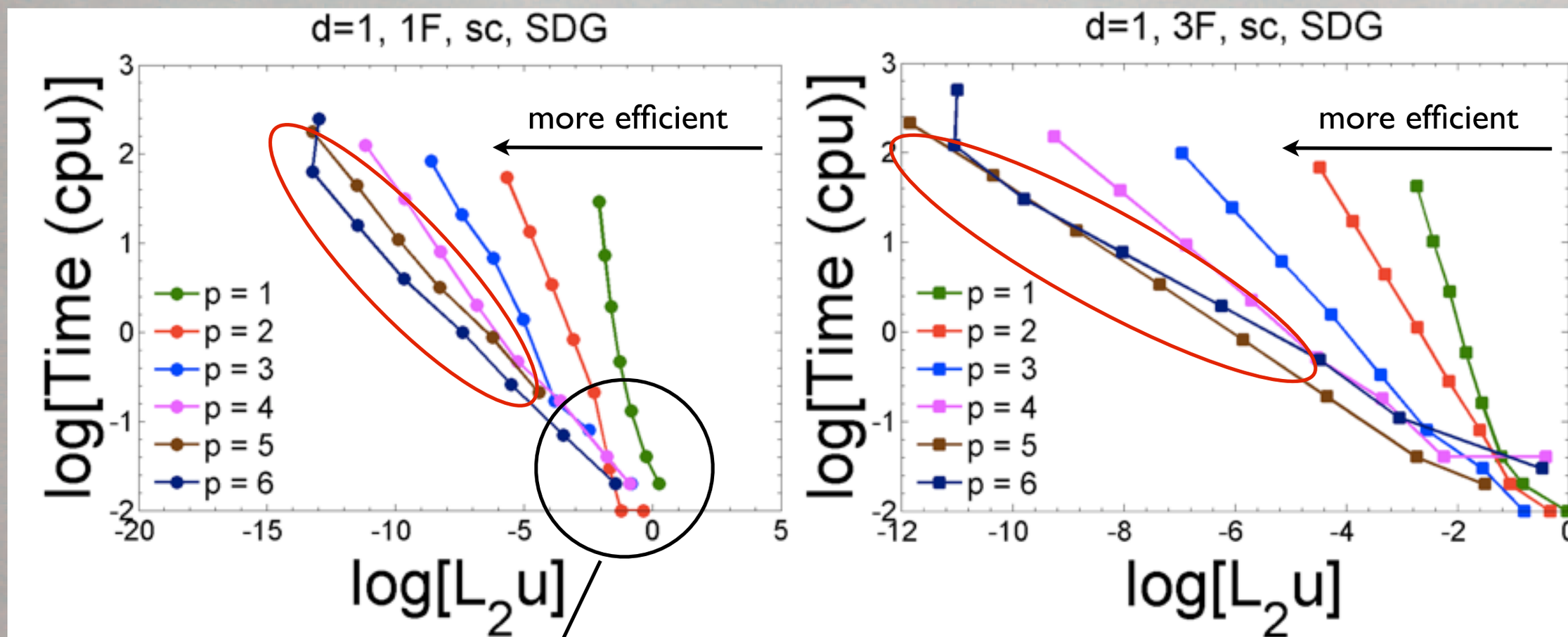
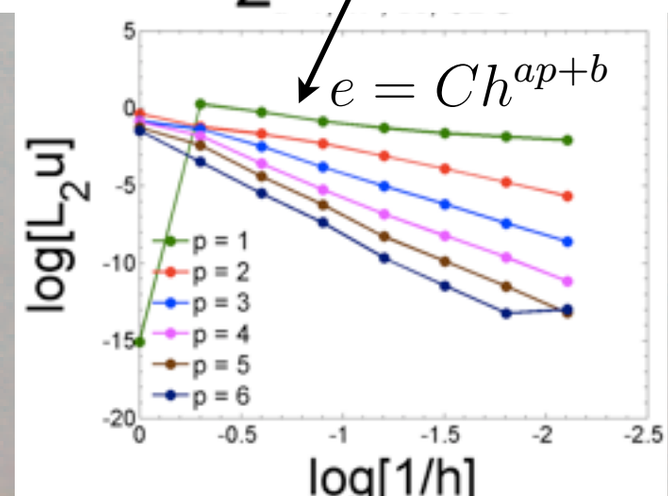


hp-Adaptive methods

- Is p-enrichment always more efficient for smooth problems?

