02/26/2014

Wednesday, February 26, 2014

8:42 AM

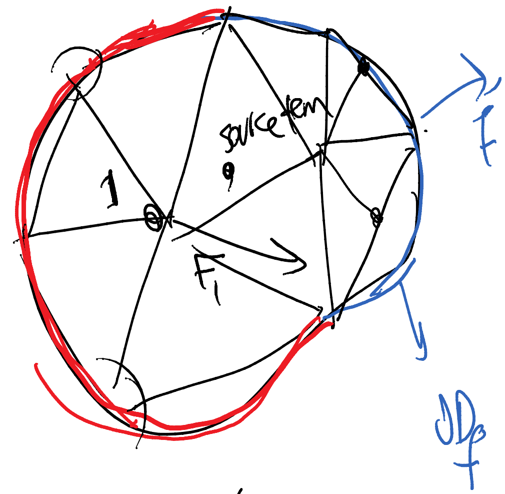
similar $\approx EA$

stiffness $K = \int_D B^T D B \, dv$

1 unknown per node

∂D_u

$u = \bar{u}$



Neumann natural BC $\quad F_N = \int_{\partial D_f} N^T F \, ds$

Dirichlet essential BC $\quad F_D = \left(\int_D B^T D \bar{B} \, dv \right) \bar{q}$

$n_p = 5$ prescribed d.o.f

$n_f = 6$

number of unknowns = free d.o.f

source term

$F_r = \int_D N^T r \, dv$

$$F_n = \begin{bmatrix} F_{n_1} \\ \vdots \\ F_{n_{n_f}} \end{bmatrix} \quad \# \text{ free d.o.f.}$$

elements n_e

$$K = \int_D B^T D B \, dv = \sum_{e=1}^{n_e} \int_{e_i} B^T D B \, dv$$

K^{e_i}

$$F_N = \int_{\partial D_f} N^T \bar{F} \, ds = \sum_{i=1}^{n_e} \int N^T \bar{F} \, ds$$

$\partial D^{e_i} = (\partial D_f \cap \partial e_i)$

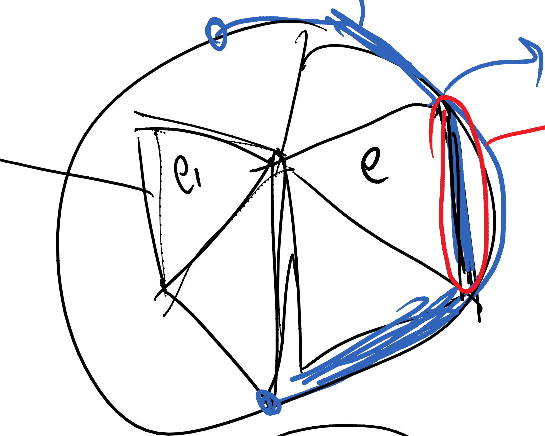
∂D_f

$\partial D_f^{e_i} = \partial D_f \cap \partial e_i$

natural BC of element e_i — boundary of the element

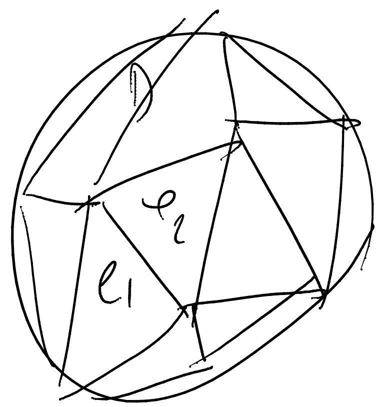
in discret sense ← ∂D_f

what is $\partial D_f^{e_1} = \phi$



∂D_f

$\partial D_f^{e_2} =$



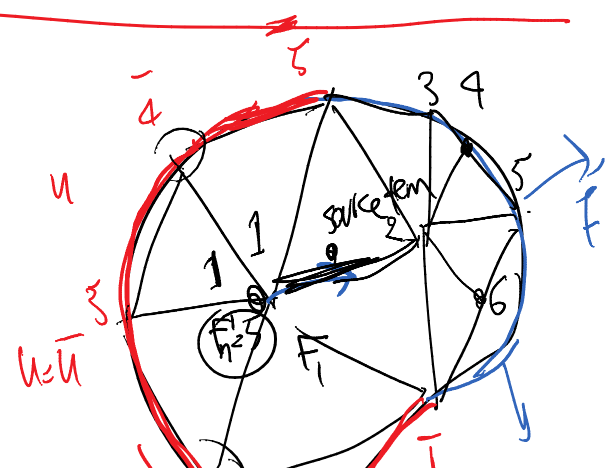
$$F_r = \int_D N^T r \, dV = \sum_{i=1}^{ne} \int_{e_i} N^T r \, dV \quad F_r^{e_i}$$

