Discretization

Discretization of the solution

- Discretization means reducing the infinite number of unknowns of the continuum problem to a discrete set of unknowns. Often, the discrete solution does not exactly match the exact solution. The accuracy of the approximation typically improves by increasing the number of unknowns.
- We denote the number of unknowns by n.
- Some common forms of discretization are:
  1. Values at a finite number of positions represent the solution:

     $$\text{Solution } u \text{ is represented by: } \{u(x_1), u(x_2), \ldots, u(x_n)\}.$$

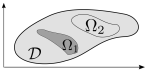
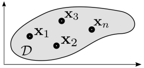
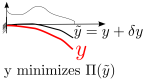
     This approach is used by Finite Difference (FD) and Finite Volume(FV) methods.
  2. Solution is represented by a finite number of functions:

     $$u^h(x) = \phi_p(x) + \sum_{i=1}^{n} a_i \phi_i(x)$$

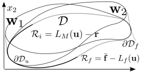
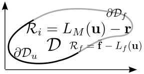
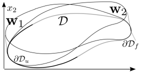
     where u^h is the symbol for discrete solution and $\phi_i(x)$ are trial or test functions. $\phi_p(x)$ is set to satisfy essential boundary conditions and will be discussed later. This approach is used by (discrete) weighted residual method, weak form, least square, and Ritz energy method.

Equations for discrete systems

- After discretizing the solution with n unknowns we need n equations to solve the discrete problem.
- The n equations are derived from different interpretation of the equations we derived so far. All these equations have a "for all" condition. In discrete systems the "for all" condition is replaced by a finite set.

Approach	Equation	Figure	Discretization	Discretization method
Balance Law (20)	$\forall \Omega \subset \mathcal{D} : \int_{\partial\Omega} (\mathbf{f}.\mathbf{n})\mathrm{ds} - \int_{\Omega} \mathbf{r}\ \mathrm{dv} = \mathbf{0}$		Change $\forall\Omega$ to $\{\Omega_1, \Omega_2, \ldots, \Omega_n\}$	Similar to subdomain method in WRM
Strong Form (23)	$\forall \mathbf{x} \in \mathcal{D} : \nabla.\mathbf{f} - \mathbf{r} = \mathbf{0}$		Change $\forall\mathbf{x}$ to $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$	Collocation method in WRM. Also FD & FV.
Energy Method (80)	$\forall \tilde{y} \in \mathcal{V} : \quad \Pi(y) \leq \Pi(\tilde{y})$		$\forall\{\tilde{a}_1, \ldots, \tilde{a}_n\}$: $\Pi(a_1, \ldots, a_n) \leq \Pi(\tilde{a}_1, \ldots, \tilde{a}_n) \Rightarrow \frac{\partial \Pi}{\partial a_1} = \cdots = \frac{\partial \Pi}{\partial a_n} = 0$	Ritz Energy Method. Also yields Weak Form.

Equations for discrete systems

Approach	Equation	Figure	Discretization	Discretization method
Weighted Resid-ual Method (45)	$\forall \mathbf{w} \in \mathcal{W}$: $\int_{\mathcal{D}} \mathbf{w}.\mathcal{R}_i \text{ dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}^f.\mathcal{R}_f \text{ ds} = \mathbf{0}$		Change $\forall \mathbf{w}$ to $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$	Weighted Residual Method (WRM)
Least Square (51)	$R^2 = \int_{\mathcal{D}} \mathcal{R}_i^2 \text{ dv} + \int_{\partial\mathcal{D}_f} \mathcal{R}_f^2 \text{ ds} = \mathbf{0}$		Change $R^2 = 0$ to $\forall\{\tilde{a}_1, \ldots, \tilde{a}_n\}$: $R^2(a_1, \ldots, a_n) \leq R^2(\tilde{a}_1, \ldots, \tilde{a}_n) \Rightarrow \frac{\partial R^2}{\partial a_1} = \cdots = \frac{\partial R^2}{\partial a_n} = 0$	Least Square method, a WRM for linear L_M (& L_f).
Weak Form (74)	$\forall \mathbf{w} \in \mathcal{W}$ $\int_{\mathcal{D}} L_m^w(\mathbf{w}) L_m(\mathbf{u}) \text{ dv} = \int_{\mathcal{D}} \mathbf{w}.\mathbf{r}\text{dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}.\bar{\mathbf{f}}\text{ds}$		Change $\forall \mathbf{w}$ to $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$	Weak For-mulation

Discrete systems

- In all approaches described, a continuum problem is discretized to a system of finite (n) unknowns. We observed how in each approach, by enforcing n discrete conditions corresponding to the continuum version, we could derive a system of n equations for n unknowns.
- The differential operators L_M (L_m for weak form) and L_f can all be nonlinear. In that case we obtain an n by n system of nonlinear equations.
- In all the approached discussed except Finite Difference and Finite Volume for the discrete version of the strong form, the solution is approximated by the summation of finite number of trial functions.

Trial functions

- As mentioned before, it is almost always beneficial (or necessary, *e.g.*, for energy method) to strongly satisfy the essential boundary conditions. That is the solution $\mathbf{u}$.

$$\mathbf{u} \in \mathcal{V} = \{\mathbf{v} \mid \mathbf{v} \in C^M(\mathcal{D}), \forall \mathbf{x} \in \partial\mathcal{D}_u \; L_u(\mathbf{u}) = \bar{\mathbf{u}}\} \tag{116}$$

We observe that for all the approached in the previous table, the solution $\mathbf{u}$ is approximated with trial functions $\boldsymbol{\phi}_i$ ($\boldsymbol{\phi}_i$ are assumed to be linearly independent):

$$\boxed{\mathbf{u} \approx \mathbf{u}^h = \sum_{j=1}^{n} a_j \boldsymbol{\phi}_j + \boldsymbol{\phi}_p} \quad \text{where} \tag{117a}$$

$$\mathbf{u} = \text{continuum (exact) solution} \tag{117b}$$

$$\mathbf{u}^h = \text{discrete (approximate) solution} \tag{117c}$$

$$\boldsymbol{\phi}_j = \text{trial (test) functions} \tag{117d}$$

$$a_j = \text{Unknown coefficients (unknowns of the discrete problem)} \tag{117e}$$

$$\boldsymbol{\phi}_p = \text{A solution satisfying essential boundary conditions} \tag{117f}$$

