02/19/2014

Wednesday, February 19, 2014
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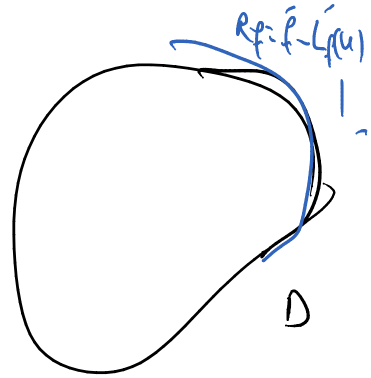
Different types of error



$u =$ exact solution

$u^h =$ approximate solution

$\Delta u = u^h - u$ error function

Types of error:

1. $L_2(\Delta u) = \sqrt{\int_D |\Delta u|^2 \, dv}$

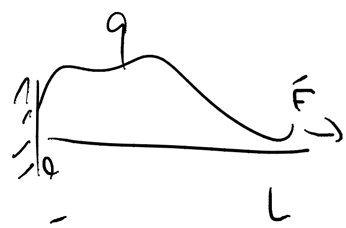
$R_i = Lu - r = \frac{d^2u}{dx^2} + q(x)$

$R_f = 1 - \frac{du}{dx}$

2. Least square error $= R^2 = \int_D R_i^2 \, dv + \int_{\partial D_f} R_f^2 \, ds$

(not a direct function of Δu)

3. Error norm from the weak statement



example

$$\int_0^L \frac{dw}{dx} EA \frac{du}{dx} \, dx - \int_0^L q u \, dx - u(L) \bar{F}$$

$$A(w,u) \quad \underbrace{\int \frac{dw}{dx} EA \frac{du}{dx} \, dx} \Longrightarrow \|w\| = \sqrt{A(w,w)}$$

Galerkin method is the optimum method to

minimize $\|\Delta u\|$ $\longrightarrow$ norm from the problem

For example in solid mechanics

it minimizes the energy of the error.

$$\phi = \{\phi_1, \phi_2, \ldots, \phi_n\} \qquad \phi_p$$

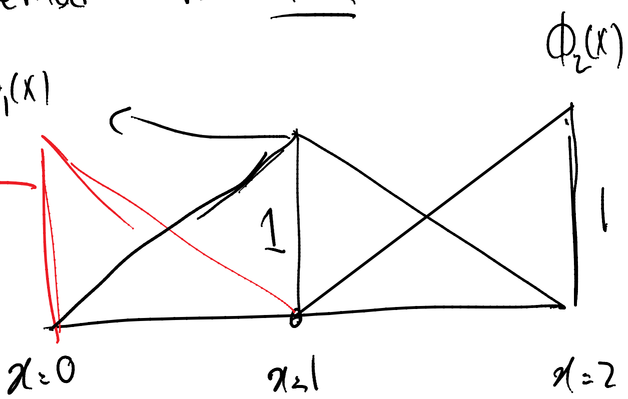
$$u^h = u^h(a_1, \ldots, a_n) = \underline{a_1\phi_1 + a_2\phi_2 + \cdots + a_n\phi_n + \phi_p}$$

for all a_i's Galerkin method minimizes

$$\| u^h - u \|$$

Galerkin Method for $\underline{n=2}$

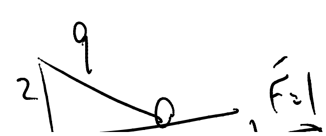
we don't use this as a trial function because $u(0)$ is known (essential BC)



$\phi_1(1) = 1 \quad \phi_1(0) = 0 \quad \phi_1(2) = 0$

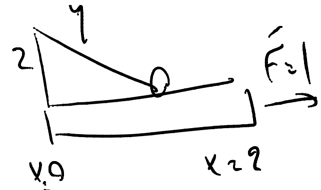
$\phi_2(2) = 1 \quad \phi_2(0) = 0 \quad \phi_2(1) = 0$