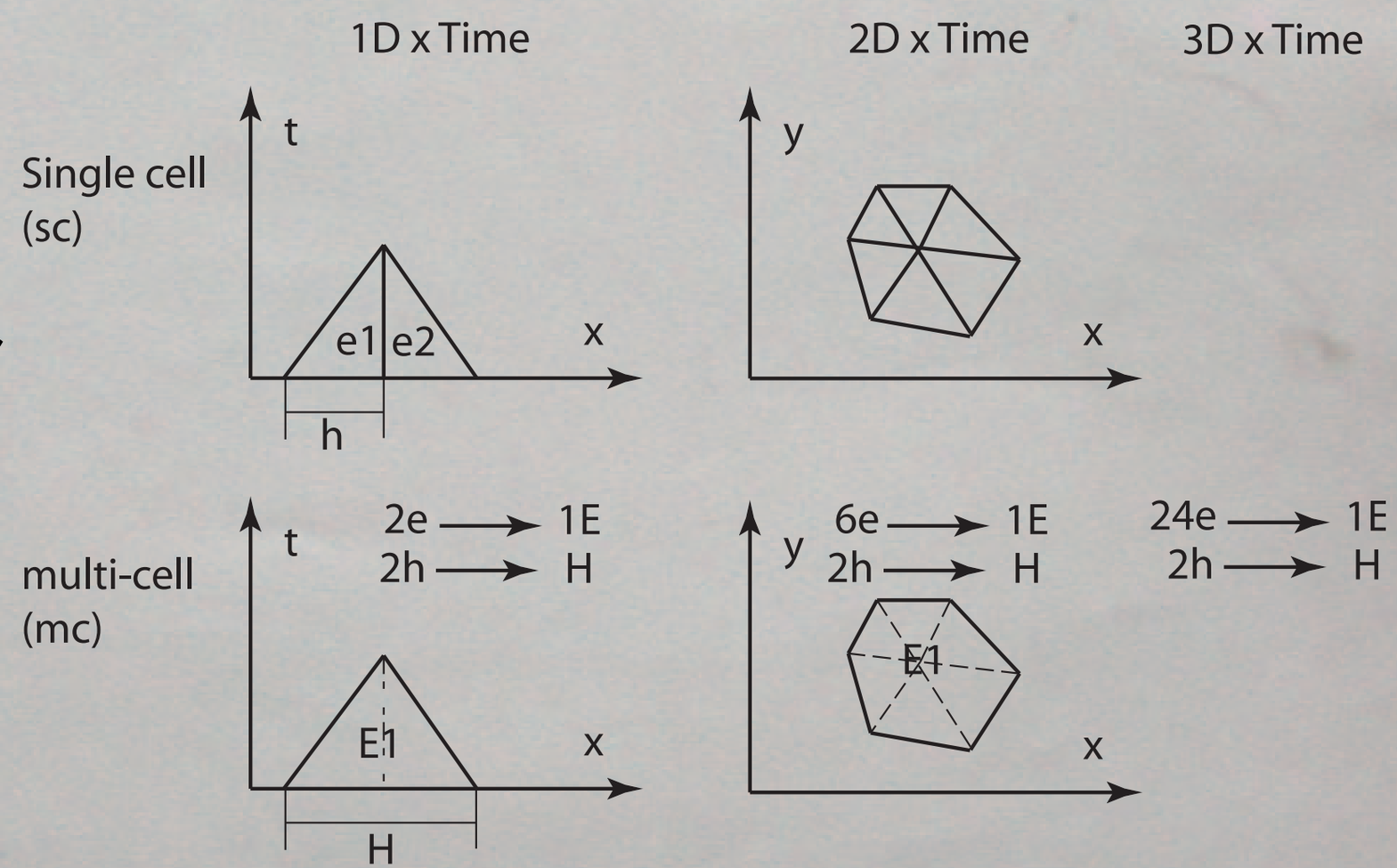




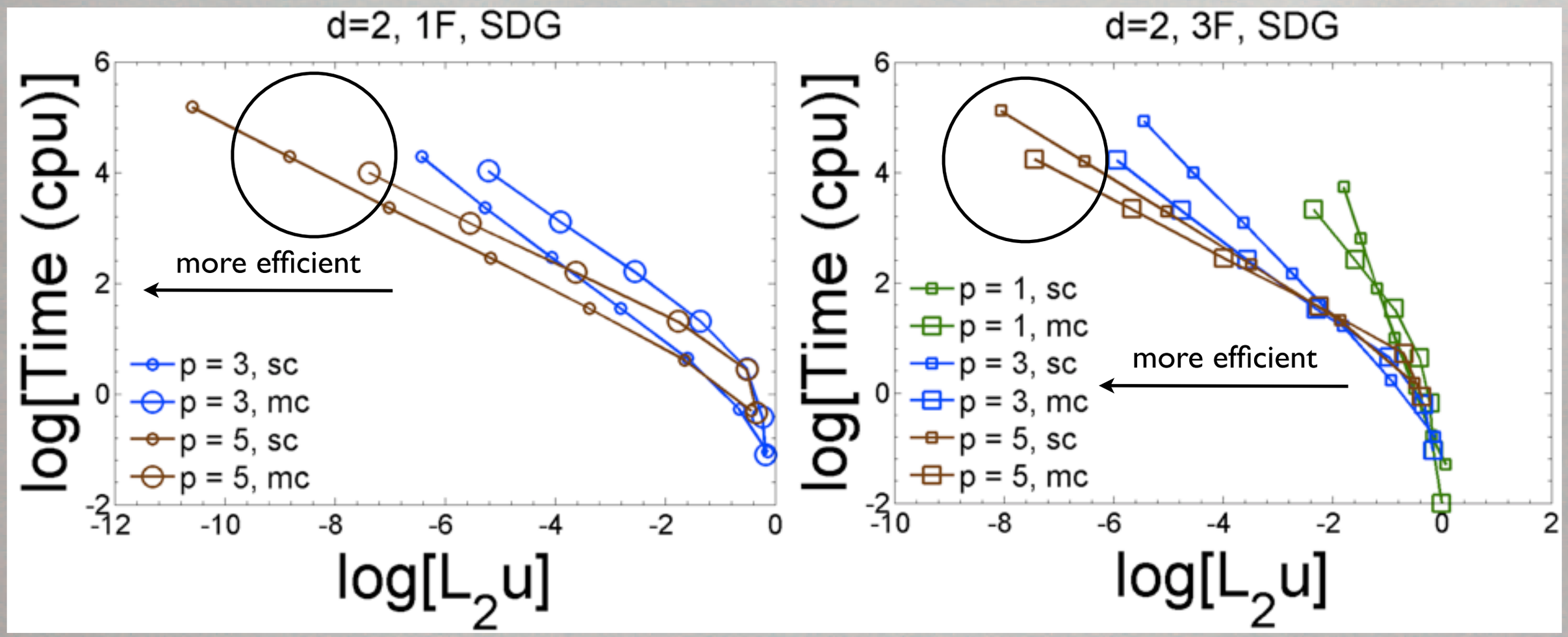
- For larger systems (3F vs 1F), higher order polynomials may not be more efficient!
- How about conntinuous Galerkin methods?

Multi-cell elements

- Eliminates complicated and costly Riemann solutions
- Cost comparison:
  - Assembly cost reduces as there are no interior facets
  - LU cost reduces substantially as there is only one element
- Error comparison:
  - Error increases as element size doubles

