Riemann solutions for spacetime discontinuous Galerkin methods

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Abstract

Spacetime discontinuous Galerkin finite element methods (cf. [1, 2, 3]) rely on 'target fluxes' on element boundaries that are computed via local one-dimensional Riemann solutions in the direction normal to the element face. In this work, we provide details of converting a space-time flux expressed in differential forms into a standard one-dimensional Riemann problem on the element interface. We then demonstrate a generalized solution procedure for linearized hyperbolic systems based on diagonalisation of the governing system of partial differential equations. The generalized procedure is particularly useful for the implementation aspects of coupled multi-physics applications. We show that source terms do not influence the Riemann solution in the spacetime setting. We provide details for implementation of coordinate transformations and Riemann solutions. Exact Riemann solutions for some linearized systems of equations are provided as examples, including an exact, semi-analytic Riemann solution for generalized thermoelasticity with one relaxation time.

Keywords:
Discontinuous Galerkin, spacetime finite element, Riemann problem, elastodynamics, non-Fourier heat conduction, generalized thermoelasticity, Euler equations

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1. Introduction

Riemann solvers or approximate Riemann solvers are commonly used in control volume based numerical methods, e.g. finite volume methods and discontinuous Galerkin methods [4, 5, 6]. Spacetime discontinuous Galerkin (SDG) methods, as described in [1, 3, 2, 7, 8, 9, 10, 11, 12, 13, 14], are one such numerical method where Riemann solutions (or Riemann fluxes) are needed on inter-elemental boundaries. In this paper, we provide the details of the Riemann solution process as we have it used in [1, 3, 2, 7, 8, 9]. A key component of the procedure is transforming the differential forms formulation into a local coordinate system in order to solve the standard Riemann problem in the normal direction. The Riemann solution procedure itself is not novel; the basic procedure can be found in articles [15] or textbooks [16, 4]. The usefulness of our current exposition lies in clarifying the transition from differential forms in spacetime to a one dimensional Riemann problem, and calculating the exact Riemann flux. The general procedure we develop can be applied to systems of hyperbolic equations, and it is not restricted to SDG methods. In addition, our semi-analytic solution structure is particularly useful for the derivation and implementation of the Riemann solutions for complicated multiphysics problems, as shown in Section 4.3 on generalized thermoelasticity.

We use differential forms and the exterior calculus on manifolds to formulate systems of hyperbolic equations and express fluxes across spacetime interfaces with arbitrary orientation. This approach yields a very concise and elegant structure for various identities such as the Stokes theorem. More importantly, it eliminates problems pertained to orthogonality and the definition of magnitude and normal vectors in classical mechanics due to the absence of an objective definition for spacetime normal vectors. While many of the mathematical statements necessary can be made with 'standard' tensor calculus notation, we find that differential forms in spacetime are extremely useful in identifying the correct spacetime fluxes, restrictions on arbitrarily oriented boundaries, and dual quantities. Differential forms for finite element methods are not quite standard, but their usage is increasing due to their compact notation and elucidation of physical and mathematical properties (see, e.g., [17, 18, 19, 20].

We follow by discussing the restrictions of differential forms to element faces and basic coordinate transformations. We then state the Riemann problem for a general linear hyperbolic system and demonstrate the solution

construction on arbitrarily oriented boundaries. Source terms in the differential system are shown to have no effect on the Riemann solution (in the SDG context). We also provide examples from linearized elastodynamics, non-Fourier heat conduction, and a form of generalized linear thermoelasticity. We use the non-linear Euler equations (inviscid flow) to demonstrate how our method can be applied after performing linearization of a the flux Jacobian.

2. Spacetime discontinuous Galerkin methods

The spacetime discontinuous Galerkin methods developed in [1, 3, 2, 7, 8, 9] are a family of discontinuous finite element methods for hyperbolic systems of equations. They utilize an advancing front mesh generation procedure that allows local 'patches' of elements to become decoupled from the global solution domain through causality. Other spacetime methods that use a more conventional 'time extrusion' or 'timeslab' approach can be found in [10, 11, 12, 13, 14]. Our formulations all share a control volume structure over a spacetime element, where volumetric changes are balanced by surface fluxes. We utilize the coordinate-free notation of differential forms, which is atypical in the computational mechanics literature but very useful on arbitrarily oriented manifolds in spacetime. The forms allow us to clearly identify and distinguish spatial and temporal fluxes, space-like and time-like manifolds, as well as circumventing the need to define a "natural" spacetime metric. The use of differential forms within the context of numerical methods has also been espoused by other authors, see e.g. [19]. Coordinate transformations are necessary on element faces, where we must solve the one-dimensional Riemann problem in the normal direction. As such, herein we provide details on the coordinate transformations used in our SDG implementations.

2.1. Differential forms notation

We use the notation of differential forms on spacetime manifolds to develop our SDG formulations. This approach supports a direct coordinate-free notation that can be used to express fluxes across spacetime interfaces with arbitrary orientation, such as element boundaries in unstructured spacetime meshes. This leads to concise representations of the governing equations that emphasize the notion of conservation on spacetime control volumes. In contrast to tensor notation, for example, the Stokes Theorem expressed in forms

notation does not require unit vectors 'normal' to spacetime d-manifolds. Such objects are not well defined, given the absence of an inner-product for spacetime vectors in classical mechanics. In this subsection, we present definitions and notations for differential forms with tensor coefficients on spacetime manifolds. See [21, 22, 23, 24] for more complete expositions of differential forms and the exterior calculus on manifolds. Our formulation is specialized to flat spacetime manifolds for simplicity.

Consider a flat spacetime manifold $\mathcal{D} \subset \mathcal{M} := \mathbb{E}^d \times \mathbb{R}$ in which d is the spatial dimension of the manifold. We use the basis $\{\mathbf{e}_i, \mathbf{e}_t\}_{i=1}^d$, in which the spatial basis $\{\mathbf{e}_i\}$ spans $\mathbb{E}^d$ and $\mathbf{e}_t$ is the temporal basis vector, to represent *vectors* in the tangent space. The tangent bundle for our flat spacetime is uniform over $\mathcal{M}$, so we denote the tangent space at all points $P \in \mathcal{M}$ simply as $\mathcal{T}$, rather than the usual $\mathcal{T}_P$. The dual basis for *covectors* in the cotangent bundle $\mathcal{T}^*$ is denoted as $\{\boldsymbol{e}^i, \boldsymbol{e}^t\}_{i=1}^d$ and is determined by the relations $\boldsymbol{e}^i(\mathbf{e}_j) = \delta^i_j$, $\boldsymbol{e}^i(\mathbf{e}_t) = 0$, $\boldsymbol{e}^t(\mathbf{e}_i) = 0$ and $\boldsymbol{e}^t(\mathbf{e}_t) = 1$. Thus, the component representation of any vector $\mathbf{a} \in \mathcal{T}$ and any covector $\boldsymbol{b} \in \mathcal{T}^*$ are $\mathbf{a} = a^i\mathbf{e}_i + a^t\mathbf{e}_t$ and $\boldsymbol{b} = b_i\boldsymbol{e}^i + b_t\boldsymbol{e}^t$ in which, and from here on, summation from 1 to d is implied for indices repeated between subscripts and superscripts, excepting the reserved index t for which no summation is implied. We use bold italic type to denote forms and covectors and bold upright type to denote vectors and tensors.