Trial functions

- We still want to enforce the essential boundary conditions for the discrete solution $\mathbf{u}$. We define,

$$\mathcal{V}_0 = \{\mathbf{v} \,|\mathbf{v} \in C^M(\mathcal{D}), \forall \mathbf{x} \in \partial\mathcal{D}_u \;\; L_u(\mathbf{u}) = 0\} \tag{118}$$

  Thus we observe,

$$\left.\begin{array}{r} \forall i \;\; \boldsymbol{\phi}_i \in \mathcal{V}_0 \\ \boldsymbol{\phi}_p \in \mathcal{V} \end{array}\right\} \Rightarrow \forall \mathbf{x} \in \partial\mathcal{D}_f \;\; \mathbf{u}^h(\mathbf{x}) = \sum_{j=1}^{n} a_j \boldsymbol{\phi}_j(\mathbf{x}) + \boldsymbol{\phi}_p(\mathbf{x}) = \mathbf{0} + \bar{\mathbf{u}} = \bar{\mathbf{u}} \tag{119}$$

  That is, the discrete solution **strongly** satisfies the essential boundary conditions.

- Function requirements for $\boldsymbol{\phi}_j$ and $\boldsymbol{\phi}_p$:

Function space	Continuity Requirement	Boundary Conditions
$\boldsymbol{\phi}_i \in \mathcal{V}_0$	$\boldsymbol{\phi}_i \in C^M(\mathcal{D})$	Trial functions satisfy homogeneous essential boundary conditions
$\boldsymbol{\phi}_p \in \mathcal{V}$	$\boldsymbol{\phi}_p \in C^M(\mathcal{D})$	A particular function satisfies the essential boundary conditions

- Note that if all the essential boundary conditions are zero, there is no need for the function $\boldsymbol{\phi}_p$ (*i.e.*, $\boldsymbol{\phi}_p$ can be chosen to be identically zero).

Discrete function spaces **(Optional)**

- The continuum function space $\mathcal{V}_0$ is infinite dimensional.
- We seek a discrete solution $\mathbf{u}^h$ of the form (117a):

$$\mathbf{u} \approx \mathbf{u}^h = \sum_{j=1}^{n} a_j \boldsymbol{\phi}_j + \boldsymbol{\phi}_p$$

  where as mentioned before all $\boldsymbol{\phi}_j \in \mathcal{V}_0$.
- The function space $\mathcal{V}_0^h$ is defined as:

$$\mathcal{V}_0^h = \{\sum_{j=1}^{n} \alpha_j \boldsymbol{\phi}_j | \; \forall \alpha_j \in \mathbb{R}\} \tag{120}$$

  This is nothing but the span of the functions $\boldsymbol{\phi}_j$:

$$\mathrm{span}(\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_n) = [\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_n] = \{\sum_{j=1}^{n} \alpha_j \boldsymbol{\phi}_j | \; \forall \alpha_j \in \mathbb{R}\} \tag{121}$$

- $\mathcal{V}_0$ and $\mathcal{V}_0^h$ are both vector spaces with $\mathcal{V}_0^h$ being a subspace of $\mathcal{V}_0$.
- Eventually, we define the discrete function space $\mathcal{V}^h$ as,

$$\mathcal{V}^h = \boldsymbol{\phi}_p + \mathcal{V}_0^h := \{\mathbf{v} = \boldsymbol{\phi}_p + \mathbf{v}_0 | \; \mathbf{v}_0 \in \mathcal{V}_0^h\} = \{\mathbf{v} = \boldsymbol{\phi}_p + \sum_{j=1}^{n} \alpha_j \boldsymbol{\phi}_j | \; \forall \alpha_j \in \mathbb{R}\} \tag{122}$$

  where $\boldsymbol{\phi}_p$ can be any arbitrary function in $\mathcal{V}$.
- The discrete solution space $\mathcal{V}^h$ is n-dimensional ($\boldsymbol{\phi}_i$ must be linearly independent).

Discrete function spaces (Optional)

- Similar to the discrete solution space, we need to formalize the discrete weight function space for the methods that utilize a weight function (weighted residual and weak statement).
- For these methods, at continuum we require a statement to hold for all weight functions in $\mathcal{W}$.
- In discrete setting, as the solution space is n dimensional, we need to form an n-dimensional *subspace* of $\mathcal{W}$.
- Thus, we choose n linearly independent weight functions $\mathbf{w}_i \in \mathcal{W}$ to define a discrete weight function space $\mathcal{W}^h$:

$$\mathcal{W}^h = \mathrm{span}(\mathbf{w}_1, \dots, \mathbf{w}_n) = [\mathbf{w}_1, \dots, \mathbf{w}_n] = \{\sum_{j=1}^{n} \alpha_j \mathbf{w}_j | \ \forall \alpha_j \in \mathbb{R}\} \tag{123}$$

- Weight functions always appear as linear differential operators in weighted residual and weak statements. Thus, we only need to enforce the discrete statement for n weight functions in $\mathcal{W}^h$ and the weak integral statement automatically holds for all $\mathbf{w} \in \mathcal{W}^h$. For example, consider the sample weighted residual statement below:

$$\int_{\mathcal{D}} \mathbf{w}.\mathcal{R}_i \,\mathrm{dv} = 0 \quad \text{then} \quad \left.\begin{array}{l} \int_{\mathcal{D}} \mathbf{w}_1.\mathcal{R}_i \,\mathrm{dv} = \mathbf{0} \\ \int_{\mathcal{D}} \mathbf{w}_2.\mathcal{R}_i \,\mathrm{dv} = \mathbf{0} \end{array}\right\} \quad \Rightarrow$$

$$\forall \alpha_1, \alpha_2 \in \mathbb{R} \ : \int_{\mathcal{D}} (\alpha_1 \mathbf{w}_1 + \alpha_1 \mathbf{w}_2).\mathcal{R}_i \,\mathrm{dv} = \alpha_1 \int_{\mathcal{D}} \mathbf{w}_1.\mathcal{R}_i \,\mathrm{dv} + \alpha_2 \int_{\mathcal{D}} \mathbf{w}_2.\mathcal{R}_i \,\mathrm{dv} = \mathbf{0}$$