$\omega = \phi$ because it's Galerkin method



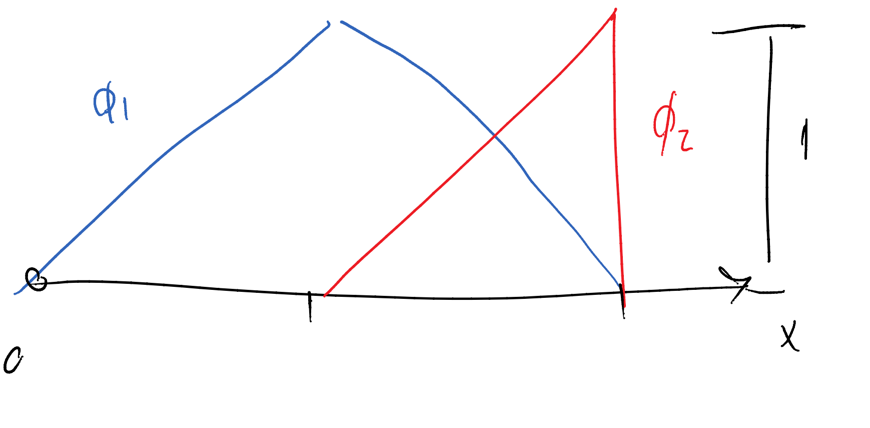
WS:

$$\int_0^2 \frac{dw}{dx} EA \frac{du}{dx}\, dx = \int_0^1 w\, q\, dx + (w\, F)\big|_{x=2}$$

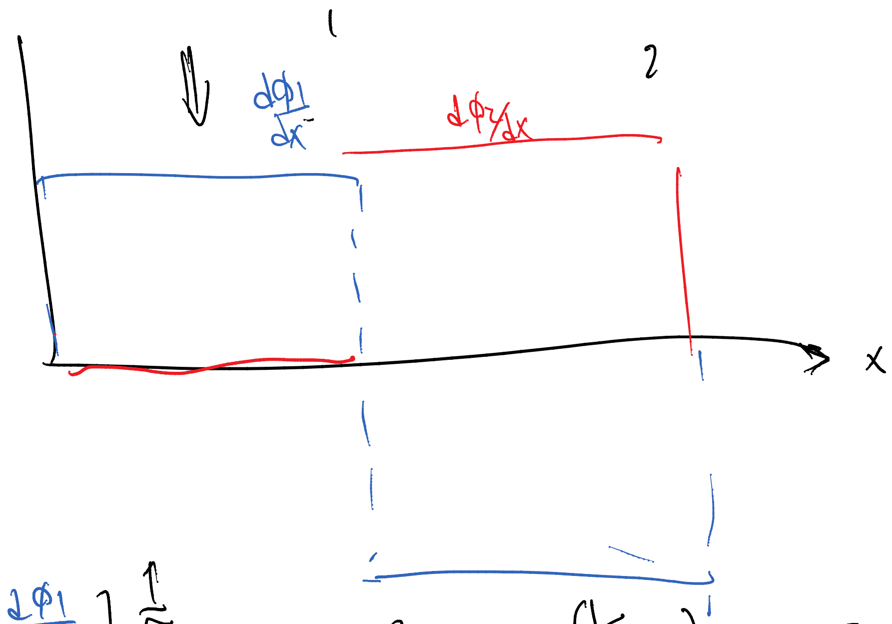


$$K = \int_0^2 \begin{bmatrix} \frac{d\phi_1}{dx} \\ \frac{d\phi_2}{dx} \end{bmatrix} EA \begin{bmatrix} \frac{d\phi_1}{dx} & \frac{d\phi_2}{dx} \end{bmatrix}$$

$$F = \int_0^1 \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \underbrace{(2-2x)}_{q}\, dx + \left(\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \cdot 1 \right)_{x=2}$$



$\frac{d\phi}{dx}$



$$K = \int_0^2 \begin{bmatrix} \frac{d\phi_1}{dx} \\ \frac{d\phi_2}{dx} \end{bmatrix} \frac{1}{EA} \begin{bmatrix} \frac{d\phi_1}{dx} & \frac{d\phi_2}{dx} \end{bmatrix} dx = \int_0^1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} dx + \int_1^2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} dx$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$F = \int_0^1 \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} (2-2x)\, dx + \left(\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \cdot 1 \right)\Bigg|_{x=2}$$

$$= \int_0^1 \begin{bmatrix} x \\ 0 \end{bmatrix} (2-2x)\, dx + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

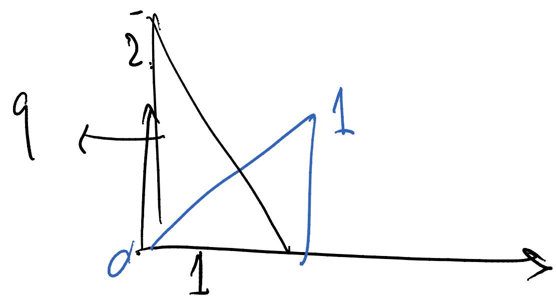
$$= \int \begin{pmatrix} 2x - 2x^2 \\ 0 \end{pmatrix} dx + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - \frac{2}{3} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix}$$

