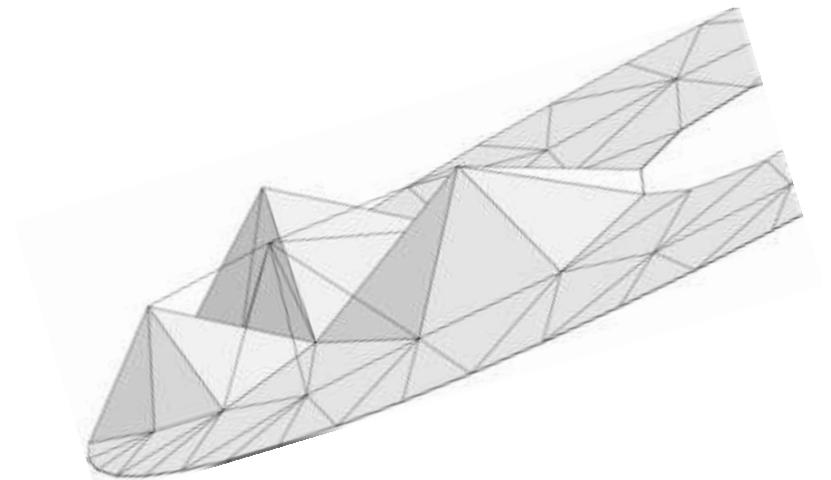
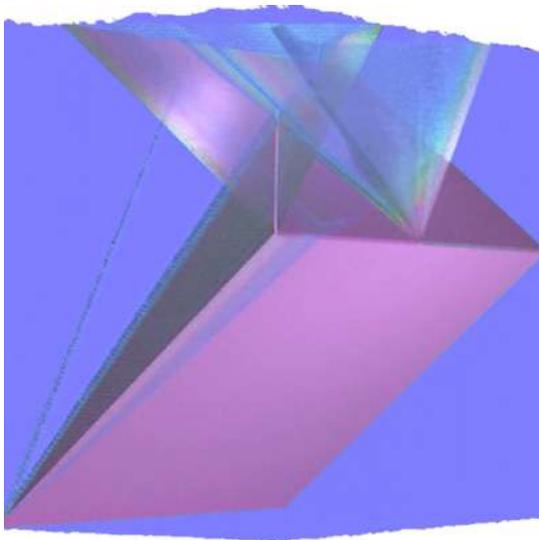


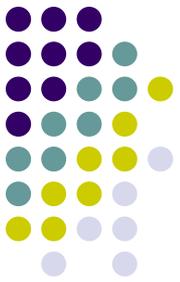
Space-time Discontinuous Galerkin Finite Element Method

Reza Abedi

Mechanical, Aerospace & Biomedical Engineering
University of Tennessee Knoxville / Space Institute

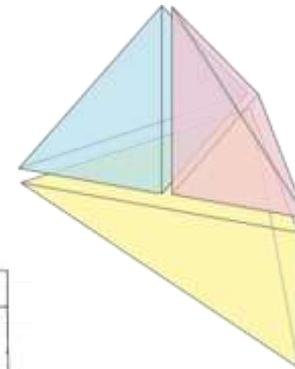


Discontinuous Galerkin (DG) Finite Element Methods



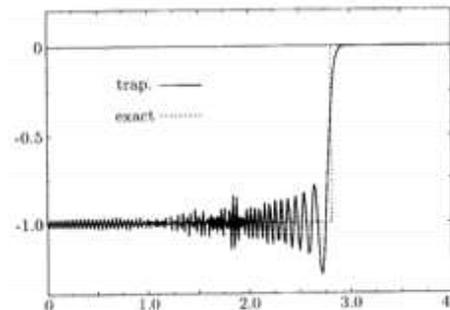
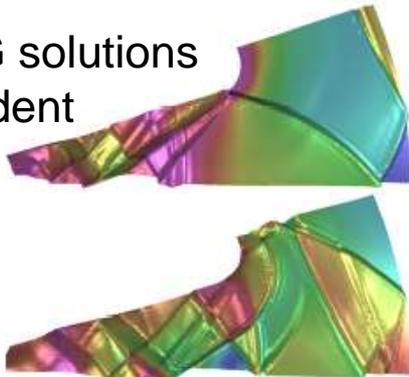
In contrast to conventional finite element methods, DG methods use discontinuous basis functions and

- Weakly enforce conservation jump conditions (e.g., Rankine–Hugoniot)
- Can recover balance properties at the element level (vs global domain)
- Support for nonconforming meshes and
- Arbitrary changes in element polynomial order } no transition elements needed
- Superior performance for resolving discontinuities (discrete solution space better resembles the continuum solution space)

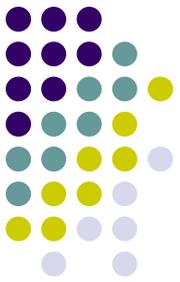


Arbitrary change in size and polynomial order

Sample DG solutions with no evident numerical artifacts



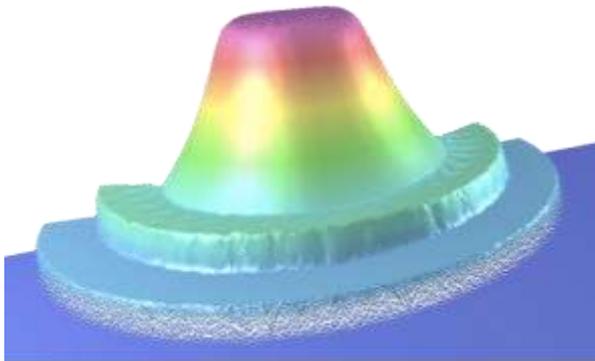
numerical artifacts generally spoil continuous FE solutions in the presence of shocks