Summary: Discrete function spaces

- To find a discrete solution of the form $\mathbf{u} \approx \mathbf{u}^h = \sum_{j=1}^{n} a_j \boldsymbol{\phi}_j + \boldsymbol{\phi}_p$, based on the space $\mathcal{V} = \{\mathbf{v} \; | \mathbf{v} \in C^M(\mathcal{D}, \forall \mathbf{x} \in \partial \mathcal{D}_f \;\; L_u(\mathbf{u}) = \bar{\mathbf{u}}\}$ (116) we define the vector space (118):

$$\mathcal{V}_0 = \{\mathbf{v} \; | \mathbf{v} \in C^M(\mathcal{D}, \forall \mathbf{x} \in \partial \mathcal{D}_u \;\; L_u(\mathbf{u}) = 0\}$$

  and choose n linearly independent trial functions $\boldsymbol{\phi}_j \in \mathcal{V}_0$ and an arbitrary $\boldsymbol{\phi}_p \in \mathcal{V}$.
- The the discrete solution space is an n-dimensional function space given by (122):

$$\mathcal{V}^h = \boldsymbol{\phi}_p + \mathcal{V}_0^h := \boldsymbol{\phi}_p + \{\sum_{j=1}^{n} \alpha_j \boldsymbol{\phi}_j | \; \forall \alpha_j \in \mathbb{R}\} = \{\mathbf{v} = \boldsymbol{\phi}_p + \sum_{j=1}^{n} \alpha_j \boldsymbol{\phi}_j | \; \forall \alpha_j \in \mathbb{R}\}$$

- For problems that are stated based on weight functions (weighted residual and weak problem), based on the continuum weight function space $\mathcal{W}$, we choose n linearly independent weight functions $\mathbf{w}_i \in \mathcal{W}$ and form the discrete weight function space (123):

$$\mathcal{W}^h = \mathrm{span}(\mathbf{w}_1, \ldots, \mathbf{w}_n) = \{\sum_{j=1}^{n} \alpha_j \mathbf{w}_j | \; \forall \alpha_j \in \mathbb{R}\}$$

- Due to linearity of the weight functions in weighted statements we only need to satisfy the discrete statement for n weight functions in $\mathcal{W}^h$ (for example $\mathbf{w}_1, \ldots, \mathbf{w}_n$) to ensure its satisfaction for all $\mathbf{w} \in \mathcal{W}^h$.

Indicial notation

For brevity we follow the following notations

$$a_j b_j = \sum_{j=1}^{n} a_j b_j \textbf{ Einstein notation} \tag{124a}$$

$$\text{statement for } a_i : \text{ statement for all } i \in \{1, 2, \ldots, n\} \tag{124b}$$

The range of indices depend on a particular problem. For example for ϕ_i, n refers to the number of test functions and for tensor expressions such as $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$ it refers to spatial dimension.

Matrix equations for discrete systems

- For all the discrete systems discussed we obtain a n by n system of equations.
- To better understand the details of each approach we assume the differential operators L_M (L_m weak form) and L_f are all linear. The same solution process with minor modification can be applied to nonlinear problems where a linear system of equation should be solved for each solution iteration.
- For linear systems, we obtain a linear matrix equation of the form:

$$\boxed{\mathbf{K}\mathbf{a} = \mathbf{F}} \tag{125}$$

  where $\mathbf{K}$ is an $n \times n$ matrix and $\mathbf{a}$ and $\mathbf{F}$ are solution coefficient and right hand side (force) vectors, respectively.
- Linear independence of trial (and weight functions when applicable) is an essential condition for the solvability of (125) (*i.e.,* $\det \mathbf{K} \neq 0$).
- Next, we obtain the matrix equation (125) for various discrete solution schemes and show that several of them can be cast into a weighted residual form.

Weighted Residual Method

We want to obtain the discrete solution corresponding to the continuum weighted residual statement (45):

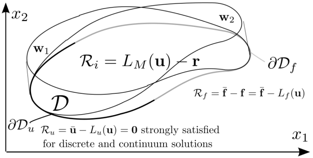
$$\text{Find } \mathbf{u} \in \mathcal{V} \text{ such that } \forall \mathbf{w} \in \mathcal{W} : \int_{\mathcal{D}} \mathbf{w}.\mathcal{R}_i(\mathbf{u})\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}^f.\mathcal{R}_f(\mathbf{u})\,\mathrm{ds} = \mathbf{0}$$

$\mathbf{w}^f$ refers to the form of the weight functions used for the boundary integrals. Often, it is the same as $\mathbf{w}$ (that is $w_i^f = w_i$) but to keep the generality, w_i^f corresponds to w_i.
By changing $\mathbf{u}$ to $\mathbf{u}^h \in \mathcal{V}^h$ and restricting $\mathbf{w}$ to $\mathcal{W}^h$:

$$\text{Find } \mathbf{u}^h \in \mathcal{V}^h \text{ such that } \forall \mathbf{w} \in \mathcal{W}^h : \int_{\mathcal{D}} \mathbf{w}.\mathcal{R}_i(\mathbf{u}^h)\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}^f.\mathcal{R}_f(\mathbf{u}^h)\,\mathrm{ds} = \mathbf{0} \qquad (126)$$

For $\mathbf{u}^h = \sum_{j=1}^{n} a_j \boldsymbol{\phi}_j + \boldsymbol{\phi}_p$ (117a) we have,

$$\left.\begin{array}{rcl} \mathcal{R}_i(\mathbf{u}) & = & L_M(\mathbf{u}) - \mathbf{r} \\ \mathcal{R}_f(\mathbf{u}) & = & \bar{\mathbf{f}} - L_f(\mathbf{u}) \\ L_M(\mathbf{u}), L_f(\mathbf{u}) & : & \text{assumed to be linear} \\ \mathbf{u}^h & = & a_j \boldsymbol{\phi}_j + \boldsymbol{\phi}_p \end{array}\right\} \Rightarrow \qquad (127)$$



$$\begin{array}{rcl} \mathcal{R}_i(\mathbf{u}^h) & = & a_j L_M(\boldsymbol{\phi}_j) + (L_M(\boldsymbol{\phi}_p) - \mathbf{r}) = [L_M(\boldsymbol{\phi})]^{\mathrm{T}}[a] + (L_M(\boldsymbol{\phi}_p) - \mathbf{r}) \\ \mathcal{R}_f(\mathbf{u}^h) & = & -a_j L_f(\boldsymbol{\phi}_j) + (\bar{\mathbf{f}} - L_f(\boldsymbol{\phi}_p)) = -[L_f(\boldsymbol{\phi})]^{\mathrm{T}}[a] + (\bar{\mathbf{f}} - L_f(\boldsymbol{\phi}_p)) \end{array} \qquad (128)$$

