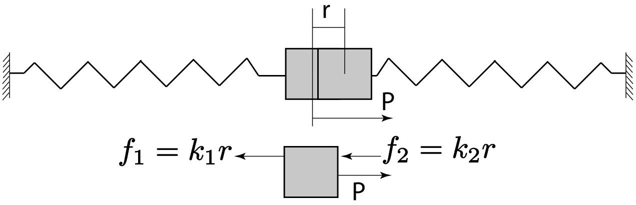
Balance laws

Motivation

- Why start with a balance law?
  - They are the actual physics laws.
  - They contain more *information* than their corresponding PDEs.
  - Larger solution space than the PDEs.
- Can we directly start the FE formulation from a PDE?
  - Yes, FE formulation starts from a differential equation.
  - A PDE may not be derived from a balance law.

Balance law for discrete systems (Statics)

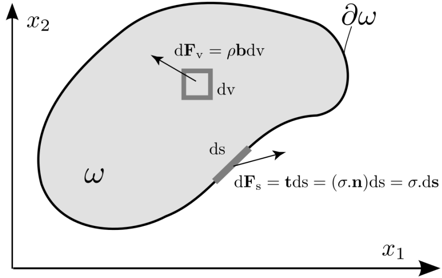


Newton's law:

$$F = \Sigma f_i = -f_1 - f_2 + P = 0 \quad \Rightarrow \quad r = \frac{P}{k_1 + k_2} \tag{1}$$

The unknowns of discrete point(s) can be obtained from the solution of one or a system of equations.

Balance law for continuum(Statics for solids)



$$\mathbf{F} = 0 \quad \Rightarrow \tag{2}$$

$$\mathbf{F} = \mathbf{F}_s + \mathbf{F}_v = \int_{\partial\omega} \mathrm{d}\mathbf{F}_{\mathrm{s}} + \int_{\omega} \mathrm{d}\mathbf{F}_{\mathrm{v}} = 0, \quad \text{where}$$

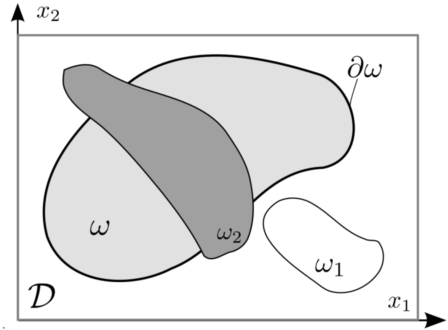
$\mathbf{F}_s =$ Sum of forces from tractions $\mathbf{t}$ (boundary flux) on the boundary of ω

$\mathbf{F}_v =$ Sum of forces from body force $\mathbf{b}$ (source term) inside ω

$\mathrm{d}\mathbf{F}_{\mathrm{s}} = \mathbf{t}\mathrm{ds} = (\sigma.\mathbf{n})\mathrm{ds} = \sigma.\mathbf{ds}$

$\mathrm{d}\mathbf{F}_{\mathrm{v}} = \rho\mathbf{b}\mathrm{dv}$ $\qquad$ $\partial\omega =$ boundary of ω

Balance law for continuum(Statics for solids)



The balance law is valid for arbitrary domains ω:

$$\forall \omega \subset \mathcal{D} : \int_{\partial\omega} \mathbf{t} \, \mathrm{ds} + \int_{\omega} \rho \mathbf{b} \, \mathrm{dv} = 0 \tag{3}$$

$$\text{that is} \quad \int_{\partial\omega} \sigma . \mathrm{d}\mathbf{s} + \int_{\omega} \rho \mathbf{b} \, \mathrm{dv} = 0$$

Note that $\mathrm{d}\mathbf{s} = \mathbf{n}\mathrm{ds}$ (boldface is used for tensor quantities)

Balance law for discrete systems (Dynamics)

$$\left.\begin{aligned} \mathbf{F} &= \tfrac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} \\ \mathbf{P} &= m\mathbf{v} = \text{linear momentum} \end{aligned}\right\} \quad \Rightarrow \quad \mathbf{F} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = ma \tag{4}$$