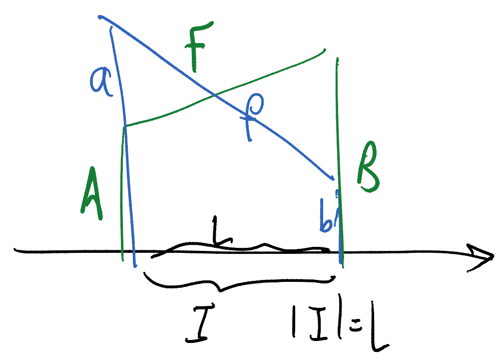
$$\Rightarrow K = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad F = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} \Rightarrow a = \begin{bmatrix} \frac{4}{3} \\ \frac{7}{3} \end{bmatrix}$$

$$\Longrightarrow \quad K = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \Longrightarrow a = \begin{bmatrix} 4/3 \\ 7/3 \end{bmatrix}$$

$$\boxed{u^h_{G2} = \tfrac{4}{3}\varphi_1(x) + \tfrac{7}{3}\varphi_2(x) + \overbrace{1}^{\phi_p}}$$

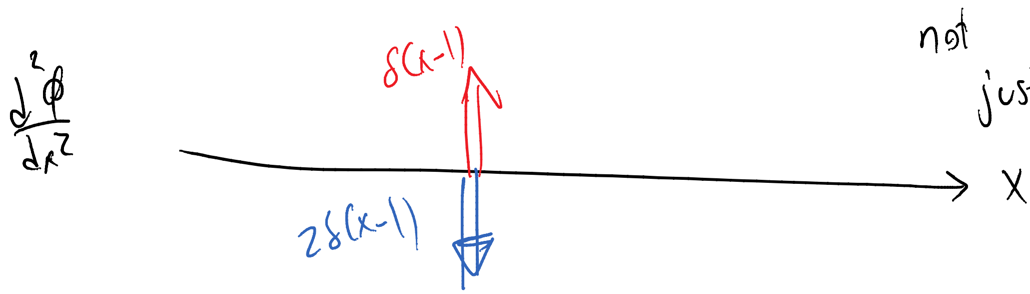
$$\int_0^1 \varphi_1 \, q \, dx$$





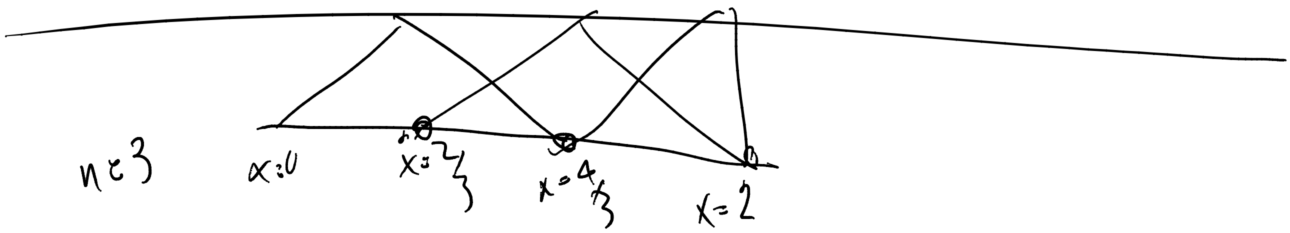
$$\int_I F f \, dx = \frac{L}{6}\left(2A.a + 2B.b + Ab + aB\right)$$

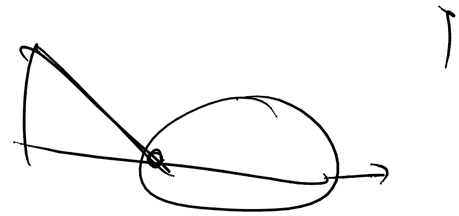
$$\int \varphi_1 q \, dx = \frac{1}{6}\left(2\times2\times0 + 2\times0\times1 + 0\times0 + 2\times1\right) = \frac{1}{3}$$



not needed here just to evaluate whether these shape functions could be used in the "Weighted Residual Statement"