Let $\mathcal{T}^r := \mathcal{T} \times \ldots \times \mathcal{T}$ (r times) be the *space of r-vectors*. The *space of r-covectors* (*i.e.,* alternating, r-linear functions on $\mathcal{T}^r$) is denoted by $\Lambda^r\mathcal{T}^*$. The standard basis for r-covectors is denoted by $\{\boldsymbol{e}^\lambda\}$, in which $\lambda = i_1 \ldots i_r$ is a strictly increasing r-index. Any r-covector $\boldsymbol{\omega} \in \Lambda^r\mathcal{T}^*$ has a unique component representation with respect to the standard basis, $\boldsymbol{\omega} = \omega_\lambda \boldsymbol{e}^\lambda$, in which summation over strictly increasing r-indices is implied.

We use "$\wedge$" to denote the usual exterior product operator and $\mathbf{d}$ to denote the exterior derivative.

A *differential r-form on $\mathcal{D}$ (with scalar coefficients)* is an r-covector field on $\mathcal{D}$; we call these *r-forms* for short. The standard basis for 1-forms is $\{\mathrm{dx}^i, \mathrm{dt}\}_{i=1}^d$, where, for our flat manifold, the dx^i are 1-forms with uniform values $\boldsymbol{e}^i$, and dt is the 1-form with uniform value $\boldsymbol{e}^t$. Thus, any one form with scalar coefficients has the unique component representation with respect to the standard basis, $\boldsymbol{\omega} = \omega_i \mathrm{dx}^i + \omega_t \mathrm{dt}$, in which ω_i and ω_t are scalar fields on $\mathcal{D}$. Top forms in spacetime are $(d+1)$-forms, for which the standard basis is the singleton set $\{\boldsymbol{\Omega}\}$, where $\boldsymbol{\Omega} = \mathrm{dx}^1 \wedge \ldots \wedge \mathrm{dx}^d \wedge \mathrm{dt}$. Thus, a top-form

$\boldsymbol{\alpha}$ with scalar coefficients is expressed as $\boldsymbol{\alpha} = \alpha\boldsymbol{\Omega}$ in which α is a scalar function on $\mathcal{D}$.

Let $\boldsymbol{\alpha}$ be an r-form with tensor coefficient $\mathbf{a}$ of order $s : s \in \mathbb{N}$, with a suitable inner product defined on the space of tensor coefficients. The *Hodge star operator* is defined by

$$\boldsymbol{\alpha} \wedge \star\boldsymbol{\alpha} = |\mathbf{a}|^2\boldsymbol{\Omega}, \tag{1}$$

in which $\star\boldsymbol{\alpha}$ is a $(d+1-r)$-form. We shall have use for d-forms, for which we define a preferred basis $\{\star\mathrm{dx}^i, \star\mathrm{dt}\}_{i=1}^d$. This implies $\mathrm{dx}^j \wedge \star\mathrm{dx}^k = \delta^{jk}\boldsymbol{\Omega}$, $\mathrm{dt} \wedge \star\mathrm{dx}^j = \mathbf{0}$ and $\mathrm{dx}^j \wedge \star\mathrm{dt} = \mathbf{0}$ for $j, k = 1, \ldots, d$.

Our formulation makes use of forms with scalar, (co)vector and (co)tensor coefficients. The usual definition of the exterior product operator addresses forms with scalar coefficients; here we extend the definition to address forms with tensor coefficients of arbitrary order. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be r- and s-forms on $\mathcal{D}$, respectively, and let $\mathbf{a}$ and $\mathbf{b}$ be tensor fields on $\mathcal{D}$ of order m and n, respectively, where $0 \leq n \leq m$. We write $\mathbf{a}\boldsymbol{\alpha}$ and $\mathbf{b}\boldsymbol{\beta}$ to describe an r-form with tensor coefficients of order m and an s-form with tensor coefficients of order n. The *exterior product of* $\mathbf{a}\boldsymbol{\alpha}$ *and* $\mathbf{b}\boldsymbol{\beta}$ is the $(r+s)$-form with tensor coefficients of order $m-n$ given by

$$\mathbf{a}\boldsymbol{\alpha} \wedge \mathbf{b}\boldsymbol{\beta} := \mathbf{a}(\mathbf{b})(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}). \tag{2}$$

where $\mathbf{a}(\mathbf{b})$ is the tensor field of order $m-n$ obtained from the linear mapping of tensor field $\mathbf{a}$ applied to tensor field $\mathbf{b}$.

We introduce a useful 1-form with vector coefficients and an associated d-form with (co)vector coefficients to facilitate our formulation:

$$\mathbf{dx} := \mathbf{e}_i\mathrm{dx}^i \tag{3a}$$

$$\star\mathbf{dx} := \boldsymbol{e}^i\star\mathrm{dx}^i. \tag{3b}$$

Given any differentiable scalar field w on $\mathcal{D}$, we find

$$\mathbf{d}(w\star\mathbf{dx}) = (\nabla w)\boldsymbol{\Omega} \tag{4a}$$

$$\mathbf{d}(w\star\mathrm{dt}) = \dot{w}\boldsymbol{\Omega}; \tag{4b}$$

for any differentiable tensor field $\mathbf{a}$ on $\mathcal{D}$ of order $m \geq 1$, we also have

$$\mathbf{d}(\mathbf{a} \wedge \star\mathbf{dx}) = (\boldsymbol{\nabla} \cdot \mathbf{a})\boldsymbol{\Omega} \tag{5a}$$

$$\mathbf{d}(\mathbf{a} \wedge \star\mathrm{dt}) = \dot{\mathbf{a}}\boldsymbol{\Omega}, \tag{5b}$$

in which ∇ and $\nabla\cdot$ denote the *spatial gradient* and *spatial divergence* operators, defined with time held fixed, and a superposed dot denotes a partial differential with respect to time.

Consider any open $Q \subset \mathcal{D}$ with a regular boundary ∂Q. *Stokes' Theorem* in differential forms notation is written as [21]

$$\int_Q \mathbf{d}\boldsymbol{\omega} = \int_{\partial Q} \boldsymbol{\omega} \tag{6}$$

in which $\boldsymbol{\omega}$ is a d-form with scalar or vector coefficients.

2.2. General SDG formulations

We write balance laws on a spacetime control volume $\mathcal{Q}$ as

$$\int_{\partial\mathcal{Q}} \mathbf{F}(\mathbf{u}) - \int_{\mathcal{Q}} \mathbf{S}(\mathbf{u}) = \mathbf{0}, \tag{7}$$

where $\mathbf{u}$ is an n-tuple of balanced quantities, $\mathbf{F}(\mathbf{u})$ is the corresponding spacetime flux, and $\mathbf{S}(\mathbf{u})$ are volumetric source terms. Note that the source term is a top form on the spacetime volume, and the flux $\mathbf{F}(\mathbf{u})$ is an n-tuple of d-form valued fluxes.

Applying Stokes' theorem and localizing, the point-valued differential system is written as

$$\mathbf{dF} - \mathbf{S} = \mathbf{0}, \tag{8}$$

where the exterior derivative must be in the sense of distributions. The terms arising due to the distributional nature of the exterior derivative are

$$[\![\mathbf{F}]\!]\big|_{\Gamma_J} := (\mathbf{F}^* - \mathbf{F})\big|_{\Gamma_J} = \mathbf{0}, \tag{9}$$

where Γ_J is the jump set over the physical domain. The *target flux* $\mathbf{F}^*$ (which is also often referred to as a numerical flux, especially in the CFD community) is precisely the quantity that we must determine via Riemann solutions.