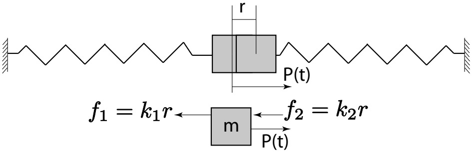
Another representation is using the "Impulse" J for the force $\mathbf{F}$ over the time interval Δt:

$$J = \int_{\Delta t} \mathbf{F}\mathrm{d}t \tag{5}$$

then Newton's second law reads as

$$J = \Delta\mathbf{P} \tag{6}$$

Advantage: $\mathbf{P}$ does not need to be differentiable

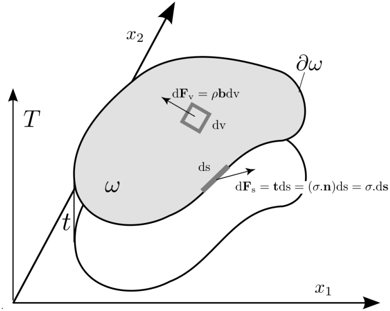


$$\mathbf{F} = P(t) - f_1 - f_2 = m\mathbf{a} \quad \Rightarrow \tag{7}$$

$$m\ddot{r} + (k_1 + k_2)r = P(t) \tag{8}$$

can solve the DE with FEM

Balance law for continuum(Dynamics for solids)



First approach (similar to $\mathbf{F} = \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t}$ for discrete systems):

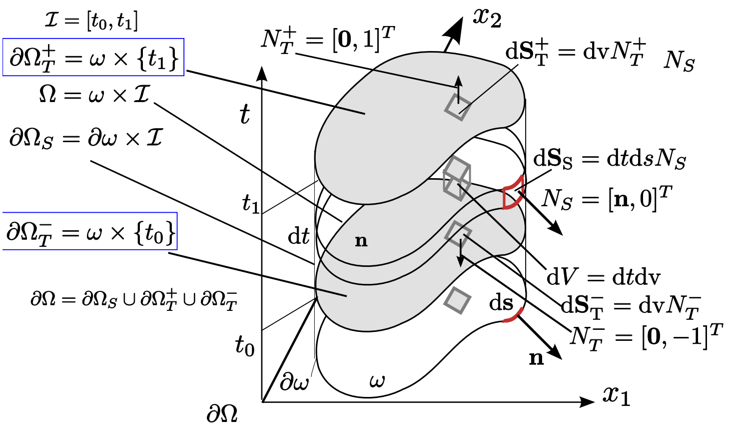
$$\forall\omega \subset \mathcal{D} : \mathbf{F} = \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} \quad \Rightarrow$$

$$\int_{\partial\omega} \sigma.\mathrm{d}\mathbf{s} + \int_{\omega} \rho\mathbf{b}\,\mathrm{dv} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\omega} \mathbf{p}\,\mathrm{dv} \tag{9}$$

note that $\mathbf{P} = \int_\omega \mathbf{p}\,\mathrm{dv}$ where $\mathbf{p}$ is the *linear momentum density* defined by:

$$\mathbf{p} = \frac{\mathrm{d}\mathbf{P}}{\mathrm{dV}} = \frac{(\mathrm{d}m)\mathbf{v}}{\mathrm{d}V} = \frac{(\rho\mathrm{d}V)\mathbf{v}}{\mathrm{d}V} = \rho\mathbf{v} \tag{10}$$

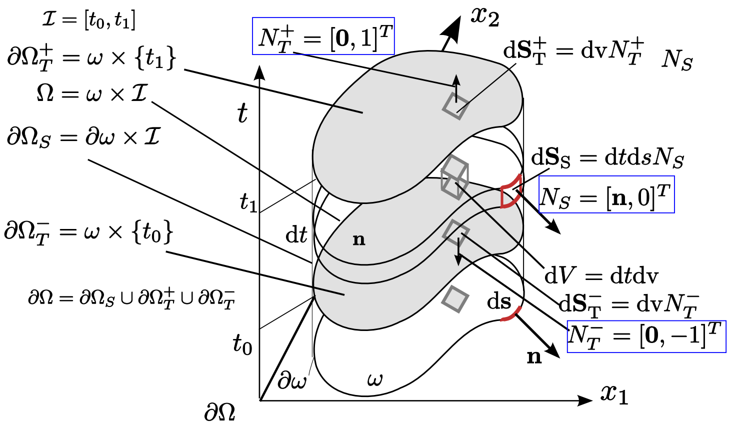
Balance law for continuum(Dynamics for solids)