Weighted Residual Method

- In (127) the row and column matrix product is:

$$[L_M(\boldsymbol{\phi})]^{\mathrm{T}}[a] = \begin{bmatrix}\boldsymbol{\phi}_1 & \boldsymbol{\phi}_2 & \cdots & \boldsymbol{\phi}_n\end{bmatrix}\begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix} = a_j\boldsymbol{\phi}_j \tag{129}$$

- We plug (127) into (126) to obtain,

$$\begin{aligned}
&\text{Find } [a] = \begin{bmatrix}a_1 & a_2 & \cdots & a_n\end{bmatrix}^{\mathrm{T}} \text{ such that}\\
&\int_{\mathcal{D}}[\mathbf{w}].\left\{[L_M(\boldsymbol{\phi})]^{\mathrm{T}}[a] + (L_M(\boldsymbol{\phi}_p) - \mathbf{r})\right\}\ \mathrm{dv}\ +\\
&\int_{\partial\mathcal{D}_f}[\mathbf{w}^f].\left\{-[L_f(\boldsymbol{\phi})]^{\mathrm{T}}[a] + (\bar{\mathbf{f}} - L_f(\boldsymbol{\phi}_p))\right\}\ \mathrm{ds} = \mathbf{0} \quad \Rightarrow\\
&\left\{\int_{\mathcal{D}}[\mathbf{w}].[L_M(\boldsymbol{\phi})]^{\mathrm{T}}\ \mathrm{dv} - \int_{\partial\mathcal{D}_f}[\mathbf{w}^f].[L_f(\boldsymbol{\phi})]^{\mathrm{T}}\ \mathrm{ds}\right\}[a] =\\
&\left\{\int_{\mathcal{D}}[\mathbf{w}].(\mathbf{r} - L_M(\boldsymbol{\phi}_p))\ \mathrm{dv} + \int_{\partial\mathcal{D}_f}[\mathbf{w}^f].(L_f(\boldsymbol{\phi}_p) - \bar{\mathbf{f}})\ \mathrm{ds}\right\}
\end{aligned} \tag{130}$$

$[\mathbf{w}]$ and $[\mathbf{w}^f]$ correspond to vectors of weight functions on $\mathcal{D}$ and $\partial\mathcal{D}_f$:

$$[\mathbf{w}] = \begin{bmatrix}\mathbf{w}_1\\\vdots\\\mathbf{w}_n\end{bmatrix} \quad \text{and} \quad [\mathbf{w}^f] = \begin{bmatrix}w_1^f\\\vdots\\w_n^f\end{bmatrix} \tag{131}$$

Weighted Residual Method

- According to equations (125), and (130), for a given ϕ_p the solution to the discrete weighted residual statement (126) is obtained from $\mathbf{K}\mathbf{a} = \mathbf{F}$:

$$\mathbf{K} = \int_{\mathcal{D}} [\mathbf{w}].[L_M(\boldsymbol{\phi})]^{\mathrm{T}}\,\mathrm{dv} - \int_{\partial\mathcal{D}_f} [\mathbf{w}^f].[L_f(\boldsymbol{\phi})]^{\mathrm{T}}\,\mathrm{ds} \tag{132a}$$

$$\mathbf{F} = \int_{\mathcal{D}} [\mathbf{w}].(\mathbf{r} - L_M(\boldsymbol{\phi}_p))\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} [\mathbf{w}^f].(L_f(\boldsymbol{\phi}_p) - \bar{\mathbf{f}})\,\mathrm{ds} \tag{132b}$$

or alternatively the individual components are given,

$$K_{ij} = \int_{\mathcal{D}} \mathbf{w}_i L_M(\boldsymbol{\phi}_j)\,\mathrm{dv} - \int_{\partial\mathcal{D}_f} \mathbf{w}_i^f L_f(\boldsymbol{\phi}_j)\,\mathrm{ds} \tag{133a}$$

$$F_i = \int_{\mathcal{D}} \mathbf{w}_i(\mathbf{r} - L_M(\boldsymbol{\phi}_p))\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}_i^f (L_f(\boldsymbol{\phi}_p) - \bar{\mathbf{f}})\,\mathrm{ds} \tag{133b}$$

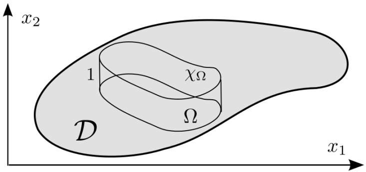
- After solving for $\mathbf{a}$ the discrete solution is obtained from $\mathbf{u}^h = \sum_{j=1}^{n} a_j \boldsymbol{\phi}_j + \boldsymbol{\phi}_p$ (117a).

Weighted Residual Method: Weight options

Weighted residual method is the basis of many numerical methods. In the following we only discuss a few of the weight function options and elaborate how they are related to the continuum principles discussed in the table at the beginning of the section:

- Subdomain method: Weight functions are the so-called characteristic functions of arbitrary sets $\Omega \subset \mathcal{D}$. As will be discussed, this choice of weighted functions resemble satisfaction of the balance law for a finite number of Ω_i.
- Collocation method: The weight functions are delta Dirac "functions". This form corresponds to satisfaction of the strong form (differential equations) at a finite number of points.
- Least Square method: For linear operators L_M and L_f we observe that least square method corresponds to a particular choice of weight function in weighed residual method.
- Galerkin method corresponds to weight function being equal to trial functions. A large group of numerical methods, including spectral and various finite element methods fall into this group.