In many cases, e.g. conservation laws, we can split the spacetime flux $\mathbf{F}$ into separate temporal and spatial fluxes as

$$\mathbf{F}(\mathbf{u}) := \mathbf{u}\star\mathrm{dt} + \mathbf{f}(\mathbf{u})\star\mathrm{dx}, \tag{10}$$

where $\mathbf{f}$ represents the spatial flux. In such cases, (8) is equivalent to

$$(\dot{\mathbf{u}} + \nabla\cdot\mathbf{f}(\mathbf{u}) - \mathbf{s}(\mathbf{u}))\,\mathrm{dx}^1\mathrm{dx}^2\mathrm{dt} = \mathbf{0}, \tag{11}$$

in the more conventional multi-dimensional calculus notation (in two spatial dimensions).

Our Bubnov-Galerkin weak formulation is obtained from a weighted residual statement, wherein (8) and (9) are weighted by the n-tuple of zero-forms $\hat{\mathbf{w}}$ and integrated over their respective domains. Application of Stokes' Theorem yields the following weak problem over the physical domain $\mathcal{P}$:

Problem 1 (General SDG weak problem)**.** *Find*
$\mathbf{u} \in \mathcal{U}$ *such that for every* $\mathcal{Q} \in \mathcal{P}$

$$-\int_{\mathcal{Q}} [\mathbf{d}\hat{\mathbf{w}} \wedge \mathbf{F}(\mathbf{u}) + \hat{\mathbf{w}} \wedge \mathbf{S}(\mathbf{u})] + \int_{\partial\mathcal{Q}} \hat{\mathbf{w}} \wedge \mathbf{F}^*(\mathbf{u}) = 0 \quad \forall\ \hat{\mathbf{w}} \in \mathcal{U}^{\mathcal{Q}}, \tag{12}$$

where $\mathcal{U}$,$\mathcal{U}^{\mathcal{Q}}$ are appropriate function spaces for the problem of interest.

2.3. Local coordinates and transformations

The flux through an arbitrary co-dimension 1 manifold in spacetime is substantially simplified once a local coordinate system is employed. In this section, we first investigate the transformation relations between two arbitrary coordinate systems. Next, we present the restriction of differential forms on a manifold employing a corresponding local coordinate system.

2.3.1. Coordinate transformation

We presented our differential form expressions using the coordinate system $\{\mathbf{e}_i, \mathbf{e}_t\}_{i=1}^d$. We choose a second coordinate system $\{\underline{\mathbf{e}}_i, \underline{\mathbf{e}}_t\}_{i=1}^d$, related to the former through,

$$\underline{\mathbf{e}} := \mathbf{Q}\mathbf{e}, \qquad \text{that is,} \qquad \underline{\mathbf{e}}_I := Q_I{}^i \mathbf{e}_i, \tag{13a}$$

$$\underline{\mathbf{e}}_t := \mathbf{e}_t. \tag{13b}$$

where $\mathbf{Q}$ is an orthogonal matrix. According to the properties of the dual basis, *cf.* section 2.1, and orthogonallity of Q we obtain,

$$\underline{\boldsymbol{e}} = \mathbf{Q}\boldsymbol{e}, \qquad \text{that is,} \qquad \underline{\boldsymbol{e}}^I = Q^I{}_i \boldsymbol{e}^i, \tag{14a}$$

$$\underline{\boldsymbol{e}}^t = \boldsymbol{e}^t, \tag{14b}$$

Since the standard basis for 1-forms take the uniform values of dual basis covectors, *cf.* section 2.1, we observe that,

$$\mathrm{d}\underline{\mathrm{x}}^I = Q^I{}_i \mathrm{dx}^i, \qquad \Rightarrow \quad \mathbf{d}\underline{\mathbf{x}} = \mathbf{dx} \tag{15a}$$

$$\mathrm{d}\underline{\mathrm{t}} = \mathrm{dt}. \tag{15b}$$

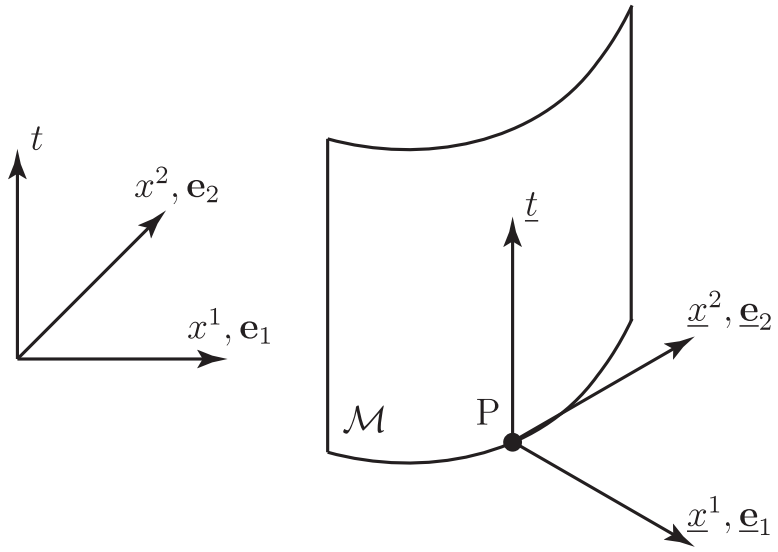

Figure 1: Local coordinate system on a vertical 2-manifold Γ.

Equation (15a) is obtained from (3a), (13a). According to the definition of the single member of the basis for top forms in section 2.1, wedge product properties and (15) we get $\underline{\boldsymbol{\Omega}} = \boldsymbol{\Omega}$. Finally, the definition of our preferred basis for d-forms yields,

$$\star \mathrm{d}\underline{\mathrm{x}}^I = Q_I{}^i \star \mathrm{dx}^i, \qquad \Rightarrow \quad \star \mathbf{d}\underline{\mathbf{x}} = \star \mathbf{dx} \tag{16a}$$

$$\star \mathrm{d}\underline{\mathrm{t}} = \star \mathrm{dt}. \tag{16b}$$

Equation (16a) is derived from (3b) and (14a). Note that $\mathbf{dx}$, $\star\mathbf{dx}$, $\star$dt, and $\boldsymbol{\Omega}$ are all objective with respect to the choice of the coordinate system, so are the tensorial parts of the tensor-valued forms introduced in 2.1. That is, as expected, our formulations are objective with respect to the choice of spatial coordinate system. Accordingly, in the following section we present a local coordinate system at a given point which in turn simplifies the subsequent developments.

2.3.2. Local coordinate system

Let $\mathcal{M}$ be an arbitrary manifold in $\mathbb{E}^2 \times \mathbb{R}$, as illustrated in figure 1. We are interested in the local solution at the arbitrary point P on $\mathcal{M}$. To facilitate a description of the restricted cotangent space $T^*\mathcal{M}$, we define on $\mathcal{M}$ a local