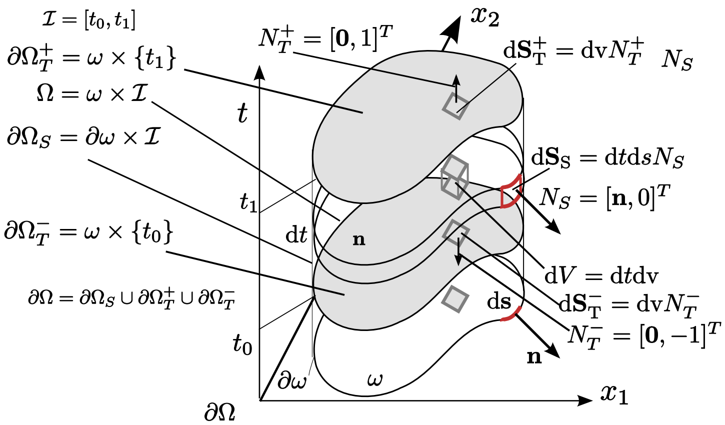
Second approach (similar to $J = \int_{\Delta t} \mathbf{F}\,\mathrm{d}t = \Delta\mathbf{P}$ for discrete systems):

$$\forall \Omega = \omega \times \mathcal{I} \subset \mathcal{D} : J = \int_{t_0}^{t_1} \mathbf{F} = \mathbf{P}(t_1) - \mathbf{P}(t_0) \quad \Rightarrow$$

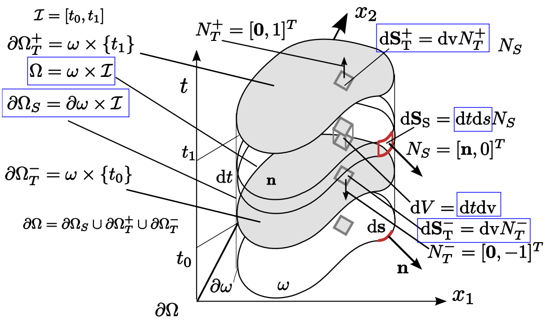
$$\int_{t_0}^{t_1} \left(\int_{\partial\omega} \sigma.\mathrm{d}\mathbf{s} \right) \mathrm{d}t + \int_{t_0}^{t_1} \left(\int_{\omega} \rho\mathbf{b}\,\mathrm{dv} \right) \mathrm{d}t = \int_{\partial\Omega_T^+} \mathbf{p}\,\mathrm{dv} - \int_{\partial\Omega_T^-} \mathbf{p}\,\mathrm{dv}$$



$$\int_{\partial\Omega_S} \sigma.\,(\mathrm{d}\mathbf{s}\mathrm{d}t) + \int_{\Omega} \rho\mathbf{b}\ (\mathrm{dv}\mathrm{d}t) = \int_{\partial\Omega_T^+} \mathbf{p}\ \mathrm{dS}_T^+ - \int_{\partial\Omega_T^-} \mathbf{p}\ \mathrm{dS}_T^-$$



$$\int_{\partial\Omega_S} (\sigma.\mathbf{n}) \underbrace{\mathrm{dS}_S}_{\mathrm{d}s\mathrm{d}t} + \int_{\Omega} \rho\mathbf{b} \underbrace{\mathrm{dV}}_{\mathrm{dv}\mathrm{d}t} = \int_{\partial\Omega_T^+} \mathbf{p}\, \mathrm{dS}_T^+ + \int_{\partial\Omega_T^-} \mathbf{p}\, (-\mathrm{dS}_T^-)$$



$$\int_{\partial\Omega_S} \underbrace{-\left[\ -\sigma \mid \mathbf{p}\ \right] . \overbrace{\left[\begin{array}{c}\mathbf{n}\\ \hline 0\end{array}\right]}^{\mathbf{N}_S}}_{\sigma.\mathbf{n}} \mathrm{dS}_S + \int_\Omega \rho\mathbf{b}\,\mathrm{dV} =$$

$$\int_{\partial\Omega_T^+} \underbrace{\left[\ -\sigma \mid \mathbf{p}\ \right] . \overbrace{\left[\begin{array}{c}\mathbf{0}\\ \hline 1\end{array}\right]}^{\mathbf{N}_T^+}}_{\mathbf{p}} \mathrm{dS}_T^+ + \int_{\partial\Omega_T^-} \underbrace{\left[\ -\sigma \mid \mathbf{p}\ \right] . \overbrace{\left[\begin{array}{c}\mathbf{0}\\ \hline -1\end{array}\right]}^{\mathbf{N}_T^-}}_{-\mathbf{p}} \mathrm{dS}_T^-$$

second approach continued ...