Weighted Residual Method: Subdomain Method



- We define the characteristic function for the set $\Omega \subset \mathcal{D}$

$$\chi_\Omega(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in \Omega \\ 0 & \text{otherwise} \end{cases} \tag{134}$$

- In subdomain method, the weight functions are n characteristic functions for sets $\Omega_1, \dots, \Omega_n$:

$$[\mathbf{w}] = \begin{bmatrix} \chi_{\Omega_1} \\ \chi_{\Omega_2} \\ \vdots \\ \chi_{\Omega_n} \end{bmatrix} \tag{135}$$

Weighted Residual Method: Subdomain Method

- According to (135) and (132) the matrix equations for the subdomain method are:

$$\mathbf{K} = \int_{\mathcal{D}} [\chi_{\Omega_i}].[L_M(\boldsymbol{\phi})]^{\mathrm{T}}\,\mathrm{dv} - \int_{\partial\mathcal{D}_f} [\chi_{\Omega_i}].(L_f(\boldsymbol{\phi}_p) - \bar{\mathbf{f}})\,\mathrm{ds} \tag{136a}$$

$$\mathbf{F} = \int_{\mathcal{D}} [\chi_{\Omega_i}].(\mathbf{r} - L_M(\boldsymbol{\phi}_p))\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} [\chi_{\Omega_i}].(L_f(\boldsymbol{\phi}_p) - \bar{\mathbf{f}})\,\mathrm{ds} \tag{136b}$$

- Based on (134) and (133) the component expressions are:

$$K_{ij} = \int_{\Omega_i} L_M(\boldsymbol{\phi}_j)\,\mathrm{dv} - \int_{\partial(\Omega_i)_f} L_f(\boldsymbol{\phi}_j)\,\mathrm{ds} \tag{137a}$$

$$F_i = \int_{\Omega_i} \mathbf{w}_i(\mathbf{r} - L_M(\boldsymbol{\phi}_p))\,\mathrm{dv} + \int_{\partial(\Omega_i)_f} (L_f(\boldsymbol{\phi}_p) - \bar{\mathbf{f}})\,\mathrm{ds} \tag{137b}$$

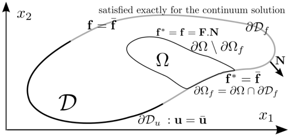
where $\partial(\Omega_i)_f := \Omega_i \cap \partial\mathcal{D}_f$ is the intersection of the boundary of Ω_i with natural boundary $\partial\mathcal{D}_f$.

Optional: Subdomain Method vs. Balance laws

- In subdomain method for a weight function $\mathbf{w} = \chi_\Omega$ the discrete solution $\mathbf{u}^h \in \mathcal{V}^h$ satisfies (126):

$$\int_\Omega \mathcal{R}_i(\mathbf{u}^h)\,\mathrm{dv} + \int_{\partial\Omega_f} \mathcal{R}_f(\mathbf{u}^h)\,\mathrm{ds} = \mathbf{0} \tag{138a}$$

$$\partial\Omega_f = \partial\Omega \cap \partial\mathcal{D}_f \tag{138b}$$



- If the residuals are obtained from a balance law we have,

$$\mathcal{R}_i(\mathbf{u}^h) = \nabla.\mathbf{F}(\mathbf{u}^h) - \mathbf{r} \qquad \text{residual of strong form} \tag{139a}$$

$$\mathcal{R}_f(\mathbf{u}^h) = \bar{\mathbf{f}} - \mathbf{f}(\mathbf{u}^h) \qquad \text{residual of natural boundary condition} \tag{139b}$$

where $\mathbf{F}, \mathbf{r}, \bar{\mathbf{f}}$ are the flux tensor, source term, and natural boundary flux for the given balance law, respectively. The vector $\mathbf{f}(\mathbf{u}^h) = \mathbf{F}(\mathbf{u}^h).\mathbf{N}$ is the flux through the boundary and $\mathbf{N}$ is the normal vector.

Optional: Subdomain Method vs. Balance laws

- We plug (139) for the weight function χ_Ω and employ divergence theorem to obtain,

$$\int_\Omega \nabla.\mathbf{F}(\mathbf{u}^h) - \mathbf{r}\,\mathrm{dv} + \int_{\partial\Omega_f} \left\{\bar{\mathbf{f}} - \mathbf{f}(\mathbf{u}^h)\right\} \mathrm{ds} = \mathbf{0} \quad \text{Divergence theorem} \;\Rightarrow$$

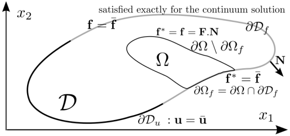
$$\int_{\partial\Omega} \underbrace{\mathbf{F}(\mathbf{u}^h).\mathbf{N}}_{\mathbf{f}(\mathbf{u}^h)} \mathrm{ds} + \int_{\partial\Omega\cap\partial\mathcal{D}_f} \left\{\bar{\mathbf{f}} - \mathbf{f}(\mathbf{u}^h)\right\} \mathrm{ds} = \int_\Omega \mathbf{r}\,\mathrm{dv} \Rightarrow$$

$$\int_{\partial\Omega\setminus\partial\mathcal{D}_f} \mathbf{f}(\mathbf{u}^h)\,\mathrm{ds} + \int_{\partial\Omega\cap\partial\mathcal{D}_f} \left\{\mathbf{f}(\mathbf{u}^h) + \left[\bar{\mathbf{f}} - \mathbf{f}(\mathbf{u}^h)\right]\right\} \mathrm{ds} = \int_\Omega \mathbf{r}\,\mathrm{dv} \quad \Rightarrow$$

$$\int_{\partial\Omega\setminus\partial\mathcal{D}_f} \mathbf{f}(\mathbf{u}^h)\,\mathrm{ds} + \int_{\partial\Omega\cap\partial\mathcal{D}_f} \bar{\mathbf{f}}\,\mathrm{ds} = \int_\Omega \mathbf{r}\,\mathrm{dv}$$

- That is if we define the numerical flux $\mathbf{f}^*$ we conclude,

$$\boxed{\mathbf{f}^*(\mathbf{u}(\mathbf{x})^h) = \begin{cases} \mathbf{f}(\mathbf{u}(\mathbf{x})^h) & \mathbf{x} \in \partial\Omega \setminus \partial\mathcal{D}_f \\ \bar{\mathbf{f}}(\mathbf{x}) & \mathbf{x} \in \partial\Omega \cap \partial\mathcal{D}_f \end{cases}} \qquad (140)$$