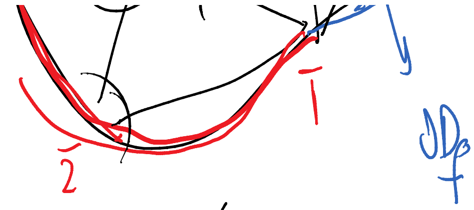
$$F_D = \left(\int_D B^T D \bar{B} \, dV\right) \bar{a} = \sum_{i=1}^{ne} \left\{ \int_{e_i} B^T D \bar{B} \, dV \, \bar{a}_{e_i} \right\} f_D^{e_i}$$

$$K a = F$$

$K_{6\times 6}$

$[a_1 \quad]$





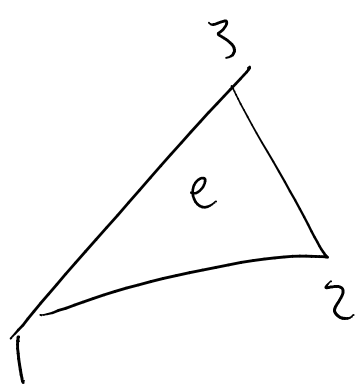
$$Q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} \quad 6\times 1$$

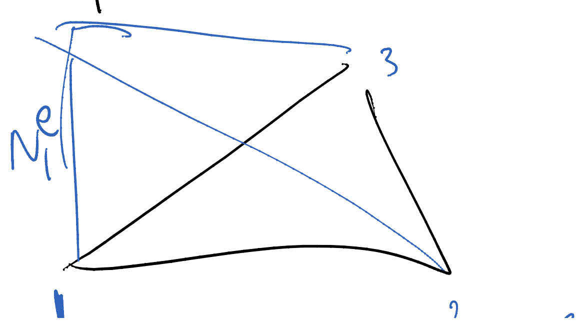
$n_f = 6$

$$F = \underbrace{\underset{\text{nodal force}}{F_n} + \underset{\text{natural BC}}{F_N} + \underset{\text{sourse term}}{f_r} - \underset{\text{essential Dirichlet BC}}{F_D}}_{\text{all are } 6\times 1}$$