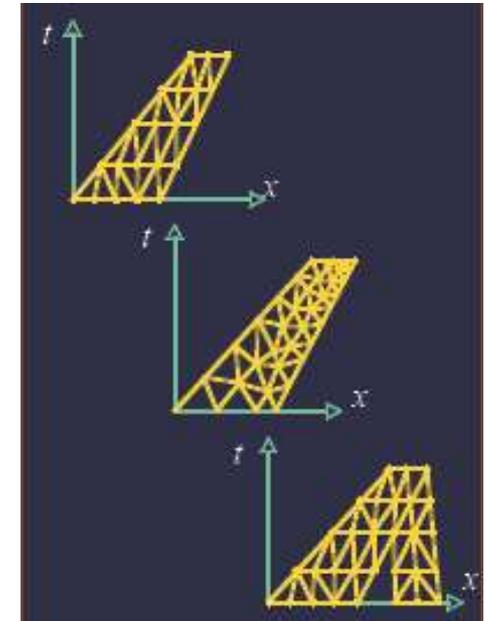
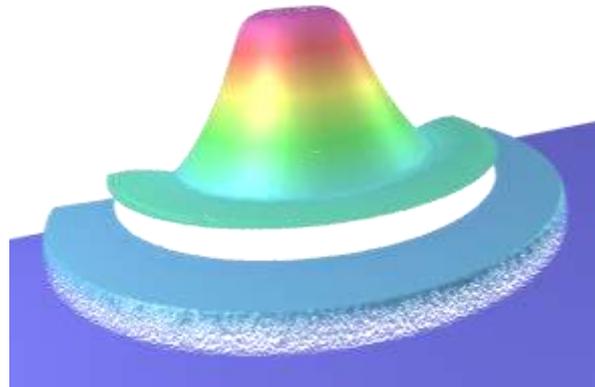
Direct discretization of spacetime

- Replaces a separate time integration; no global time step constraint
- Unstructured meshes in spacetime
- No tangling in moving boundaries
- Arbitrarily high and local order of accuracy in time
- Unambiguous numerical framework for boundary conditions

shock capturing more expensive, less accurate



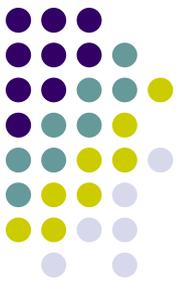
Shock tracking in spacetime: more accurate and efficient



Results by [Scott Miller](#)

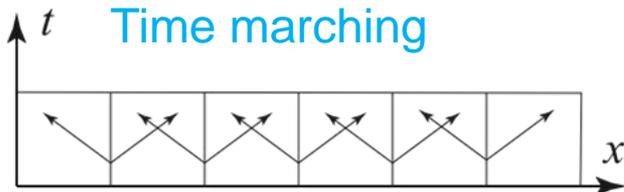


Spacetime Discontinuous Galerkin (SDG) Finite Element Method

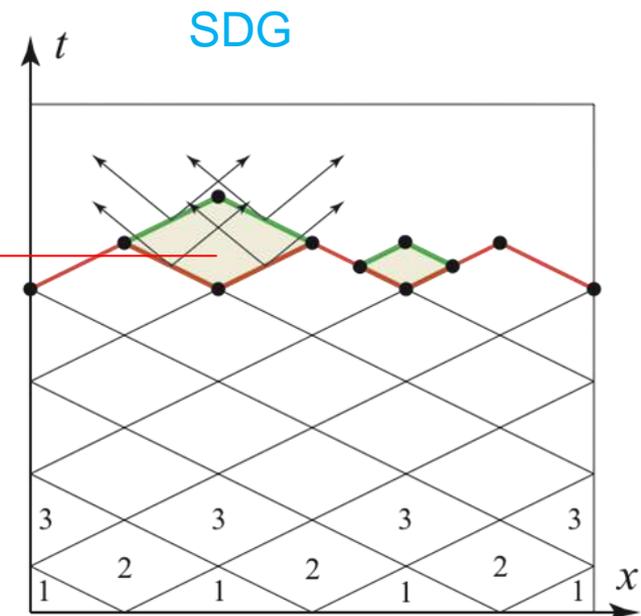


DG + spacetime meshing + causal meshes for hyperbolic problems:

- Local solution property
- $O(N)$ complexity (solution cost scales linearly vs. number of elements N)
- Asynchronous patch-by-patch solver
 - incoming characteristics on red boundaries
 - outgoing characteristics on green boundaries
 - The element can be solved as soon as inflow data on red boundary is obtained \Rightarrow
 - partial ordering & local solution property
 - elements of the same level can be solved in parallel