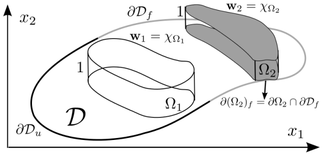
Subdomain WR method for Ω is equivalent to the balance law with respect to $\mathbf{f}^*$

$$\boxed{\int_{\partial\Omega} \mathbf{f}^*(\mathbf{u}^h)\,\mathrm{ds} = \int_\Omega \mathbf{r}} \qquad (141)$$

Summary: Subdomain Method vs. Balance laws

- We mentioned that for the solution for the discrete problem, we seek for an "approximate solution" of the form $\mathbf{u}^h = \sum_{j=1}^{n} a_j \boldsymbol{\phi}_j + \boldsymbol{\phi}_p$ (117a) in the n-dimensional space $\mathcal{V}^h$.
- The n equations for the solutions of the unknowns a_i can be formed by,

Subdomain method

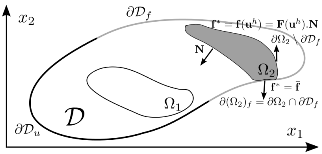


- For weight functions $\mathbf{w}_i = \chi_{\Omega_i}$ the weighted residual statement is $\forall i \in \{1, \ldots, n\}$ (*cf.* (126) and (138)):

$$\int_{\mathcal{D}} \mathbf{w}_i . \mathcal{R}_i(\mathbf{u}^h)\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}_i^f . \mathcal{R}_f(\mathbf{u}^h)\,\mathrm{ds} = \mathbf{0}$$

$$\int_{\Omega_i} \mathcal{R}_i(\mathbf{u}^h)\,\mathrm{dv} + \int_{\partial(\Omega_i)_f} \mathcal{R}_f(\mathbf{u}^h)\,\mathrm{ds} = \mathbf{0}$$

Balance law approach



- We demonstrated that the subdomain weighted residual statement is equivalent to satisfying the balance law for $\forall i \in \{1, \ldots, n\}$ (*cf.* (141)):

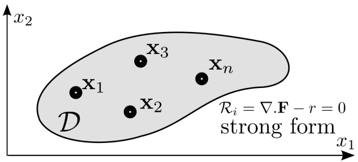
$$\int_{\partial\Omega_i} \mathbf{f}^*(\mathbf{u}^h)\,\mathrm{ds} = \int_{\Omega_i} \mathbf{r}$$

- The numerical flux $\mathbf{f}^*$ is defined by (140):

$$\mathbf{f}^*(\mathbf{u}(\mathbf{x})^h) = \begin{cases} \mathbf{f}(\mathbf{u}(\mathbf{x})^h) & \mathbf{x} \in \partial\Omega_i \setminus \partial\mathcal{D}_f \\ \bar{\mathbf{f}}(\mathbf{x}) & \mathbf{x} \in \partial\Omega_i \cap \partial\mathcal{D}_f \end{cases}$$

Optional: While for continuum solution $\mathbf{f}^* = \mathbf{f}$ everywhere (because natural BCs are strongly satisfied) in discrete setting the definition of $\mathbf{f}^*$ ensures the statement of the balance law with respect to the correct flux $\bar{\mathbf{f}}$ on $\partial\mathcal{D}_f$.

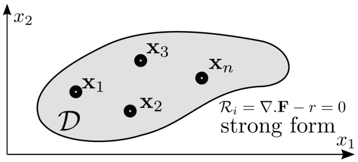
Weighted Residual Method: Collocation Method



- While subdomain method corresponds to discrete version of balance laws, the collocation method corresponds to discrete satisfaction of the strong form
- For simplicity, we assume that ϕ_p not only satisfies both essential and natural boundary conditions while functions ϕ_i satisfy the homogeneous version of both boundary conditions. Thus, the discrete solution $\mathbf{u}^h$ strongly satisfies all the boundary conditions. We want to demonstrate how the collocation method satisfies the strong form at finite number of points.
- With the temporary change the discrete weighted residual (126) simplifies to,

$$\text{Find } \mathbf{u}^h \in \mathcal{V}^h \text{ such that } \forall \mathbf{w} \in \mathcal{W}^h : \int_{\mathcal{D}} \mathbf{w}.\mathcal{R}_i(\mathbf{u}^h)\,\mathrm{dv} = \int_{\mathcal{D}} \mathbf{w}.\{L_M(\mathbf{u}^h(\mathbf{x})) - \mathbf{r}\}\,\mathrm{dv} = \mathbf{0} \tag{142}$$

Weighted Residual Method: Collocation Method



- In collocation method, the weight functions are defined as,

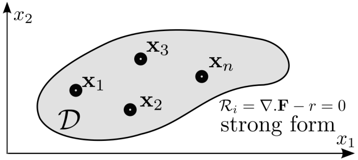
$$\boxed{\mathbf{w}_i(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_i) \text{ for } i \in \{1, 2, \ldots, n\}} \tag{143}$$

  where $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are n arbitrary points in $\mathcal{D}$. Note that $\mathbf{w}_i$ are not even "functions" let alone the $C^0(\mathcal{D})$ condition that we typically stipulate for weight functions in WRM! However, all the integrals with these weight functions can be evaluated. This would be further commented later.

- We plug (143) into (142) to obtain,

$$\text{Find } \mathbf{u}^h \in \mathcal{V}^h \text{ such that } \forall i \in \{1, 2, \ldots, n\} \int_{\mathcal{D}} \mathbf{w}_i(\mathbf{x}).\mathcal{R}_i(\mathbf{u}^h(\mathbf{x}))\,\mathrm{dv} = \int_{\mathcal{D}} \delta(\mathbf{x} - \mathbf{x}_i).\mathcal{R}_i(\mathbf{u}^h(\mathbf{x}))\,\mathrm{dv} = \mathcal{R}_i(\mathbf{u}^h(\mathbf{x}_i)) = \mathbf{0} \tag{144}$$

Weighted Residual Method: Collocation Method



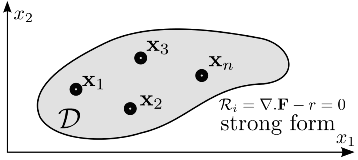
- According to (144) in collocation method we exactly satisfy the strong form at n discrete points:

$$\text{Solution } \mathbf{u}^h = a_j \boldsymbol{\phi}_j + \boldsymbol{\phi}_p \in \mathcal{V}^h \text{ satisfies} \tag{145}$$
$$\forall i \in \{1, 2, \ldots, n\} \; \mathcal{R}_i(\mathbf{u}^h(\mathbf{x}_i)) = L_M(\mathbf{u}^h(\mathbf{x}_i)) - \mathbf{r}(\mathbf{x}_i) = \mathbf{0}$$