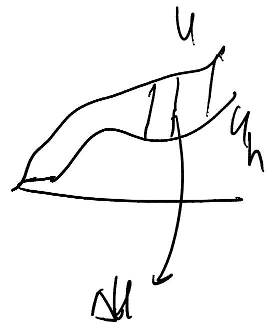
$n=3$



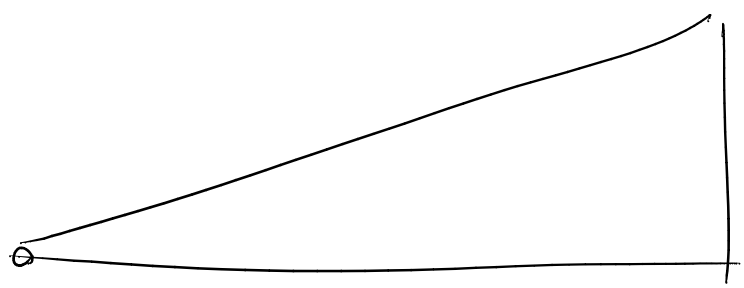


$$\Delta u = u^h - u$$

$$L_2 \|\Delta h\| = \sqrt{\int_0^2 (u_h - u)^2 \, dx}$$

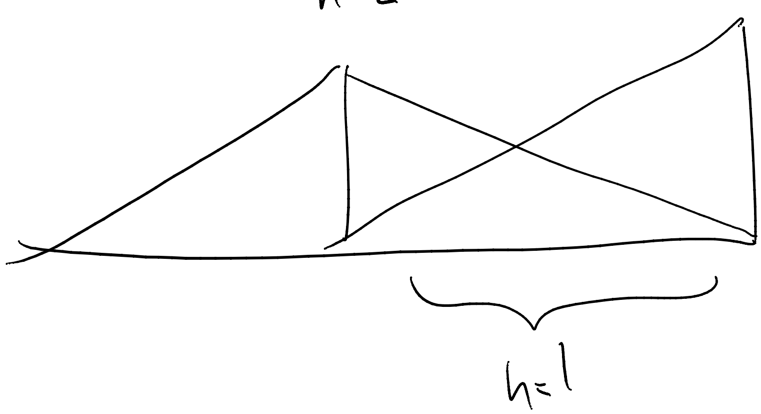


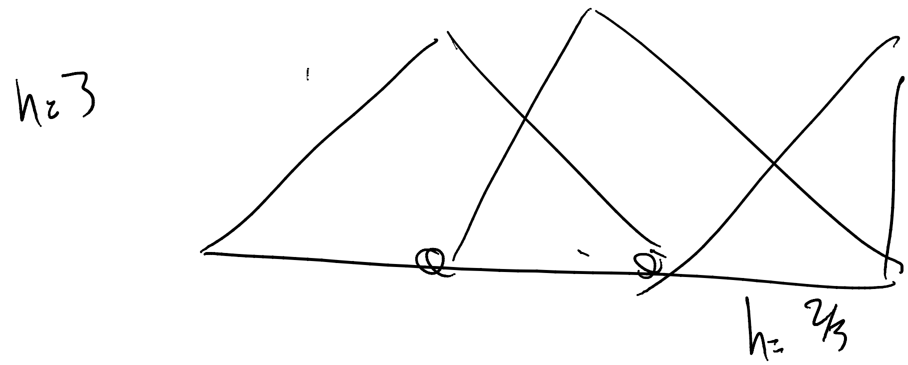
$n=1$



$h=2$

$n=2$



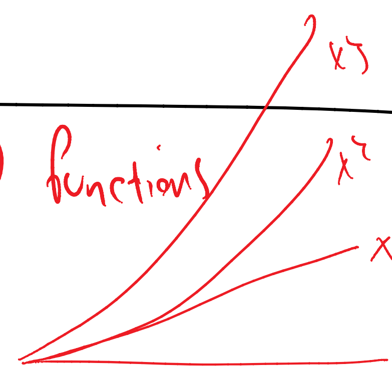


$$e = Ch^{\alpha}$$

$$\underbrace{\log(e)}_{Y} = \log C + \underbrace{\alpha}_{\text{slope}} \underbrace{\log h}_{X}$$