frame, $\{\underline{\boldsymbol{e}}_i, \boldsymbol{e}_t\}_{i=1}^d : \underline{\boldsymbol{e}}_1 \perp T^*\mathcal{M}|_{\mathbb{E}^d}$, with local coordinates $\{\underline{x}^i, t\}_{i=1}^d$, in which underlined symbols denote items referred to the local frame. In contrast to the $(d+1)$-manifold, $\mathcal{D}$, where the standard basis for d-forms in local coordinates is $\{\star\mathbf{d}\underline{\mathbf{x}}^k, \star\mathrm{dt}\}_{k=1}^d$, the basis for d-forms on the d-manifold $\mathcal{M}$ is the singleton set, $\{\star\mathrm{d}\underline{x}^1\}$ as $\mathrm{d}\underline{x}^k|_{\mathcal{M}} = \star\mathrm{dt}|_{\mathcal{M}} = \mathbf{0}$ for $k \neq 1$, since $\mathrm{d}\underline{x}^1|_{\mathcal{M}} = 0$. Thus, only the normal component of the spatial flux $\mathbf{F}(\mathbf{u})$ is pertinent when restricted to $\mathcal{M}$; *cf.* (8). For example for the second order stress tensor $\mathbf{s}$, *cf.* section 4.1, the form $\mathbf{s} \star\mathbf{dx}|_{\mathcal{M}} = \mathbf{t} \star \mathrm{d}\underline{x}^1|_{\mathcal{M}}$, where $\mathbf{t} = s_{i1}\underline{\boldsymbol{e}}_1$ is the traction vector acting on $\mathcal{M}$. Similarly, for the heat flux vector $\mathbf{q}$, *cf.* section 4.2, we have $\mathbf{q} \star\mathbf{dx}|_{\mathcal{M}} = q_1 \star \mathrm{d}\underline{x}^1|_{\mathcal{M}}$, which is the normal component of the flux. In short, once a local coordinate system is employed only the computation of the normal components of the flux are required.

The choice of local coordinate system also correlates well with the one-dimensional Riemann problem. In the Riemann problem we have two distinct sets of uniform initial conditions on opposite sides of $\mathcal{M}$ which correspond to the zeroth order terms of the local solution. The jumps of initial conditions across the interface contribute to a nonzero differential of the normal fluxes while in all tangential directions the flux has zero total differential. Consequently, in equation (17), only the normal part of the flux differential $\mathbf{A}\boldsymbol{u}_{,n}$ is present and there is no need to compute flux matrices for tangential directions.

3. Riemann solution for linear hyperbolic systems

Problem 2 (One-dimensional Riemann problem for hyperbolic systems)**.** *The one-dimensional Riemann problem consists of a hyperbolic system*

$$\boldsymbol{u}_{,t} + \mathbf{A}\boldsymbol{u}_{,n} = \mathbf{0}, \quad \boldsymbol{u} = \{u_1, u_2, \ldots, u_n\}^T, \tag{17}$$

with step data that is piecewise constant but discontinuous along the local $\underline{x}_1$ *coordinate direction:*

$$\mathbf{u} = \begin{cases} \mathbf{u}_\alpha, & \underline{x}_1 < 0 \\ \mathbf{u}_\beta, & \underline{x}_1 > 0, \end{cases} \tag{18}$$

The flux Jacobian matrix $\mathbf{A}$ is smoothly varying across the domain. The subscript "$_{,n}$" denotes the derivative in the normal direction. Source terms have been dropped in deriving (17) from the hyperbolic system (6); see Remark 1.

Remark 1. *Source terms, whether or not they are solution dependent, do not affect the Riemann solution in the SDG context. Using the eigenvalue decomposition* $\mathbf{A} = \mathbf{\Gamma\Lambda\Gamma^{-1}}$*, and defining the characteristic variables by* $\boldsymbol{w} = \mathbf{\Gamma}^{-1}\boldsymbol{u}$ *the Riemann problem* $\boldsymbol{u}_{,t} + \mathbf{A}\boldsymbol{u}_{,n} = \boldsymbol{S}$ *is transformed into* $\boldsymbol{w}_{,t} + \mathbf{\Lambda}\boldsymbol{w}_{,n} = \boldsymbol{s}$*, where* $\boldsymbol{s} = \mathit{\Gamma}^{-1}\boldsymbol{S}$*; cf.* (21a), (22) *and section 3.2. Accordingly, for characteristic variable* w_i*, we have* $w_{i,t} + \lambda_i w_{i,n} = s_i$*. That is,* $w_i = w_i(0) + \int_0^L s_i \mathrm{d}l/\sqrt{1+\lambda_i^2}$*, where* $w_i(0)$ *is the initial condition at* $\underline{t} = 0$ *and the integration is carried over the ray that connects initial condition to the point where we seek the Riemann solution along the characteristic direction. However, in the SDG context, we are looking at the limiting case when the point of interest is approaching* $\underline{t} = 0$*. Thus, the solution to these differential equations is entirely determined by the initial values of characteristic variables and the accumulative contribution of source terms does not influence our Riemann solutions.*

In the semi-discrete setting of most finite volume methods, treatment of the source term within the (approximate) Riemann solver in not trivial; see, e.g., [25]. Integrating the flux over both the time step and the cell face necessitates using an approximated source term that may (or may not) end up being consistent with the inviscid flux discretization. Our SDG methods with Riemann solver do not suffer from this disparity; inter-elemental fluxes are by definition discretized in a manner identical to the cell volumes. Numerical difficulties such as source terms balancing flux gradients are handled naturally.

3.1. Orientation of surfaces of discontinuity

Let us assume that the solution of the local Riemann problem suffers a discontinuity along a ray moving with the speed c in $(\underline{x}_1, t)$ plane. Let Q be a rectangle with long edges on two sides and parallel to the ray. By letting the short edges approaching zero length and employing (7) and the fact that $\star\mathrm{dt} = -c \star \mathrm{d}\underline{x}^1$ along the ray we obtain,

$$\mathbf{A}^- \mathbf{u}^- - c\,\mathbf{u}^- = \mathbf{A}^+ \mathbf{u}^+ - c\,\mathbf{u}^+ \tag{19}$$

where $^-$ and $^+$ superscripts denote the immediate traces to the left and right of the ray. If the flux Jacobian matrix $\mathbf{A}$ varies smoothly in space and time it is continuous along the discontinuity and we obtain,

$$\mathbf{A}(\mathbf{u}^+ - \mathbf{u}^-) = c(\mathbf{u}^+ - \mathbf{u}^-). \tag{20}$$

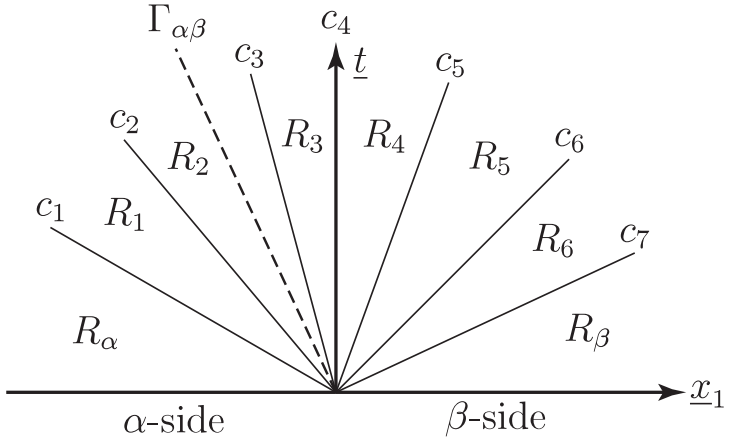

Figure 2: Eigenstructure for a one-dimensional Riemann problem in the $\{\underline{x}_1, \underline{t}\}$ plane. Initial step data is given on the $\{\alpha, \beta\}$ sides. The non-causal interface is represented by the $\Gamma_{\alpha\beta}$ ray. The solution regions $\{R_\alpha, R_i, R_\beta\}_{i=1}^{6}$ partition the spacetime domain into 8 unique solution regions, each separated by a ray corresponding to a characteristic direction.
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That is the orientation of the discontinuity c should be an eigenvalue of $\mathbf{A}$ and the jump of the solution $\mathbf{u}^+ - \mathbf{u}^-$ is the corresponding eigenvector for c.