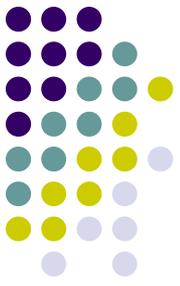


Time marching or the use of extruded meshes imposes a global coupling that is not intrinsic to a hyperbolic problem

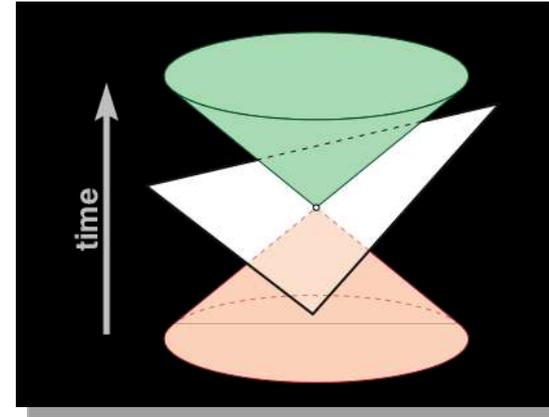


Elements labeled 1 can be solved in parallel from initial conditions; elements 2 can be solved from their inflow element 1 solutions and so forth.

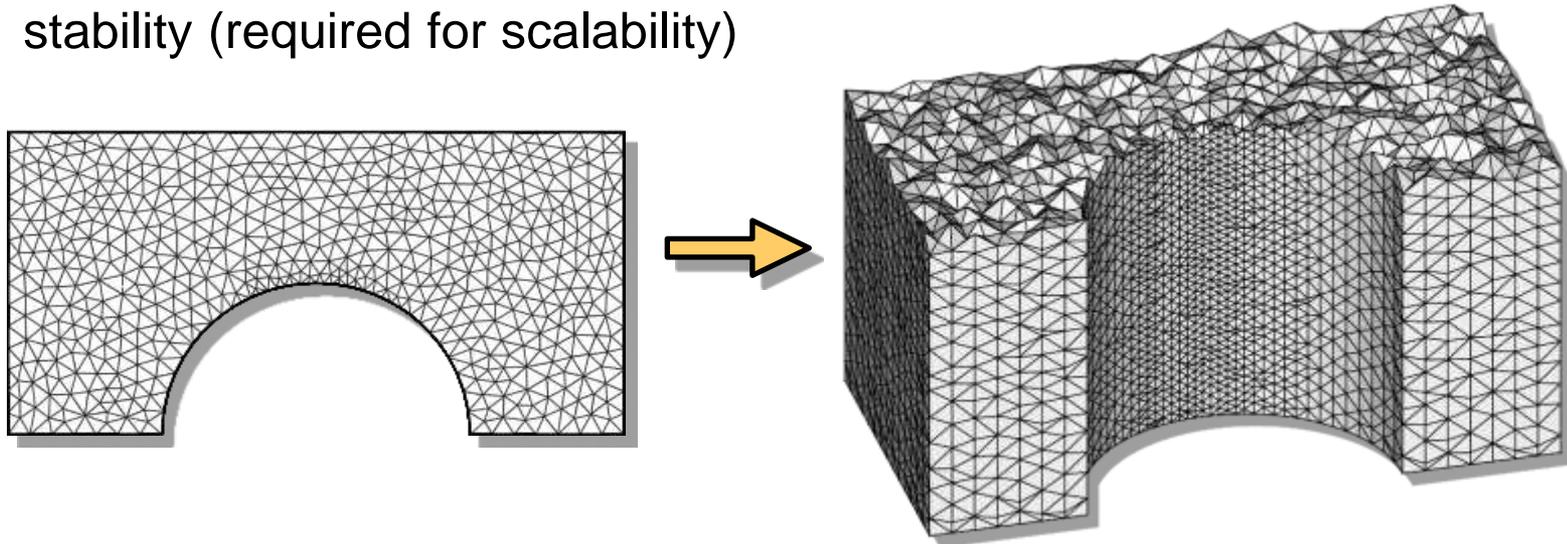
Tent Pitcher: Causal spacetime meshing



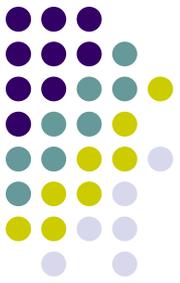
- Given a space mesh, Tent Pitcher constructs a spacetime mesh such that the slope of every facet on a sequence of advancing fronts is bounded by a causality constraint
- Similar to CFL condition, except entirely local and not related to stability (required for scalability)