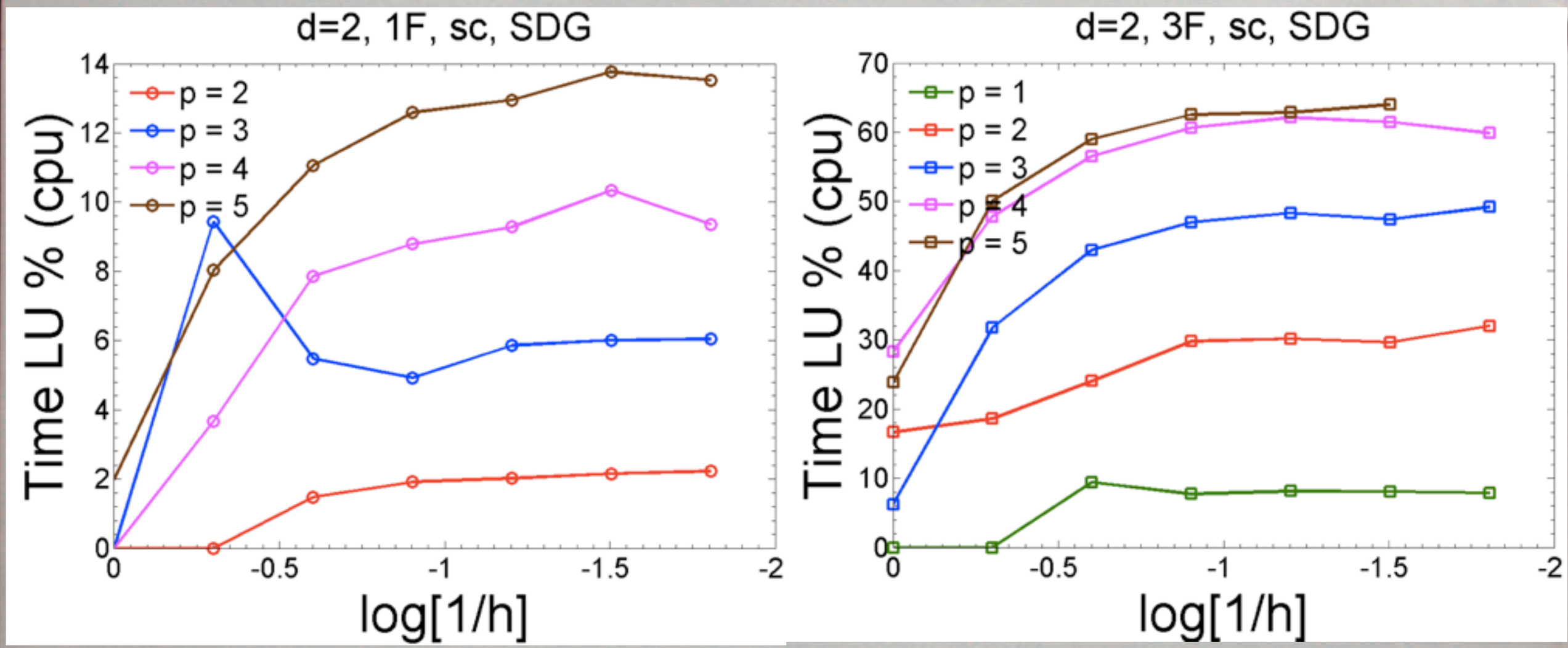


Is multi-cell formulation more efficient?