3.2. Diagonalisation of the flux matrix

Figure 2 depicts the eigenstructure for a typical hyperbolic system. Information propagates along the characteristic directions with speed c. The regions between characteristic directions are denoted by R_i. The interface $\Gamma_{\alpha\beta}$ is said to be *causal* if it lies in either region R_α or R_β. Otherwise, it is *non-causal*. Riemann solutions on causal manifolds are trivial; they are simply the earlier-in-time values. Non-causal interfaces pose non-trivial Riemann solutions, and it is that family of solutions we investigate next.

The flux Jacobian matrix $\mathbf{A}$ is assumed to be smoothly varying; as such, it is single-valued on any submanifold $\mathcal{M}$. Also, it arises from a purely hyperbolic system, so we are assured that it is diagonalisable [26, 27, 28].

The matrix is diagonalized as

$$\mathbf{A} = \mathbf{\Gamma \Lambda \Gamma^{-1}} \tag{21a}$$

$$\mathbf{\Gamma} = [\gamma_1 | \cdots | \gamma_n] \tag{21b}$$

$$\mathbf{\Lambda} = \begin{pmatrix} c_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & c_n \end{pmatrix} \tag{21c}$$

where c's are the eigenvalues (ordered smallest to largest) and the γ's are the corresponding eigenvectors. If the algebraic multiplicity of an eigenvalue c is $n > 1$, its geometric multiplicity is also equal to n for hyperbolic systems and the system is still diagonalizable. Then the eigenvalue c is repeated n times in the list $\{c_1, c_2, ..., c_n\}$.

We can also introduce a change of variables by letting $\boldsymbol{w} = \mathbf{\Gamma}^{-1}\boldsymbol{u}$. Rewriting (17) in terms of $\boldsymbol{w}$, we have

$$\boldsymbol{w}_{,t} + \mathbf{\Lambda}\boldsymbol{w}_{,n} = \mathbf{0}, \quad \boldsymbol{w} = \{w_1, w_2, \dots, w_n\}^T. \tag{22}$$

3.3. Riemann solution

We shall consider the local one-dimensional Riemann problem with step data given by

$$\boldsymbol{u}(\underline{x}_1) = \begin{cases} \boldsymbol{u}_\alpha, & \underline{x}_1 < 0, \\ \boldsymbol{u}_\beta, & \underline{x}_1 > 0. \end{cases} \tag{23}$$

Assuming there are $(m+1)$ distinct eigenvalues $(m < n)$, there are then m regions between c_1 and c_n separated by the remaining eigenvalues. In each of these regions, the solution $\boldsymbol{u}_m$ is determined by constructing and solving the following system:

$$\boldsymbol{u}_1 - \boldsymbol{u}_\alpha = a^1 \gamma_\alpha^1 \tag{24a}$$

$$\boldsymbol{u}_2 - \boldsymbol{u}_1 = a^2 \gamma_\alpha^2 \tag{24b}$$

$$\vdots$$

$$\boldsymbol{u}_p - \boldsymbol{u}_{p-1} = a^p \gamma_\alpha^p \tag{24c}$$

$$\boldsymbol{u}_{p+1} - \boldsymbol{u}_p = a^{p+1} \gamma_\beta^{p+1} \tag{24d}$$

$$\vdots$$

$$\boldsymbol{u}_\beta - \boldsymbol{u}_m = a^m \gamma_\beta^m, \tag{24e}$$

where $c^p \leq c_{\alpha\beta} \leq c^{p+1}$, and $c_{\alpha\beta}$ corresponds to the speed of the interface $\Gamma_{\alpha\beta}$. Note that we have purposely delineated between the eigenvectors from each side of the interface. For non-smoothly varying flux Jacobians (material interfaces or non-linearities), these become important; however, in the present context, $\gamma^i_\alpha = \gamma^i_\beta \; \forall \, i$. Also, as mentioned above, if the multiplicity of an eigenvalue is larger than one, then there are the same number of eigenvectors, hence jump conditions, corresponding to it. Thus, (24) is always a square system on n equations.

We can rewrite system (24) as the jump in the step data by summing them:

$$\boldsymbol{u}_\beta - \boldsymbol{u}_\alpha = \left[\gamma^1_\alpha | \ldots | \gamma^p_\alpha | \gamma^{p+1}_\beta | \ldots | \gamma^m_\beta\right] \boldsymbol{a} = \boldsymbol{\Gamma a}. \tag{25}$$

Solving for $\boldsymbol{a}$,

$$\boldsymbol{a} = \boldsymbol{\Gamma}^{-1} [\![\boldsymbol{u}]\!]. \tag{26}$$

Now, we utilize the solution structure that the value of $\boldsymbol{u}$ is constant in any given 'wedge' of space that is separated by the characteristics. As such, we have the piecewise Riemann values in each region given by Eqs. (24)–(26), e.g. $\boldsymbol{u}^* = \boldsymbol{u}^p$. The flux at the interface for a linear system of equations is given by

$$\mathbf{F}^* = \boldsymbol{u}^* \star \mathrm{d}t + \mathbf{A}\boldsymbol{u}^* \star \mathrm{d}\underline{x}^1. \tag{27}$$

3.4. Semi-analytic Riemann solution

In this section we detail how to calculate the Riemann values and the corresponding fluxes (and flux Jacobians) without obtaining a closed-form expression. The only requirement for this method is knowledge of the eigenvalues and eigenvectors of the matrix $\mathbf{A}$, which is many times much easier than computing the full Riemann solution. The computational cost of this method is non-trivial, as it does involve a matrix inversion. However, if the matrix inverse can be obtained analytically, this method is much simpler to implement than analytic expressions.

We begin by transforming the quantities $\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta$ into $\boldsymbol{w}_\alpha, \boldsymbol{w}_\beta$ via (22) as:

$$\boldsymbol{w}_\alpha = \boldsymbol{\Gamma}^{-1}\boldsymbol{u}_\alpha, \quad \boldsymbol{w}_\beta = \boldsymbol{\Gamma}^{-1}\boldsymbol{u}_\beta. \tag{28}$$

The solution of this decoupled differential equations is trivial [26, 27]:

$$\boldsymbol{w}^* = \left\{w^1_\beta, \ldots, w^p_\beta, w^{p+1}_\alpha, \ldots, w^m_\alpha\right\}^T. \tag{29}$$

We can now invert our transformation of variables and write the Riemann values as

$$\boldsymbol{u}^* = \boldsymbol{\Gamma}\boldsymbol{w}. \tag{30}$$

In index notation, this becomes

$$u_i^* = \left(\sum_{j=1}^{p} \Gamma_{ij} (\Gamma^{-1})_{jk} u_\beta^k \right) + \left(\sum_{j=p+1}^{m} \Gamma_{ij} (\Gamma^{-1})_{jk} u_\alpha^k \right). \tag{31}$$

We can combine equations (27) and (30) to obtain a much simpler expression for the spatial component of the Riemann flux:

$$\mathbf{A}\boldsymbol{u}^* = \left(\boldsymbol{\Gamma}\boldsymbol{\Lambda}\boldsymbol{\Gamma}^{-1}\right) . \left(\boldsymbol{\Gamma}\boldsymbol{w}\right), \tag{32}$$

which simplifies to

$$\mathbf{A}\boldsymbol{u}^* = \boldsymbol{\Gamma}\boldsymbol{\Lambda}\boldsymbol{w} \tag{33}$$

Since $\Lambda_{ij} = \delta_{ij} c_i$ from (21c), we can simplify the expression as

$$A_{ij} u_j^* = \left(\sum_{j=1}^{p} \Gamma_{ij} \lambda^j (\Gamma^{-1})_{jk} u_\beta^k \right) + \left(\sum_{j=p+1}^{m} \Gamma_{ij} \lambda^j (\Gamma^{-1})_{jk} u_\alpha^k \right). \tag{34}$$

Note that the spatial flux computation given by (34) is extremely similar to the expression for u_i^* in (31). In an implementation, these terms can usually be computed within a single loop structure to obtain the total spacetime flux $\mathbf{F}^*$.