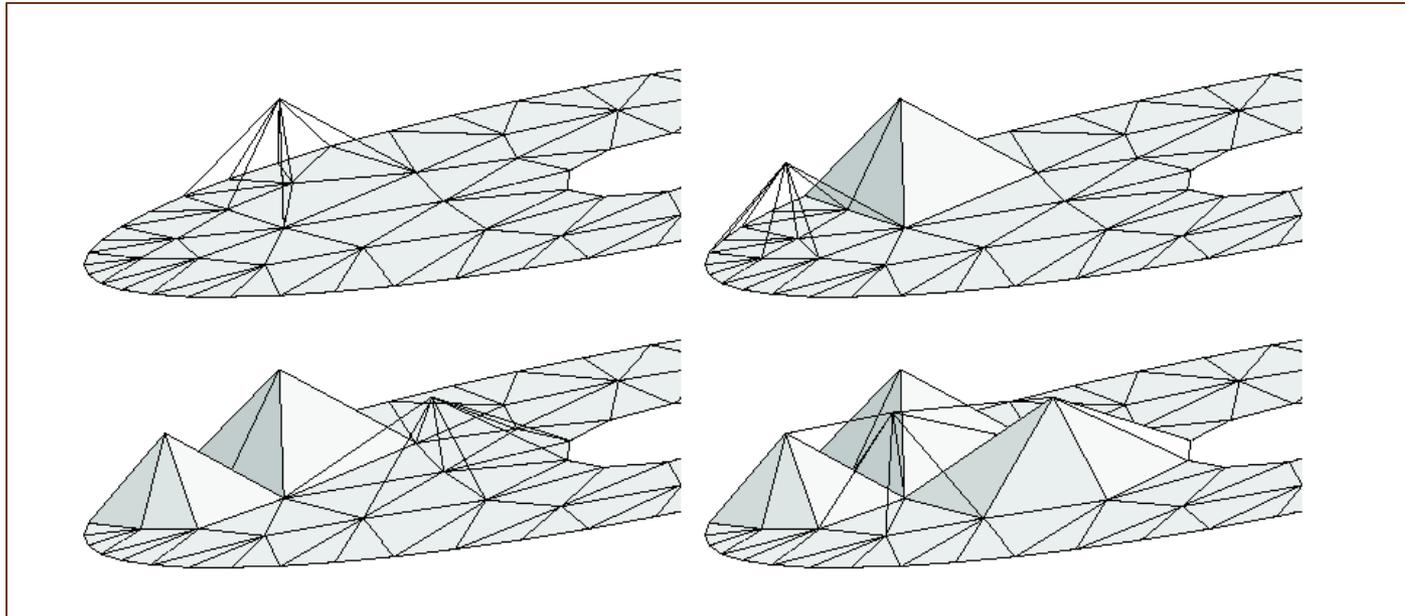
causality constraint



Tent Pitcher: Patch-by-patch meshing



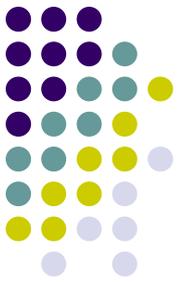
- meshing and solution are interleaved
 - patches ('tents') of tetrahedra are solve immediately \Rightarrow $O(N)$ property
 - rich parallel structure: patches can be created and solved in parallel



tent-pitching sequence

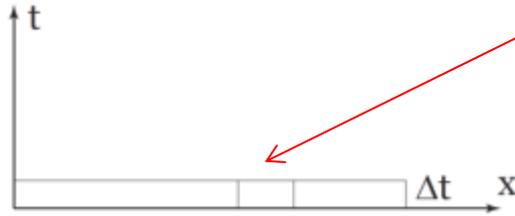
SDG vs time-marching methods

Multi-scale features



Time-marching methods

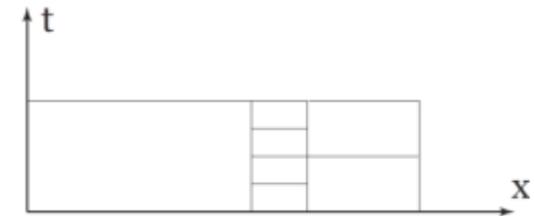
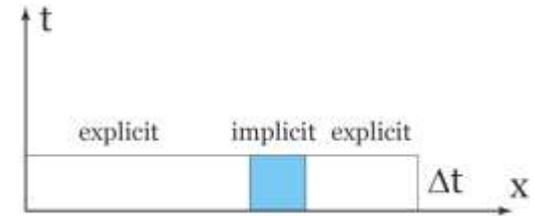
- Time step is limited by smallest elements



Explicit: stability
Implicit: Accuracy

- Improvements:

- **Implicit-Explicit (IMEX)** methods: increase the time step by using implicit integration for small elements
- **Local time-stepping**: subcycling for smaller elements enables using larger global time steps



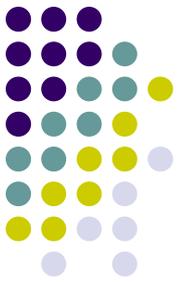
SDGFEM

- Small elements locally have smaller progress in time (no global time step constrains)
- None of the complicated “improvements” of time marching methods needed

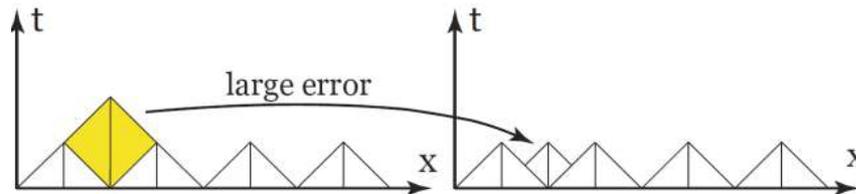
SDGFEM graciously and efficiently handles highly multiscale domains