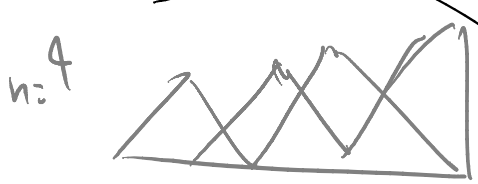
Why monomial ~~mial~~ functions

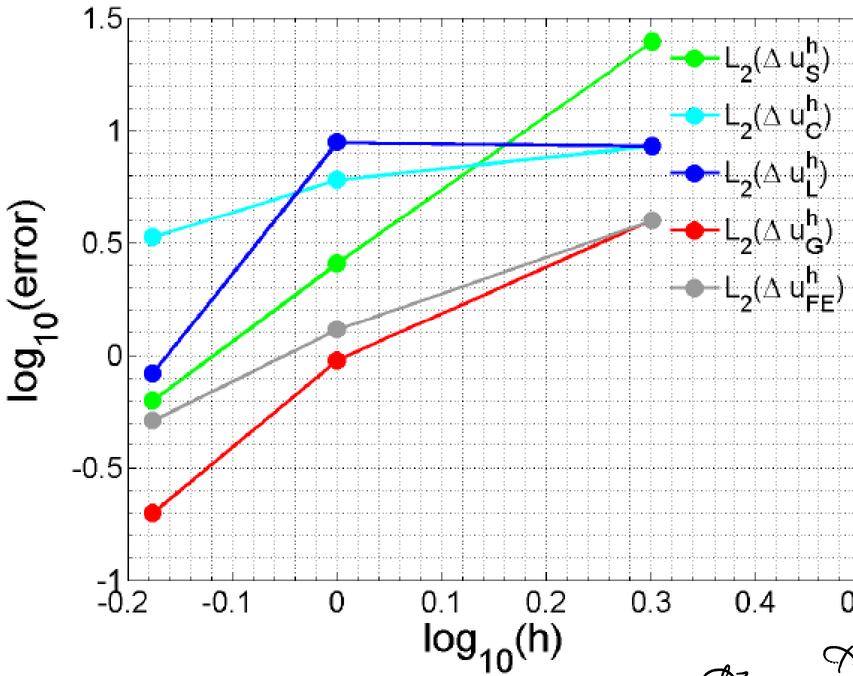
$\{x, x^2, x^3, \ldots\}$

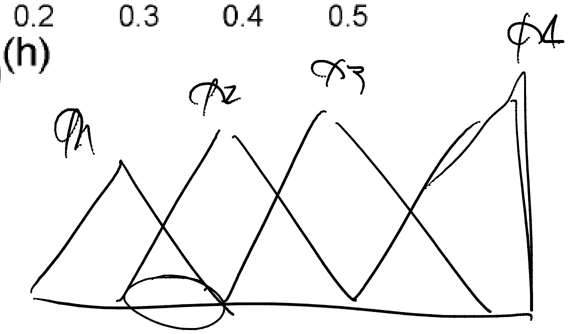


perform better than finite element?

continued improvement to "convergence rate"?

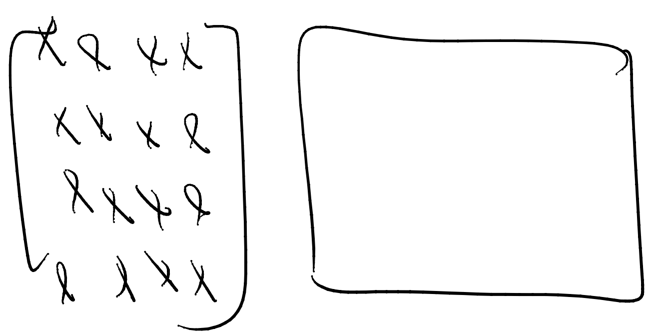






ϕ_1 interacts only ϕ_2

Getting a diagonal matrix for spectral methods?



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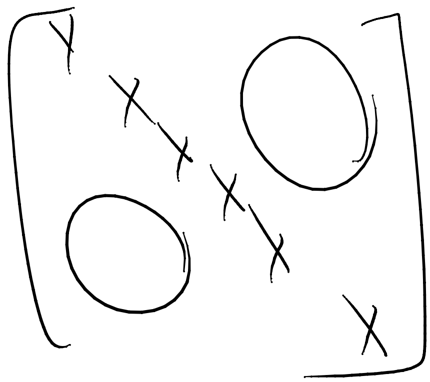
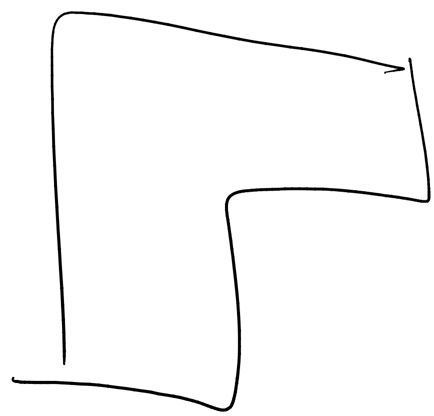
orthogonal Trial functions



very easy to solve

If the computational domain is complicated

It's very difficult or practically impossible to form orthonormal basis!



What is the inner product corresponding to our problem?

example solid bar

$$\mathcal{A}(u,w) = \int_0^L \frac{du}{dx} AE \frac{dw}{dx}\, dx$$

inner product $\qquad K_{ij} = \mathcal{A}(\phi_i, \phi_j)$