- For linear operator L_M (*cf.* (128)):

$$L_M(\mathbf{u}^h(\mathbf{x}_i)) = a_j L_M(\boldsymbol{\phi}_j)(\mathbf{x}_i) + L_M(\boldsymbol{\phi}_p)(\mathbf{x}_i) \tag{146}$$

Weighted Residual Method: Collocation Method

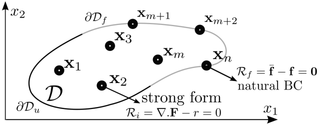


- From equations (145) and (146) the linear system $\mathbf{Ka} = \mathbf{F}$ (*cf.* (125), (132), and (133)) takes the form:

$$K_{ij} = \int_{\mathcal{D}} \mathbf{w}_i L_M(\boldsymbol{\phi}_j)\,\mathrm{dv} = L_M(\boldsymbol{\phi}_j)(\mathbf{x}_i) \tag{147a}$$

$$F_i = \int_{\mathcal{D}} \mathbf{w}_i(\mathbf{r} - L_M(\boldsymbol{\phi}_p))\,\mathrm{dv} = \mathbf{r}(\mathbf{x}_i) - L_M(\boldsymbol{\phi}_j)(\mathbf{x}_i) \tag{147b}$$

Optional: Weighted Residual Method: Collocation Method



- Without going to detail, equations for more practical case where natural boundary conditions are not strongly satisfied are provided. In this case, we solve the usual discrete system (126)

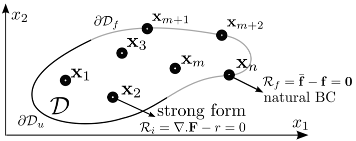
$$\text{Find } \mathbf{u}^h \in \mathcal{V}^h \text{ such that } \forall \mathbf{w} \in \mathcal{W}^h : \int_{\mathcal{D}} \mathbf{w}.\mathcal{R}_i(\mathbf{u}^h)\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}^f.\mathcal{R}_f(\mathbf{u}^h)\,\mathrm{ds} = \mathbf{0}$$

- As shown in the figure

$$\text{For } i = 1, \ldots, m,\ \mathbf{x}_i \in \mathcal{D} \setminus \partial\mathcal{D} \text{ (interior of } \mathcal{D}) \tag{148}$$
$$\text{For } i = m+1, \ldots, n,\ \mathbf{x}_i \in \partial\mathcal{D}_f$$

- $\mathbf{w}_i$ are set to delta Dirac function on set $\mathcal{D}$ for $\mathbf{x}_1, \ldots, \mathbf{x}_m$ and on set $\partial\mathcal{D}_f$ for $\mathbf{x}_{m+1}, \ldots, \mathbf{x}_n$.
- For $i = 1, \ldots, m$ interior residuals, $\mathcal{R}_i(\mathbf{x}_i)$ will be activated, while for $m+1, \ldots, n$ $\mathcal{R}_f(\mathbf{x}_i)$ will be activated.

Optional: Weighted Residual Method: Collocation Method

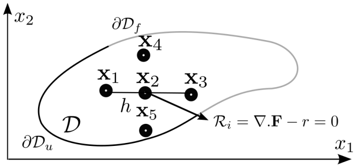


- From the previous discussion the linear system $\mathbf{Ka} = \mathbf{F}$ (*cf.* (125), (132), and (133)) takes the form:

$$K_{ij} = \begin{cases} \int_{\mathcal{D}} \mathbf{w}_i L_M(\boldsymbol{\phi}_j)\,\mathrm{dv} = L_M(\boldsymbol{\phi}_j)(\mathbf{x}_i) & i \le m \\ -\int_{\partial\mathcal{D}_f} \mathbf{w}_i^f . L_f(\boldsymbol{\phi}_j)\,\mathrm{ds} = -L_f(\boldsymbol{\phi}_j)(\mathbf{x}_i) & m < i \le n \end{cases} \tag{149a}$$

$$F_i = \begin{cases} \int_{\mathcal{D}} \mathbf{w}_i(\mathbf{r} - L_M(\boldsymbol{\phi}_p))\,\mathrm{dv} = \mathbf{r}(\mathbf{x}_i) - L_M(\boldsymbol{\phi}_j)(\mathbf{x}_i) & i \le m \\ \int_{\partial\mathcal{D}_f} \mathbf{w}_i^f .(L_f(\boldsymbol{\phi}_p) - \bar{\mathbf{f}})\,\mathrm{ds} = L_f(\boldsymbol{\phi}_p)(\mathbf{x}_i) - \bar{\mathbf{f}}(\mathbf{x}_i) & m < i \le n \end{cases} \tag{149b}$$

Collocation method versus Finite Difference

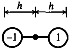
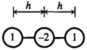
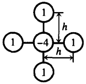
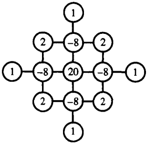


- Both Collocation and Finite Difference methods directly work with the strong form and boundary conditions.
- Collocation method is a particular class of weighted residual method where the solution is interpolated as $\mathbf{u}^h = a_j \boldsymbol{\phi}_j + \boldsymbol{\phi}_p$.
- Finite Difference does not interpolate the solution with trial function. Rather, it uses discrete values of the function on often regular grids to approximate differential operators.
- Differential operators in Finite Difference method are approximate, where as in collocation method the solution $\mathbf{u}^h$ exactly satisfies the strong form at $\mathbf{x}_i$.
- As an example, let us assume the differential operator L_M in $\mathcal{R}_i$ includes a Laplacian operator $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$. The finite difference approximation of Laplacian on a uniform grid with size h would be,

$$\Delta u(\mathbf{x}_2) = \frac{1}{h^2}\left(u(\mathbf{x}_1) + u(\mathbf{x}_3) + u(\mathbf{x}_4) + u(\mathbf{x}_5) - 4u(\mathbf{x}_2)\right) \tag{150}$$