Newton-Raphson type solution schemes will require the flux Jacobian matrix in order to solve the resulting linear system. It is straightforward to compute the spatial flux Jacobians from (34):

$$\frac{\partial A_{ij} u_j^*}{\partial u_\beta^k} = \sum_{j=1}^{p} \Gamma_{ij} \lambda^j (\Gamma^{-1})_{jk}, \tag{35}$$

$$\frac{\partial A_{ij} u_j^*}{\partial u_\alpha^k} = \sum_{j=p+1}^{m} \Gamma_{ij} \lambda^j (\Gamma^{-1})_{jk}. \tag{36}$$

The temporal flux Jacobian follows by analogy.

It is worth noting that the semi-analytical approach for Riemann solvers is incredibly useful. Computer algebra packages can deliver the eigenvalues

and eigenvectors for a given system fairly easily, even printing the results directly into specified code languages. The ease of implementation and the automation help to eliminate bugs from code. For large, tightly coupled systems of equations, our semi-analytical approach has the benefit of providing the Jacobian matrices directly rather than relying on the more computationally intensive numerical Jacobians or the more human intensive analytical Jacobians.

4. Riemann solution examples

Here, we provide some examples of Riemann solution calculations. We give our results for spatial dimension $d = 2$; results for $d = \{1, 3\}$ are straightforward extensions. In the following examples, subscripts n and t denote the normal and tangential components of a field. Numerical results obtained by using the Riemann solutions detailed herein have been previously published [1, 2, 3].

4.1. Linearized elastodynamics

The Riemann solution for linearized elastodynamics in standard vector calculus notation can be found in, for instance, [1, 4, 29, 30], and references therein.

Let $\mathbf{u}$ denote the *displacement* vector field. The *velocity* $\mathbf{v}$, and *linearized strain* $\mathbf{E}$ are given by $\mathbf{v} = \dot{\mathbf{u}}$ and $\mathbf{E} = (\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}})/2$. That is, $v_i = \dot{u}_i$ and $E_{ij} = (u_{i,j} + u_{j,i})/2$.

The spacetime flux and source terms corresponding to balance of linear momentum are written as,

$$\mathbf{F}_p = \mathbf{p} \star \mathrm{dt} - \mathbf{s} \wedge \star\mathbf{dx}, \qquad \mathbf{S}_p = \rho\boldsymbol{b}\Omega \tag{37}$$

where $\boldsymbol{p}$, $\boldsymbol{s}$, and $\boldsymbol{b}$ are *linear momentum density*, *stress*, and *body force per unit mass* forms, respectively. The *material density per unit volume* is denoted by ρ. In absence of discontinuities, the application of Stokes theorem on (37) yields,

$$\nabla \cdot \mathbf{s} + \rho\mathbf{b} - \dot{\mathbf{p}} = 0 \tag{38}$$

which is the familiar *equation of motion*.

The force-like fields $\mathbf{s}$ and $\mathbf{p}$ are related to kinematic fields through constitutive equations. For isotropic materials we have $s_{ij} = \lambda\delta_{ij}E_{kk} + 2\mu E_{ij}$,

where λ and μ are the Lamé parameters. Furthermore, the linear momentum density is given by $p_i = \rho v_i$.

We augment the balance law (37) with compatibility equations so that the elastodynamics problem is cast in the form of a system of first order equations. This enables us to derive the $\mathbf{A}$ matrix in (17). The spacetime flux and source term corresponding to the compatibility of displacement and velocity fields is given by,

$$\mathbf{F}_u = \mathbf{u} \star \mathrm{dt}, \qquad \mathbf{S}_u = \mathbf{v}\Omega \tag{39}$$

Finally, the compatibility between strain and velocity is stated as,

$$\dot{E_{ij}} - \frac{1}{2}(v_{i,j} + v_{j,i}) = 0 \tag{40}$$

which can be expressed as a balance law with temporal flux of $\{E_{11}, E_{12}, E_{22}\}$ and zero source term. Accordingly, the spatial flux is obtained from appropriate terms of velocity field.

Equations (37), (39), and (40) comprise the balance laws for the linearized elastodynamics problem. We define the 3-tuple of balance fields

$$\boldsymbol{u} = \begin{Bmatrix} \mathbf{u} \\ \mathbf{p} \\ \mathbf{E} \end{Bmatrix}. \tag{41}$$

It is often convenient to refer to the components of (41) as

$$\{u_i\}_{i=1}^{7} = \{\mathrm{u}_n, \mathrm{u}_t, \rho v_n, \rho v_t, E_{nn}, E_{nt}, E_{tt}\}. \tag{42}$$

The eigenvalues are given by,

$$\{c\}_{i=1}^{7} = \{-c_{\mathrm{D}}, -c_{\mathrm{S}}, 0, 0, 0, c_{\mathrm{S}}, c_{\mathrm{D}}\}, \tag{43}$$

where c_{D} and c_{S} are longitudinal and shear wave speeds, respectively,

$$c_{\mathrm{D}} = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_{\mathrm{S}} = \sqrt{\frac{\mu}{\rho}}. \tag{44}$$

The corresponding eigenvectors are

$$\gamma_1 = \{0, 0, \rho c_{\mathrm{D}}, 0, 1, 0, 0\}, \tag{45a}$$
$$\gamma_2 = \{0, 0, 0, 2\rho c_{\mathrm{S}}, 0, 1, 0\}, \tag{45b}$$
$$\gamma_3 = \{1, 0, 0, 0, 0, 0, 0\} \tag{45c}$$
$$\gamma_4 = \{0, 1, 0, 0, 0, 0, 0\} \tag{45d}$$
$$\gamma_5 = \{0, 0, 0, 0, -\lambda, 0, \lambda + 2\mu\} \tag{45e}$$
$$\gamma_6 = \{0, 0, 0, -2\rho c_{\mathrm{S}}, 0, 1, 0\}, \tag{45f}$$
$$\gamma_7 = \{0, 0, -\rho c_{\mathrm{D}}, 0, 1, 0, 0\}. \tag{45g}$$