Multi-cell option makes only 3-field formulation more efficient

Multi-cell formulation: solution cost break down

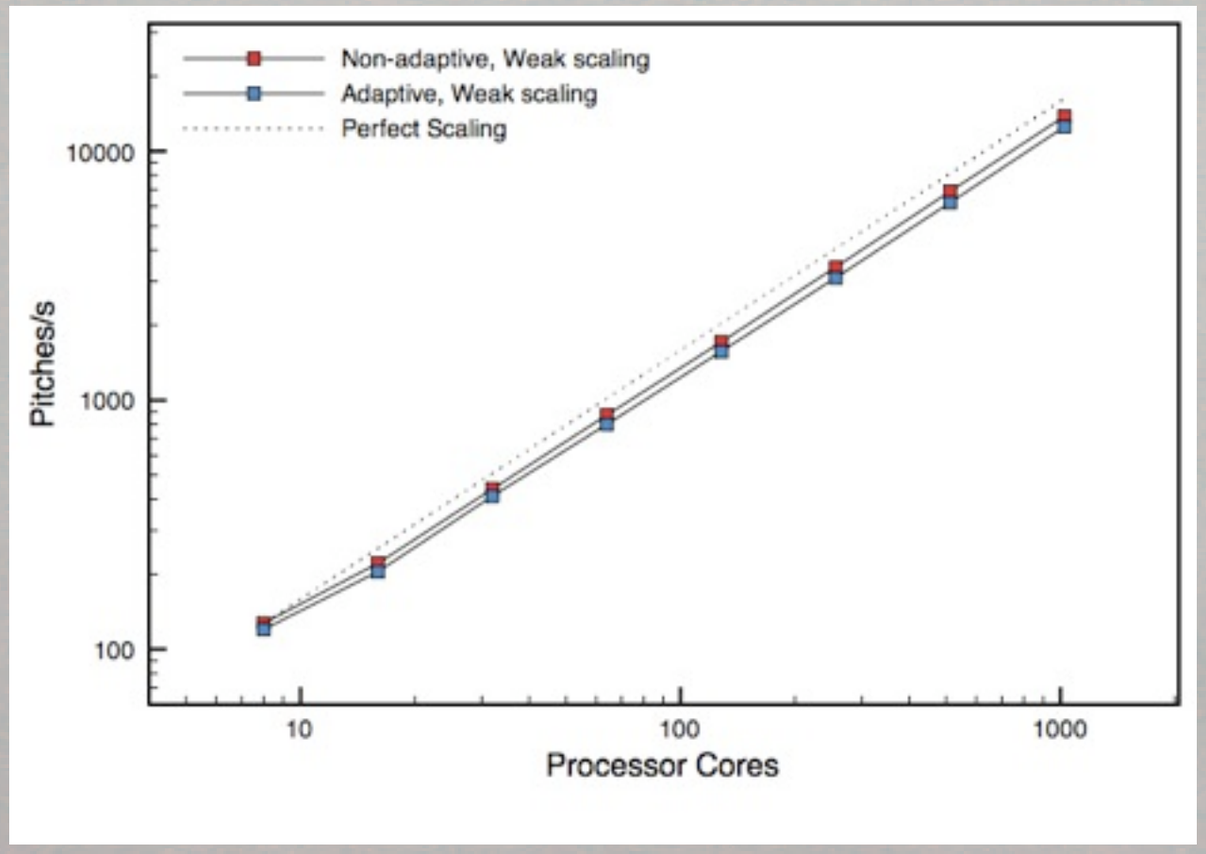


- The computation cost reduction is mostly on LU factorization.
- The cost reduction should compensate for the increase in element size (*h*)

3-field formulation (u, v, E) becomes more efficient due to more dominant LU part