Unique properties of SDGFEM for adaptive simulations

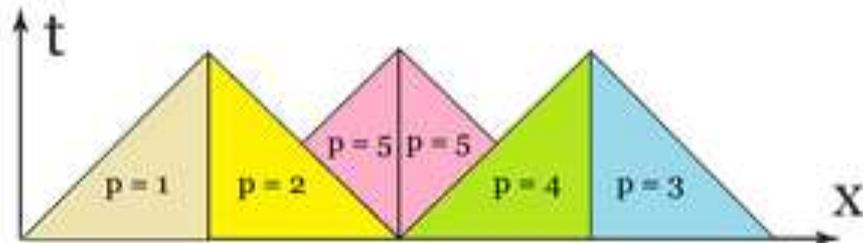
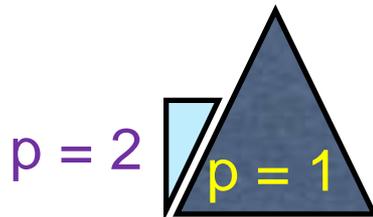


- **Local adaptive operations:** no need for reanalysis of the entire domain



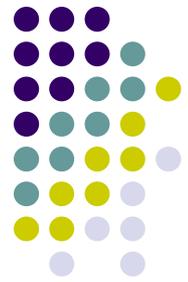
- **Arbitrary h-refinement**
- **Arbitrary p-enrichment**

} Similar to other DG methods, it does not require transition elements to change h or p

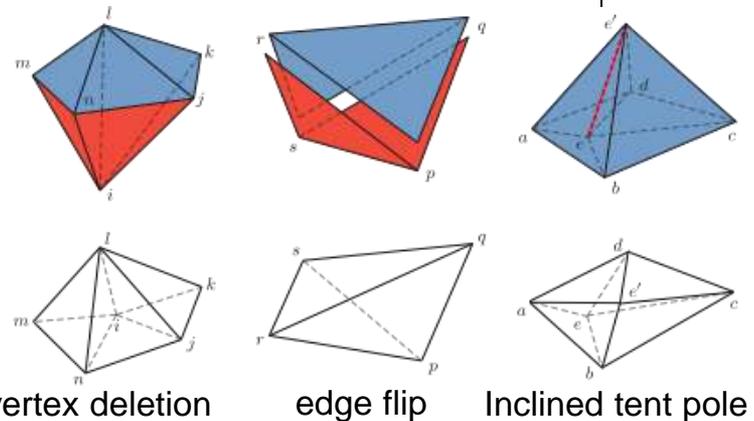


- **Arbitrarily high & local resolution in time**
- **SDGFEM ideal for multiscale meshes:** meshes generated by adaptive operations are highly multiscale, with the latter being a major concern in time marching schemes (previous slide)

Unique properties of SDGFEM for adaptive simulations



- **Spacetime operations eliminate projection errors:** Unlike time-marching methods operations such as vertex deletion, edge flip, and moving vertices do not require projection from old to new meshes

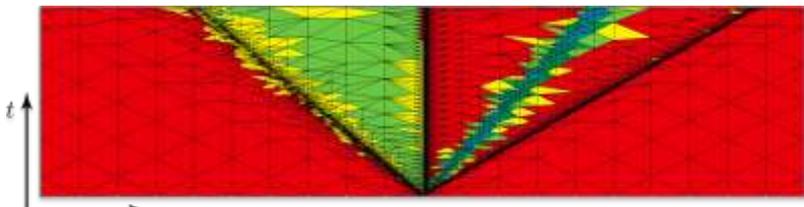


Some applications with these spacetime operations

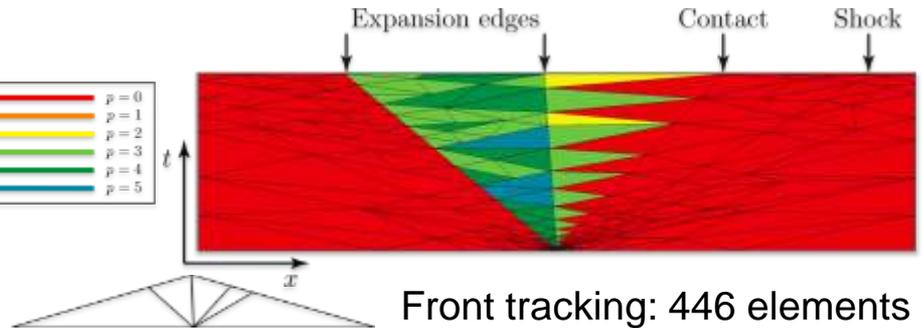
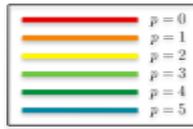
- Front tracking, shock capturing, *hp*-adaptivity
 - *hp*-adaptivity better than *h*-adaptivity
 - Front-tracking better than shock capturing

Results by [Scott Miller](#)