Let us define the spacetime flux $\mathbf{M}$ by:

$$\mathbf{M} = \left[\; -\sigma \; \middle| \; \mathbf{p} \; \right] \tag{11a}$$

$$\mathbf{p} = \text{linear momentum} = \text{temporal flux} = \text{conserved quantity} \tag{11b}$$

$$-\sigma = -\text{stress} = \text{total } \textit{outward} \text{ spatial flux} \tag{11c}$$

then

$$\left. \begin{aligned} \int_{\partial\Omega_T^+} \mathbf{M}.\mathbf{N}_T^+ \,\mathrm{dS}_T^+ + \int_{\partial\Omega_T^-} \mathbf{M}.\mathbf{N}_T^- \,\mathrm{dS}_T^- & \\ + \int_{\partial\Omega_S} \mathbf{M}.\mathbf{N}_S \,\mathrm{dS}_S &= \int_\Omega \rho \mathbf{b} \,\mathrm{dV} \\ \partial\Omega_T^+ \cup \partial\Omega_T^- \cup \partial\Omega_S &= \partial\Omega \quad \text{disjoint union} \end{aligned} \right\} \Rightarrow$$

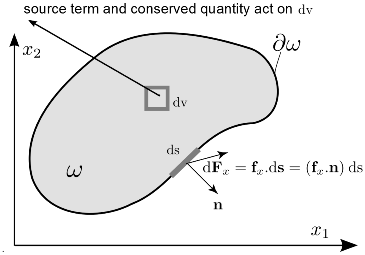
$$\boxed{\int_{\partial\Omega} \mathbf{M}.\mathrm{d}\mathbf{S} = \int_{\partial\Omega} \mathbf{M}.\mathbf{N} \,\mathrm{dS} = \int_\Omega \rho \mathbf{b} \,\mathrm{dV}} \tag{12}$$

General form of balance laws

For a general conservation law let:

- $\mathbf{f}_t$: conserved quantity = temporal flux
- $\mathbf{f}_x$: total outward spatial flux
- $\mathbf{r}$: source term

then the balance law for dynamics reads:



$$\boxed{\forall \omega \subset \mathcal{D} \wedge \forall t : \int_\omega \mathbf{r}\,\mathrm{dv} - \int_{\partial\omega} \mathbf{f}_x.\mathrm{ds} = \int_\omega \mathbf{r}\,\mathrm{dv} - \int_{\partial\omega} (\mathbf{f}_x.\mathbf{n})\,\mathrm{ds} = \frac{\mathrm{d}}{\mathrm{d}t}\int_\omega \mathbf{f}_t\,\mathrm{dv}} \tag{13}$$

For static case the RHS is zero (*i.e.*, the quantity $\int_\omega \mathbf{f}_t\,\mathrm{dv}$ remains constant). The static balance law reads:

$$\boxed{\forall \omega \subset \mathcal{D} : \int_\omega \mathbf{r}\,\mathrm{dv} - \int_{\partial\omega} \mathbf{f}_x.\mathrm{ds} = \int_\omega \mathbf{r}\,\mathrm{dv} - \int_{\partial\omega} (\mathbf{f}_x.\mathbf{n})\,\mathrm{ds} = \mathbf{0}} \tag{14}$$

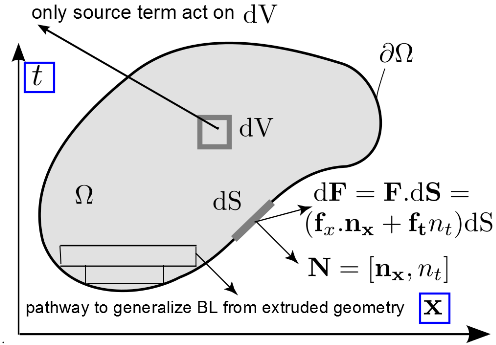
These can be directly compared to $\mathbf{F} = \mathrm{d}\mathbf{P}/\mathrm{d}t$ and $\mathbf{F} = 0$ in previous discrete examples.

General form of balance laws using spacetime flux

Using the same definitions from previous page we define the spacetime flux by

$$\mathbf{F} = [\mathbf{f}_x | \mathbf{f}_t] \tag{15}$$

then the balance law for dynamics reads:



$$\boxed{\forall \Omega \subset \mathcal{D} : \int_{\partial\Omega} \mathbf{F}.\mathrm{dS} - \int_{\Omega} \mathbf{r}\, \mathrm{dV} = \int_{\partial\Omega} (\mathbf{f}_x.\mathbf{n_x} + \mathbf{f_t} n_t)\mathrm{dS} - \int_{\Omega} \mathbf{r}\, \mathrm{dV} = \mathbf{0}} \tag{16}$$

This can be directly compared to $\int_{\Delta t} \mathbf{F} = \Delta \mathbf{P}$ in previous discrete examples.