According to (43) the longitudinal and shear wave speeds plus zero wave speed divide the *non-causal* region into four regions of R_1 to R_4. The Riemann solutions are given by,

$$u_1^* = \begin{cases} f_1^\alpha & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \text{ or } R_2 \\ f_1^\beta & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_3 \text{ or } R_4 \end{cases} \tag{46a}$$

$$u_2^* = \begin{cases} f_2^\alpha & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \text{ or } R_2 \\ f_2^\beta & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_3 \text{ or } R_4 \end{cases} \tag{46b}$$

$$u_3^* = \langle\!\langle f_3 \rangle\!\rangle + \frac{\rho c_{\mathrm{D}}}{2} \left([\![f_5]\!] + \frac{\lambda}{\lambda + 2\mu} [\![f_7]\!] \right), \quad \text{for all regions} \tag{46c}$$

$$u_4^* = \begin{cases} f_4^\alpha & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \\ \langle\!\langle f_4 \rangle\!\rangle + \rho c_{\mathrm{S}} [\![f_6]\!] & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_2 \text{ or } R_3 \\ f_4^\beta & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_4 \end{cases} \tag{46d}$$

$$u_5^* = \frac{1}{2\rho c_{\mathrm{D}}} [\![f_3]\!] + \langle\!\langle f_5 \rangle\!\rangle + \frac{\lambda}{2(\lambda + 2\mu)} [\![f_7]\!], \quad \text{for all regions} \tag{46e}$$

$$u_6^* = \begin{cases} f_6^\alpha & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \\ \frac{1}{4\rho c_{\mathrm{S}}} [\![f_4]\!] + \langle\!\langle f_6 \rangle\!\rangle & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_2 \text{ or } R_3 \\ f_6^\beta & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_4 \end{cases} \tag{46f}$$

$$u_7^* = \begin{cases} f_7^\alpha & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \text{ or } R_2 \\ f_7^\beta & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_3 \text{ or } R_4 \end{cases} \tag{46g}$$

where $[\![(\cdot)]\!] := (\cdot)_\beta - (\cdot)_\alpha$ and $\langle\!\langle(\cdot)\rangle\!\rangle := \{(\cdot)_\alpha + (\cdot)_\beta\}/2$ are the jump and average operators.

4.2. Non-Fourier heat conduction

Non-Fourier heat conduction models do not use the typical Fourier law $\mathbf{q} = -\kappa\nabla T$, where $\mathbf{q}, \kappa, T$ are the heat flux, thermal conductivity, and temperature, respectively. We will use a modified heat conduction relation, known as the Maxwell-Cattaneo-Vernotte (MCV) model [3], that relates the heat flux to the temperature gradient through the relation

$$\frac{1}{\kappa}\left(\tau\dot{\mathbf{q}} + \mathbf{q}\right) + \nabla T = 0, \tag{47}$$

where τ is the thermal relaxation time. Note that for simplicity, but without loss of generality, we have assumed an isotropic medium; this reduces the second-order conduction tensor field κ to a scalar field. The exact Riemann solution for this problem can be found in [31, 3].

We combine the constitutive relationship (47) with the balance of energy equation to define the non-Fourier thermal problem. We introduce the 2-tuple of balanced fields

$$\mathbf{u} = \begin{Bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{Bmatrix} = \begin{Bmatrix} CT \\ (\tau/\kappa)\mathbf{q}\end{Bmatrix}, \tag{48}$$

with components

$$\{u_1, u_2, u_3\} = \left\{CT, \frac{\tau}{\kappa}q_n, \frac{\tau}{\kappa}q_t\right\}. \tag{49}$$

The spacetime flux and source terms are written as

$$\mathbf{F}(\mathbf{u}) = \begin{Bmatrix}\mathbf{F}_1\\ \mathbf{F}_2\end{Bmatrix} = \begin{Bmatrix} CT\star\mathrm{dt} + \mathbf{q}\wedge\star\mathbf{dx} \\ (\tau/\kappa)\mathbf{q}\star\mathrm{dt} + T\star\mathbf{dx}\end{Bmatrix}, \quad \mathbf{S}(\mathbf{u}) = \begin{Bmatrix}\mathbf{S}_1\\ \mathbf{S}_2\end{Bmatrix} = \begin{Bmatrix} 0 \\ (1/\kappa)\mathbf{q}\Omega\end{Bmatrix}. \tag{50}$$

The eigenvalues for the hyperbolic MCV system are

$$\{c\}_{i=1}^{3} = \{-c_T, 0, c_T\}, \quad c_T = \sqrt{\frac{\kappa}{C\tau}}. \tag{51}$$

The corresponding eigenvectors are

$$\gamma_1 = \{-\sqrt{\frac{C\kappa}{\tau}}, 1, 0\}, \tag{52a}$$

$$\gamma_2 = \{0, 0, 1\}, \tag{52b}$$

$$\gamma_3 = \{\sqrt{\frac{C\kappa}{\tau}}, 1, 0\}. \tag{52c}$$

At this point, the Riemann solution and face fluxes can be assembled according to the procedure detailed in section 3. For this simple case, the analytic Riemann solution is

$$u_1^* = \langle\!\langle u^1 \rangle\!\rangle - \frac{1}{2}\xi \left[\!\left[u^2 \right]\!\right] \quad \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \text{ or } R_2 \tag{53a}$$

$$u_2^* = \langle\!\langle u_2 \rangle\!\rangle - \frac{1}{2\xi} \left[\!\left[u_1 \right]\!\right] \quad \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \text{ or } R_2 \tag{53b}$$

$$u_3^* = \begin{cases} u_3^\alpha & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_1 \\ u_3^\beta & \text{if } \Gamma_{\alpha\beta} \text{ is in } R_2 \end{cases} \tag{53c}$$

where $\xi := \sqrt{C\kappa/\tau}$.

4.3. Generalized linear thermoelasticity

One form of generalized linear thermoelasticity is obtained by combining the MCV equations in 4.2 with linearized elastodynamics in 4.1 through the stress constitutive equation and mechanical energy into the energy balance equation. The resulting thermomechanical theory is known as generalized thermoelasticity with one relaxation time. A full discussion on thermoelasticity with finite wave speeds can be found in [32]. The stress now depends on the temperature as

$$s_{ij} = \lambda \delta_{ij} E_{kk} + 2\mu E_{ij} + \delta_{ij} k, \tag{54}$$

where '$k\,\mathbf{id}$' is the isotropic stress-temperature tensor. The energy equation is augmented by the term "$(T_0 k)\nabla \cdot \mathbf{v}$" [33], where T_0 is the reference temperature.

The Fourier conduction law has been replaced with a hyperbolic version, which renders the coupled system entirely hyperbolic. The fully coupled wavespeeds can be written in terms of the uncoupled wavespeeds as

$$c_{\mathrm{d}} = \left[\frac{1}{2} \left(c_{\mathrm{D}}^2 + c_T^2 + \beta - \sqrt{-4c_{\mathrm{D}}^2 c_T^2 + (c_{\mathrm{D}}^2 + c_T^2 + \beta)^2} \right) \right]^{\frac{1}{2}} \tag{55a}$$

$$c_t = \left[\frac{1}{2} \left(c_{\mathrm{D}}^2 + c_T^2 + \beta + \sqrt{-4c_{\mathrm{D}}^2 c_T^2 + (c_{\mathrm{D}}^2 + c_T^2 + \beta)^2} \right) \right]^{\frac{1}{2}}, \tag{55b}$$

where $\beta = T_0 k^2 / C\rho$.