Sod's shock tube problem

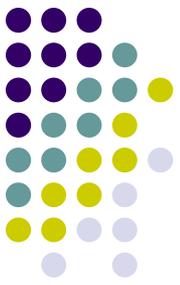


Shock capturing: 473K elements



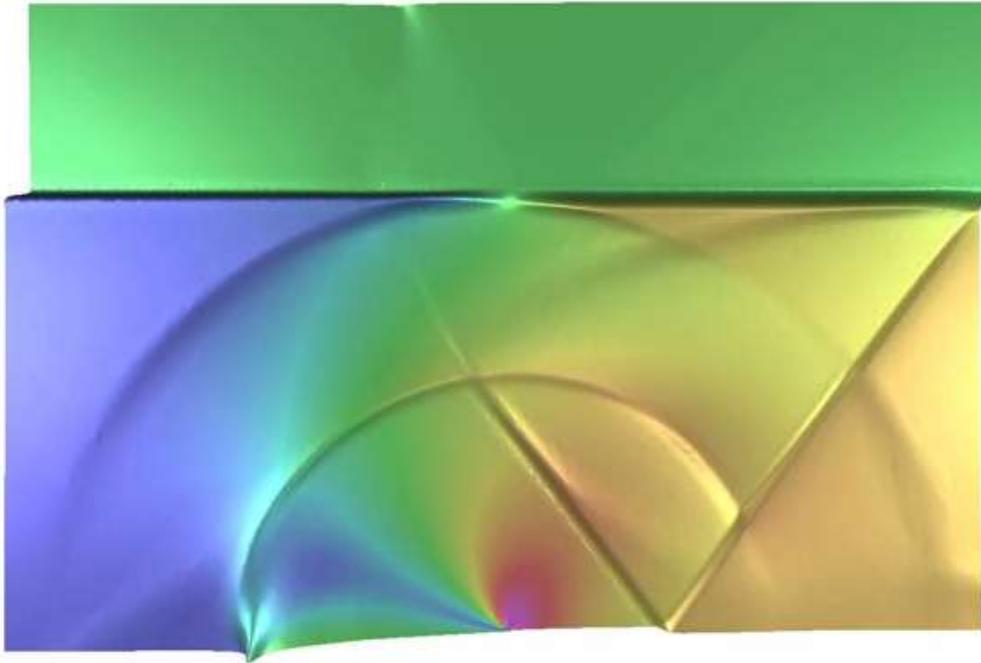
Front tracking: 446 elements

Sample adaptive operations: highly multiscale grids in spacetime



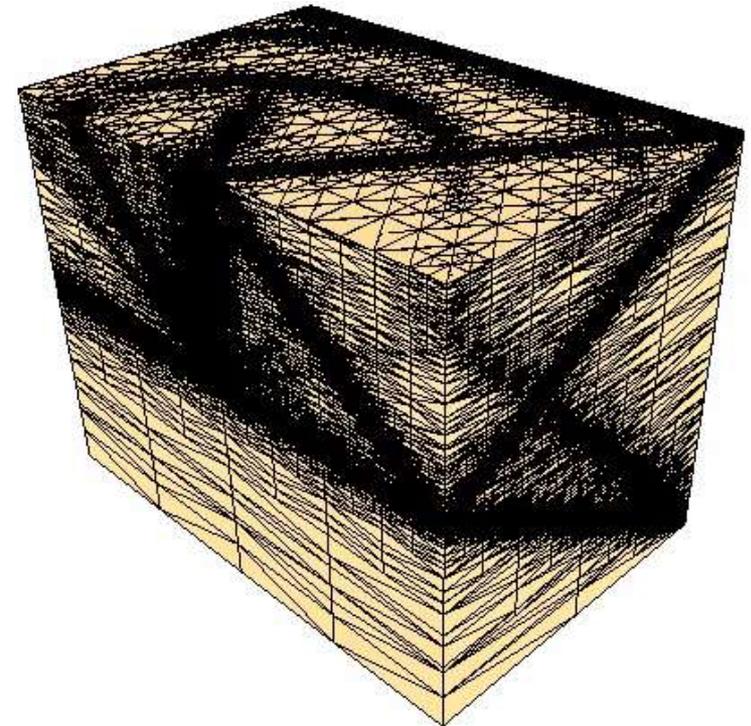
These meshes for a crack-tip wave scattering problem are generated by adaptive operations. Refinement ratio smaller than 10^{-4} .

[click to play movie](#)



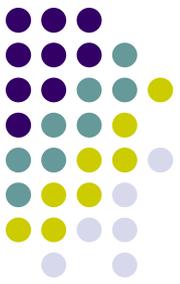
Color: $\log(\text{strain energy})$; Height: velocity

[click to play movie](#)