$$F_n = \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

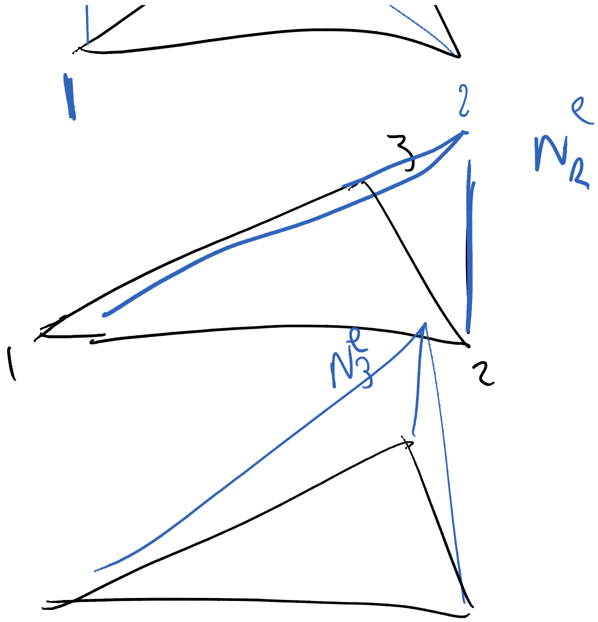
element level

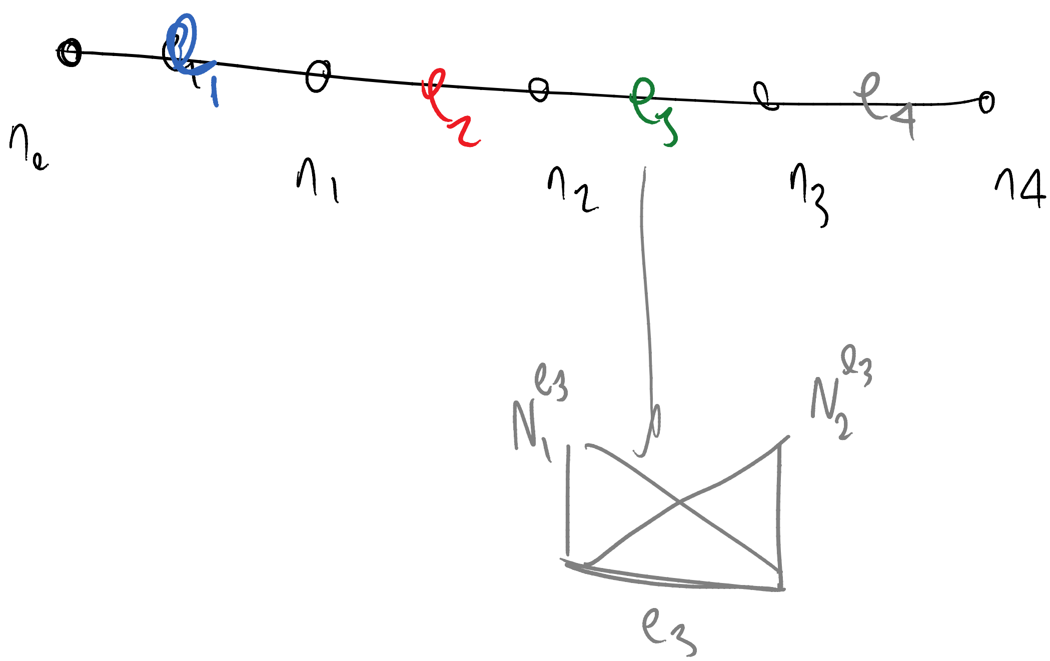




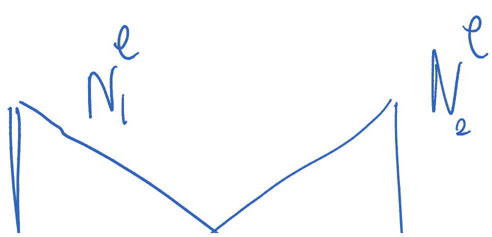
$$K^e = \int B^{e^T} D B^e \, dV$$

$$K^e = \int_e B^{e^T} D B^e \, dV$$





how to calculate element level stiffness matrix?



$N_1^e(\xi) = a\xi + b$

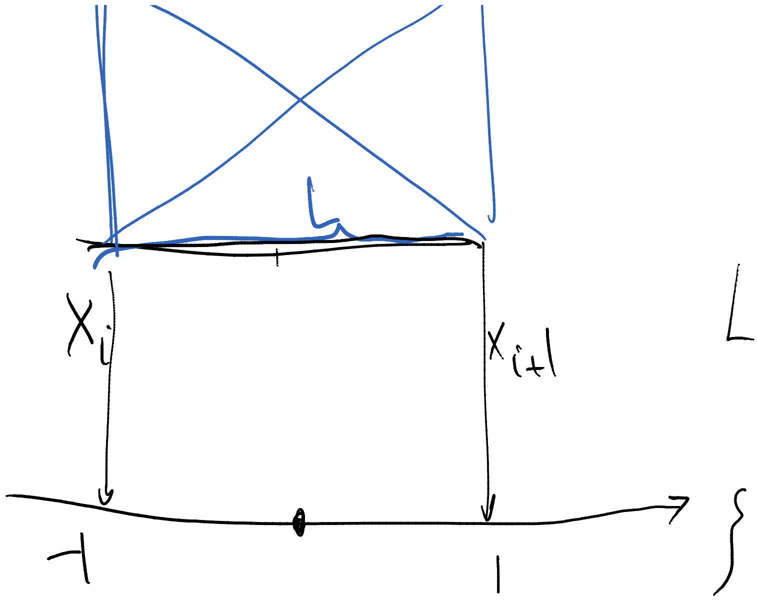
$$N_1^e(\xi) = a\xi + b$$

$$N_1^e(-1) = 1$$

$$N_1^e(1) = 0$$

$$-a + b = 1$$

$$a + b = 0$$



$$L = x_{i+1} - x_i$$

$$b = \frac{1}{2} \qquad a = \frac{-1}{2}$$

$$\boxed{\begin{aligned} N_1^e(\xi) &= \frac{-1}{2}\xi + \frac{1}{2} \\ N_2^e(\xi) &= \frac{1}{2}\xi + \frac{1}{2} \end{aligned}}$$