We write the system of coupled equations in terms of the fields $\{\mathbf{u}, \rho\mathbf{v}, \mathbf{E}, CT, \tau\mathbf{q}/\kappa\}$. The scalar components of our system are then

$$\boldsymbol{u} = \{u_i\}_{i=1}^{10} = \left\{\mathrm{u}_n, \mathrm{u}_t, \rho v_n, \rho v_t, E_{nn}, E_{nt}, E_{tt}, CT, \frac{\tau}{\kappa}q_n, \frac{\tau}{\kappa}q_t\right\} \tag{56}$$

The eigenvectors of the system can be simplified and written as

$$\gamma_1 = \{0, 0, \rho c_{\mathrm{D}}^2(c_t - c_T)(c_t + c_T), 0, c_t(c_D^2 - c_d^2), 0, 0, -c_D^2 c_t C\theta, cD^2\theta, 0\}, \tag{57a}$$

$$\gamma_2 = \{0, 0, \rho c_{\mathrm{D}}^2(c_d - c_T)(c_d + c_T), c_d(c_D - c_t)(c_D + c_t), 0, 0, -c_D^2 c_d C\theta, c_D^2\theta, 0\}, \tag{57b}$$

$$\gamma_3 = \{0, 0, 0, 2\rho c_{\mathrm{S}}, 0, 1, 0, 0, 0, 0\}, \tag{57c}$$

$$\gamma_4 = \{0, 1, 0, 0, 0, 0, 0, 0, 0, 0\}, \tag{57d}$$

$$\gamma_5 = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 1\}, \tag{57e}$$

$$\gamma_6 = \{0, 0, 0, 0, -\lambda, 0, \rho c_D^2, 0, 0, 0\}, \tag{57f}$$

$$\gamma_7 = \{1, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \tag{57g}$$

$$\gamma_8 = \{0, 0, 0, -2\rho c_{\mathrm{S}}, 0, 1, 0, 0, 0, 0\}, \tag{57h}$$

$$\gamma_9 = \{0, 0, \rho c_D^2(c_d - c_T)(c_d + c_T), c_d(c_t^2 - c_D^2), 0, 0, C c_d c_D^2\theta, c_D^2\theta, 0\}, \tag{57i}$$

$$\gamma_{10} = \{0, 0, \rho c_D^2(c_t - c_T)(c_t + c_T), c_t(c_d - c_D)(c_d + c_D), 0, 0, C c_D^2 c_t\theta, c_D^2\theta, 0\}. \tag{57j}$$

where $\theta = T_0 K/C$. Using these eigenvalues (i.e. the wavespeeds) and eigenvectors, the Riemann solution is obtainable via the procedure described in Section 3. To the authors' knowledge, this is the first work to provide the Riemann solutions for generalized thermoelasticity with one relaxation time.

We would like to note that this example is one in which the coupling between solution fields makes it extremely tedious and time consuming to write out the closed-form analytic expression for the Riemann solution. As an indication of how complicated the analytic solutions can become, consider a homogeneous, isotropic medium, such that there are no jumps in material properties. Then, in region R_4, the *simplest* non-trivial solution is

$$(u_8)^* = CT^* = \langle\!\langle u_8 \rangle\!\rangle - \frac{1}{2\rho(c_d + c_t)}\left(\rho C(c_d c_t + c_T^2)\,[\![u_9]\!] + kT_0\,[\![u_3]\!]\right) \tag{58}$$

In general, the non-trivial solutions are comprised of an averaged quantity and a linear combination of jumps in all other quantities. However, the

eigenvectors are simple enough that the solution procedure of 3 makes the implementation of an exact (albeit semi-analytic) Riemann solver tractable, if not easy.

4.4. Inviscid gas dynamics

The Euler equations of inviscid gas dynamics are a highly non-linear set of equations. As such, the textbook diagonalisation method we have applied is insufficient for solving the Riemann problem. The exact solution of the Riemann problem requires (Newton-Raphson) iterations, making it computationally expensive when evaluating it at every quadrature point on all non-causal faces. It is commonplace to substitute an approximate Riemann solution. We shall discuss one such approximate Riemann solution technique which allows us to use the diagonalisation procedure after linearizing the system. There are many choices of approximate Riemann solvers for this problem; see [28] for a general discussion.

The balanced quantities of mass, momentum, and energy density are denoted as

$$\mathbf{u} = \{\rho, \mathbf{m}, \mathcal{E}\}. \tag{59}$$

We write the spacetime fluxes as

$$\mathbf{F}(\mathbf{u}) = \begin{Bmatrix} \mathbf{F}_\rho \\ \mathbf{F}_\mathbf{m} \\ \mathbf{F}_\mathcal{E} \end{Bmatrix} = \begin{Bmatrix} \rho \star \mathrm{dt} + \mathbf{m} \wedge \star\mathbf{dx} \\ \mathbf{m} \star \mathrm{dt} + \frac{1}{\rho}\mathbf{m} \otimes \mathbf{m} + p\boldsymbol{I} \star \mathbf{dx} \\ \frac{1}{\rho}\left[\mathcal{E}\mathbf{m} + \boldsymbol{\sigma}(\mathbf{m})\right] \star \mathbf{dx} \end{Bmatrix}. \tag{60}$$

where the pressure is given by

$$p = \frac{1}{\gamma - 1}\left(\mathcal{E} - \frac{1}{2\rho}(\mathbf{m} \cdot \mathbf{m})\right) \tag{61}$$

The scalar components of our system in local coordinates are then

$$\boldsymbol{u} = \{u_i\}_{i=1}^4 = \{\rho, m_n, m_t, \mathcal{E}\} \tag{62}$$

The eigenvalues of the non-linear flux Jacobian are

$$\{c\}_{i=1}^4 = \{u - c_E, u, u, u + c_E\}, \quad c_E = \left(\frac{\gamma p}{\rho}\right)^{\frac{1}{2}}, \quad u = \frac{m_n}{\rho}. \tag{63}$$

The corresponding eigenvectors simplified and written as

$$\gamma_1 = \{2\rho^2, 2\rho(m_n - \rho c), 2\rho m_t, m_n^2 + m_t^2 - 2\rho m_n c + \frac{2\rho\gamma p}{\gamma - 1}\}, \tag{64a}$$

$$\gamma_2 = \{2\rho^2, 2\rho m_n, 0, m_n^2 - m_t^2\}, \tag{64b}$$

$$\gamma_3 = \{2\rho m_t, 2m_n m_t, m_t^2 - m_n^2, 0\}, \tag{64c}$$

$$\gamma_4 = \{2\rho^2, 2\rho(m_n + +\rho c), 2\rho m_t, m_n^2 + m_t^2 - 2\rho m_n c + \frac{2\rho\gamma p}{\gamma - 1}\} \tag{64d}$$

Note that the wavespeeds in (63) and the eigenvectors in (64) are dependent on the primitive variables, contrary to our previous examples.

The Vijayasundaram flux [34] has been used previously within the context discontinuous Galerkin methods (e.g., cf. [35, 36, 37, 38, 39]). On the boundary surface of each element (non-causal in our case), the eigenvalues/vectors are evaluated using the average of the primary variables (62) at the previous timestep or iteration. The implicit flux, which is now linear, is obtained through application of (34). Non-arithmetic averaging (e.g. Roe averaging) procedures can also be used to obtain approximate Riemann solutions/fluxes that can be computed in a similar manner.

5. Conclusions

In this paper, we have reviewed a Riemann solution process as it pertains to our SDG methodology [1, 3, 2]. The review of differential forms and coordinate transformations provides other finite element/numerical methods researchers a clear picture of the details behind our SDG methods. The generalized Riemann solution procedure relies on a standard diagonalization of the linearized flux Jacobian, and it is exact for linear systems of equations. Our semi-analytic Riemann solution method shows that implementing and using an exact Riemann solver need not be overly complicated for SDG methods. We have provided complete Riemann solutions for linear elastodynamics, hyperbolic heat conduction, generalized thermoelasticity, and inviscid gas dynamics.

Exact Riemann problems for non-linear systems of equations share a very similar solution structure with the linear systems discussed herein [4]. The methodology presented herein can be used if the flux Jacobian is linearized, as demonstrated in 4.4; however, the resulting Riemann solution is approximate. In future work we will discuss an extension and implementation of a similar

solution method for the exact Riemann solution of non-linear hyperbolic systems. We will also extend our methodology to material interfaces, where material properties are not necessarily continuous.

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