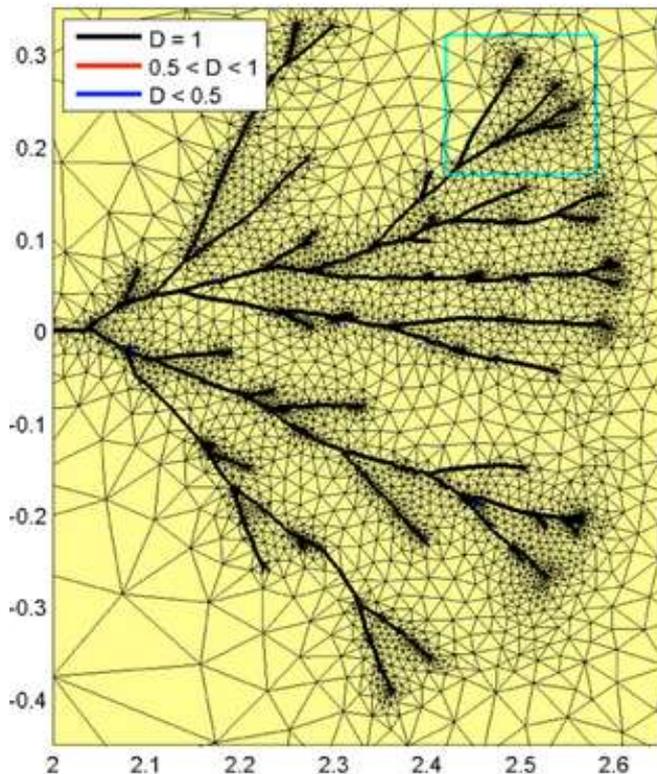
Time in up direction

Sample adaptive operations to track evolving boundaries in spacetime

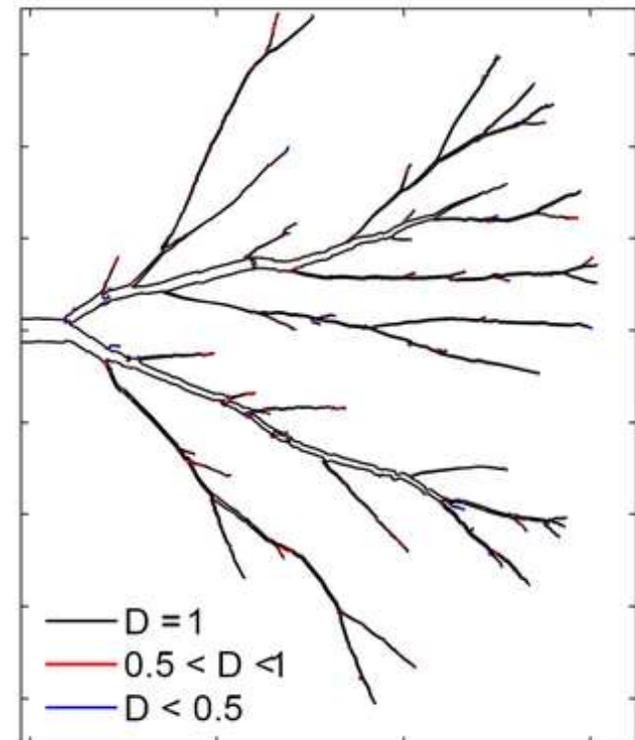


Space front (left) and displaced shape for a solution-dependent crack propagation problem. Adaptive operations exactly track crack directions in spacetime. Refinement ratio smaller than 10^{-6} .

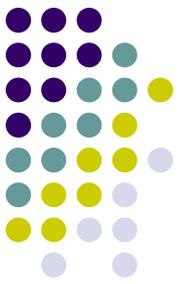
[click to play movie](#)



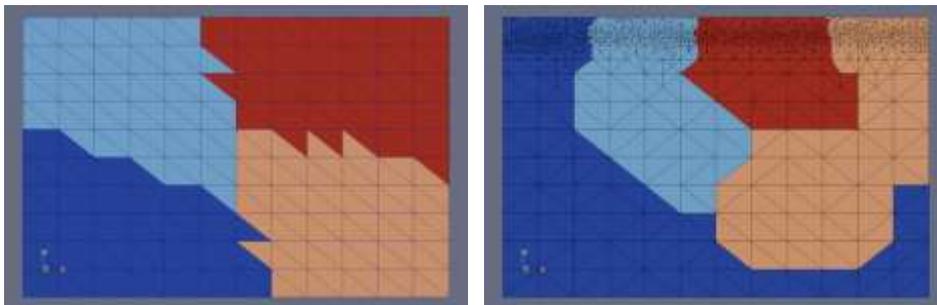
[click to play movie](#)



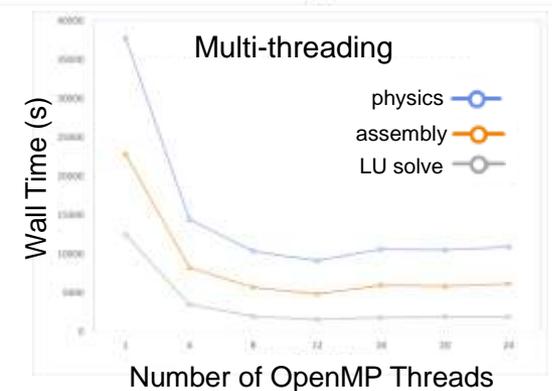
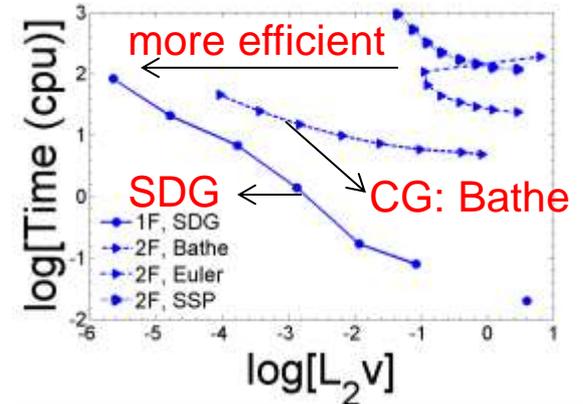
SDGFEM Efficiency: Parallel operations