$$k^e = \int_{e} B^{eT} E A B^e \, dx$$

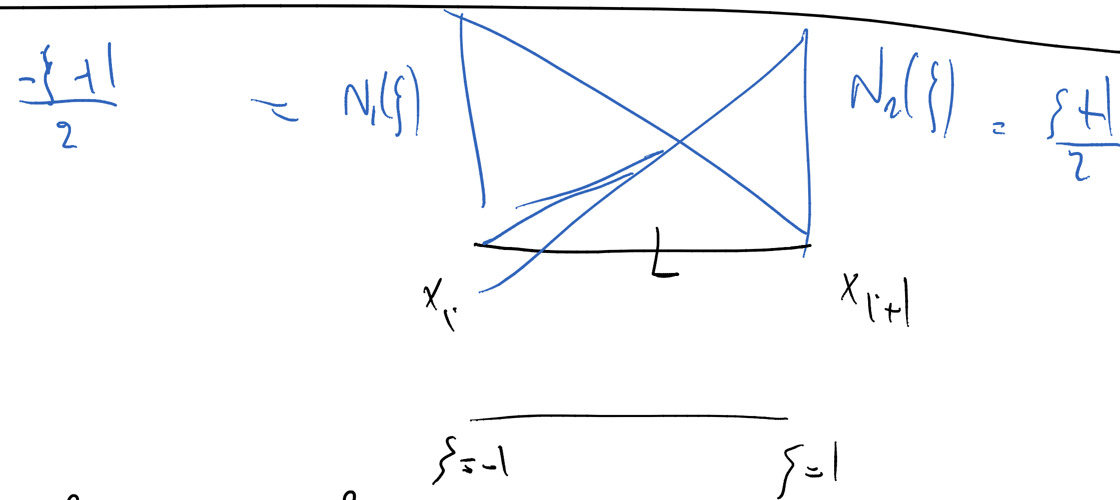
1. Integral is done over the element
2. $B = \left[B_1^e \quad B_2^e \right] = \left[\frac{dN_1^e}{dx} \quad \frac{dN_2^e}{dx} \right]$

all derivatives are still w.r.t. global system coordinate

3. $$\int_e B^{e^T} EA B^e \, d\cancel{x}$$

ξ

again have to use global differential



$$K_e = \int_e \begin{bmatrix} \frac{dN_1^e}{dx} \\ \frac{dN_2^e}{dx} \end{bmatrix} AE \begin{bmatrix} \frac{dN_1^e}{dx} & \frac{dN_2^e}{dx} \end{bmatrix} dx$$

another way more difficult

$\xi = -1 \longrightarrow x_i$

$\xi = 1 \longrightarrow x_{i+1}$

$x = A\xi + B$

$A(-1) + B = x_i$

$A(1) + B = x_{i+1}$

$$x = x_i N_1^e(\xi) + x_{i+1} N_2^e(\xi)$$

if $\xi = -1$ $N_1^e(-1) = 1$ $N_2^e(-1) = 0$

$$x = x_i \cdot 1 + x_{i+1} \cdot 0$$

$$x = x_i N_1^e(\xi) + x_{i+1} N_2^e(\xi) = x_i \left(\frac{-\xi+1}{2}\right) + x_{i+1}\left(\frac{\xi+1}{2}\right)$$

$$x = \left(\frac{\overbrace{x_{i+1} - x_i}^{L}}{2}\right)\xi + \left(\frac{x_i + x_{i+1}}{2}\right)$$

$$\boxed{x = \frac{L}{2}\xi + x_m}$$

$$\boxed{dx = \underbrace{\frac{L}{2}}_{J} d\xi \quad \text{Jacobian}} \qquad \boxed{\frac{dx}{d\xi} = \frac{L}{2}}$$

$$\frac{dN_1^e}{dx} = \frac{dN_1^e}{d\xi}\frac{d\xi}{dx} = \frac{\frac{dN_1^e}{d\xi}}{\frac{dx}{d\xi}} = \frac{\frac{d\left(\frac{-\xi+1}{2}\right)}{d\xi}}{\frac{L}{2}} = \frac{-1}{L}$$

$$\frac{dN_2^e}{dx} = \frac{1}{L}$$

$$K^e = \int \begin{bmatrix} \frac{dN_1^e}{dx} \\ \frac{dN_2^e}{dx} \end{bmatrix} EA \begin{bmatrix} \frac{dN_1^e}{dx} & \frac{dN_2^e}{dx} \end{bmatrix} dx$$

$$= \int_{-1}^{1} \begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} EA \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \left(\frac{L}{2} d\xi \right) \qquad J d\xi$$

$$= \int_{-1}^{1} \frac{AE}{2L} \begin{bmatrix} +1 & -1 \\ -1 & 1 \end{bmatrix} d\xi$$

$$= 2 \frac{AE}{2L} \begin{bmatrix} +1 & -1 \\ -1 & 1 \end{bmatrix} \rightarrow \boxed{k_e = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$