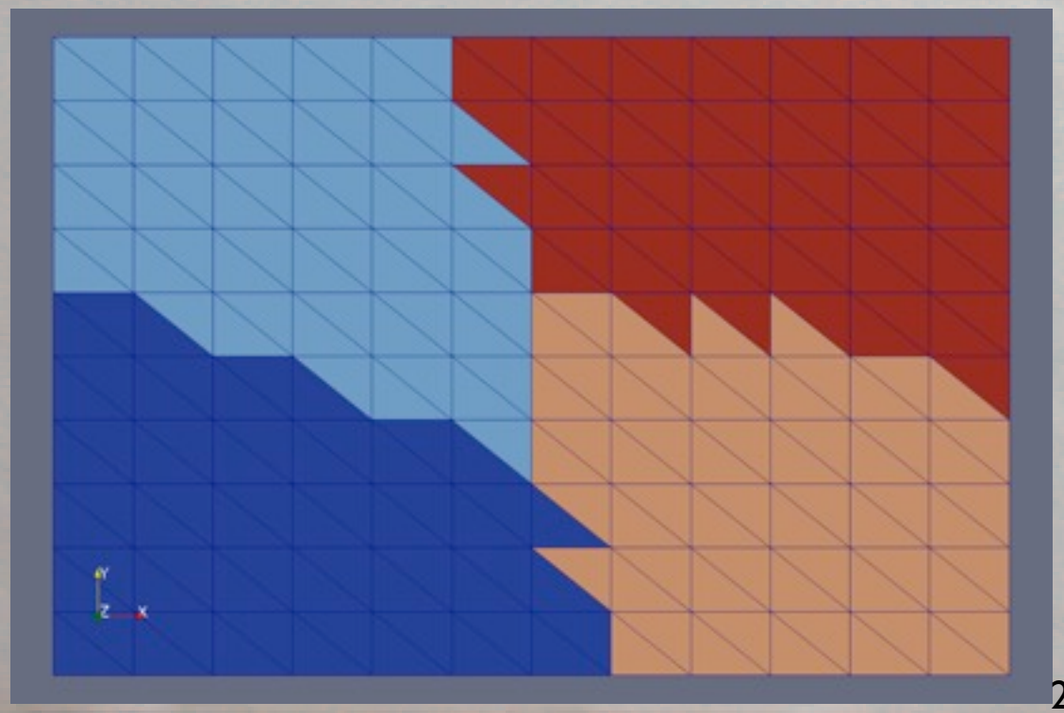
Parallel version of SDG

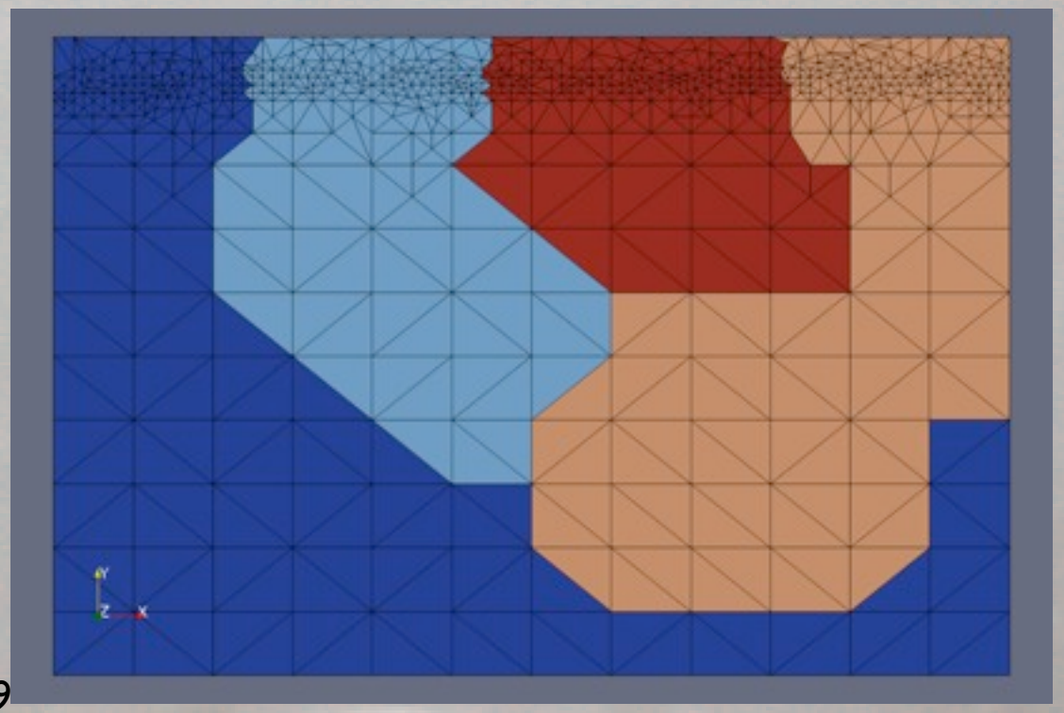
- Near-perfect scaling with static load balancing
  - Parallelize the meshing process (vs. FE solution)
  - Hierarchical and hetrogeneous parallel structure (in progress)



Asynchronous load balancing

- load balancing (new diffusion method) Novel, asynchronous continuous load balancing scheme
  - Processor cores continually compare work-load; shed or acquire elements to/from neighboring partitions
  - keeps all operations local, asynchronous and opportunistic
- Heuristic control of inter-processor boundary quality
- So far, 95% efficiency for strong adaptive refinement





Conclusions

- SDG method can achieve arbitrary high convergence rates in space and time.
- The locally implicit solution scheme yields a linear solution complexity.
- SDG method was substantially more efficient than a few time marching schemes tested.
- Local and asynchronous solution strategy lends itself to adaptive and parallel simulations.
- Future work: Numerical and Mathematical analysis of the SDG method and various other Finite Element Methods.