- Outstanding base properties of serial mode
 - $O(N)$ Complexity
 - Favors highest polynomial order
 - Favors multi-field over single-field FEMs
- Asynchronous
- Nested hierarchical structure for HPC:
 - 1.patches, 2.elements/cells, 3.quadrature points
 - 4,5.rows & columns of matrices
- Domain decomposition at **patch** level:
 - Near perfect scaling for non-adaptive case
 - 95% scaling for strong adaptive refinement
 - Diffusion-like asynchronous load balancing



SDGFEM more efficient than time marching methods

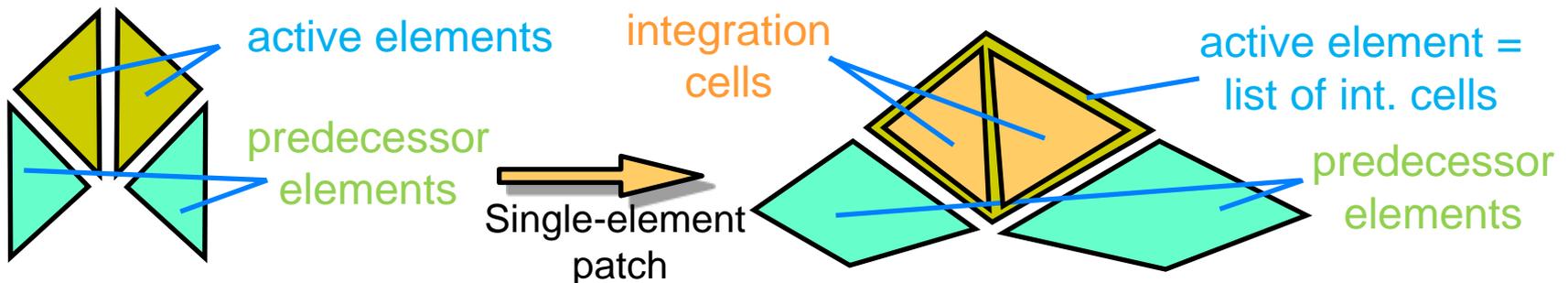


- Multi-threading & Vectorization for **patch** solver at [UIUC](https://www.uiuc.edu)

SDGFEM software features: modularity, extensibility, and efficiency



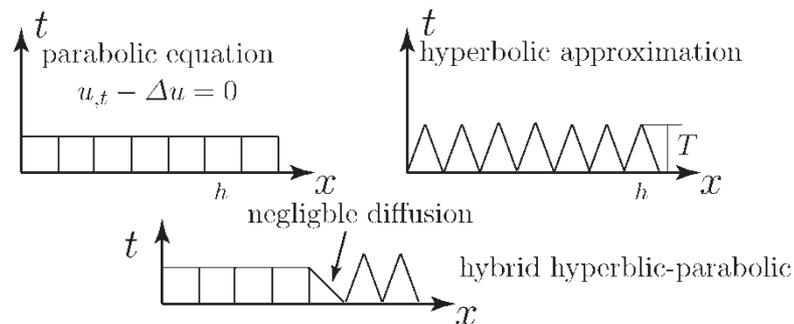
- **Symbolic programming** for weighted residual and error analysis directly translates formulations to computer code
- **Multiphysics coupling**: Unified framework for bulk and interfacial coupling
- **Spatial dimensions 1 to 3** are supported and can coexist in one domain
- **Single-element patches**:
 - Eliminate complex and computationally expensive Riemann solutions of DG and Finite Volume methods
 - More efficient: reduce d.o.f./patch
 - 6:1 & 24:1 for 2d & 3d x time





Conclusion

- The exceptional accuracy of SDGFEM stems from its element-level balance property and intrinsic strength in dealing with discontinuous solution features.
- Local solution property and causal spacetime meshing \Rightarrow
 - Linear solution complexity (very efficient especially for large problem sizes)
 - Asynchronous method \Rightarrow Ideal for parallel computing
- More efficiency gains and flexibility for adaptive simulations:
 - Local-effect adaptive operations (no reanalysis)
 - Arbitrary changes in h and p (no transition elements needed)
 - Locally adaptive and high order of accuracy in time
 - Natural treatment of multiscale features of adaptive discretizations
 - Spacetime adaptive operations can track solution features
- ongoing research: Consistent mathematical approximations alter parabolic problems or hyperbolic systems with vastly different wave speeds (e.g. electromagnetics coupled with solid or fluid problems)



Acknowledgments



- This work is in close collaboration with the past/present members of the SDG team at the University of Illinois at Urbana-Champaign:
 - Faculty (past and present):
[Robert Haber](#), [Jeff Erickson](#), [Laxmikant Kale](#), [Michael Garland](#),
[Robert Gerrard](#), [John Sullivan](#), [Duane Johnson](#)
 - Students:
Kartik Marwah, Ian McNamara, Raj Kumar Pal
 - Past students:
[Scott Miller](#), [Boris Petrakovici](#), Alex Mont, Aaron Becker, Shuo-Heng Chung, Yong Fan, Morgan Hawker, Jayandran Palaniappan, Brent Kraczek, [Shripad Thite](#), Yuan Zhou
 - NCSA Staff Member:
[Valodymyr Kindratenko](#)