Finite Difference Stencils

TABLE 3.1 *Finite difference approximations for various differentiations*

Differentiation	Finite difference approximation	Molecules
$\left.\dfrac{dw}{dx}\right\|_i$	$\dfrac{w_{i+1} - w_{i-1}}{2h}$	
$\left.\dfrac{d^2w}{dx^2}\right\|_i$	$\dfrac{w_{i+1} - 2w_i + w_{i-1}}{h^2}$	
$\left.\dfrac{d^3w}{dx^3}\right\|_i$	$\dfrac{w_{i+2} - 2w_{i+1} + 2w_{i-1} - w_{i-2}}{2h^3}$	
$\left.\dfrac{d^4w}{dx^4}\right\|_i$	$\dfrac{w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2}}{h^4}$	
$\nabla^2 w\|_{i,j}$	$\dfrac{-4w_{i,j} + w_{i+1,j} + w_{i,j+1} + w_{i-1,j} + w_{i,j-1}}{h^2}$	
$\nabla^4 w\|_{i,j}$	$[20w_{i,j} - 8(w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1}) + 2(w_{i+1,j+1} + w_{i-1,j+1} + w_{i-1,j-1} + w_{i+1,j-1}) + w_{i+2,j} + w_{i-2,j} + w_{i,j+2}\ w_{i,j-2}]/h^4$	

Source:Bathe's book, section 3.3.5.

Weighted Residual Method: Galerkin Method

- In Galerkin method weight functions are set to be the same as trial functions:

$$\boxed{\forall i \in \{1, 2, \ldots, n\} : \ \mathbf{w}_i = \boldsymbol{\phi}_i \ \Rightarrow [\mathbf{w}] = [\boldsymbol{\phi}]} \tag{151}$$

- The matrix equation $\mathbf{Ka} = \mathbf{F}$ components for Galerkin method are obtained from (132) and (133) by substituting $\boldsymbol{\phi}$ for $\mathbf{w}$:

$$\mathbf{K} = \int_{\mathcal{D}} [\boldsymbol{\phi}].[L_M(\boldsymbol{\phi})]^{\mathrm{T}} \, \mathrm{dv} - \int_{\partial\mathcal{D}_f} [\boldsymbol{\phi}].[L_f(\boldsymbol{\phi})]^{\mathrm{T}} \, \mathrm{ds} \tag{152a}$$

$$\mathbf{F} = \int_{\mathcal{D}} [\boldsymbol{\phi}].(\mathbf{r} - L_M(\boldsymbol{\phi}_p)) \, \mathrm{dv} + \int_{\partial\mathcal{D}_f} [\boldsymbol{\phi}].(L_f(\boldsymbol{\phi}_p) - \bar{\mathbf{f}}) \, \mathrm{ds} \tag{152b}$$

or alternatively the individual components are given,

$$K_{ij} = \int_{\mathcal{D}} \boldsymbol{\phi}_i L_M(\boldsymbol{\phi}_j) \, \mathrm{dv} - \int_{\partial\mathcal{D}_f} \boldsymbol{\phi}_i L_f(\boldsymbol{\phi}_j) \, \mathrm{ds} \tag{153a}$$

$$F_i = \int_{\mathcal{D}} \boldsymbol{\phi}_i(\mathbf{r} - L_M(\boldsymbol{\phi}_p)) \, \mathrm{dv} + \int_{\partial\mathcal{D}_f} \boldsymbol{\phi}_i(L_f(\boldsymbol{\phi}_p) - \bar{\mathbf{f}}) \, \mathrm{ds} \tag{153b}$$

Least Square Method

We want to obtain the discrete solution corresponding to the continuum least square method (51):

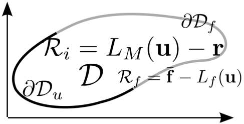
$$\text{Find } \mathbf{u} \in \mathcal{V} = \{\mathbf{v}\ |\mathbf{v} \in C^M(\mathcal{D}, L_u(\mathbf{u}) = \bar{\mathbf{u}}\} \text{ such that}$$
$$\begin{aligned} \int_{\mathcal{D}} \mathcal{R}_i^2(\mathbf{u})\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathcal{R}_f^2(\mathbf{u})\,\mathrm{ds} &= \\ \int_{\mathcal{D}} (L_M(\mathbf{u}) - \mathbf{r})^2\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} (\bar{\mathbf{f}} - L_f(\mathbf{u}))^2\,\mathrm{ds} &= 0 \end{aligned}$$

By changing $\mathbf{u}$ to $\mathbf{u}^h \in \mathcal{V}^h$ and minimizing R^2 with respect to solution coefficients $\mathbf{a}$ instead of continuum condition $R^2 = 0$, we have

$$\text{Find } \mathbf{u}^h \in \mathcal{V}^h \text{ such that} \tag{154}$$
$$\forall \tilde{\mathbf{u}}^h \in \mathcal{V}^h: \quad R^2(\mathbf{u}^h) \leq R^2(\tilde{\mathbf{u}}^h) \text{ where}$$
$$R^2(\tilde{\mathbf{u}}^h) = \int_{\mathcal{D}} \mathcal{R}_i^2(\tilde{\mathbf{u}}^h)\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathcal{R}_f^2(\tilde{\mathbf{u}}^h)\,\mathrm{ds} = \int_{\mathcal{D}} (L_M(\tilde{\mathbf{u}}^h) - \mathbf{r})^2\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} (\bar{\mathbf{f}} - L_f(\tilde{\mathbf{u}}^h))^2\,\mathrm{ds}$$

Noting that $\mathcal{V}^h$ is an n-dimensional space (122) and $\mathbf{u}^h = \sum_{j=1}^n a_j \boldsymbol{\phi}_j + \boldsymbol{\phi}_p$ (117a) the minimum condition can be expressed as,

$$\text{Find } [\mathbf{a}] \in \mathbb{R}^n \text{ such that} \tag{155}$$
$$\forall [\tilde{\mathbf{a}}] \in \mathbb{R}^n: \quad R^2([\mathbf{a}]) \leq R^2([\tilde{\mathbf{a}}]) \text{ where}$$
$$R^2([\tilde{\mathbf{a}}]) = R^2(\tilde{\mathbf{u}}^h) \text{ for } \tilde{\mathbf{u}}^h = [\boldsymbol{\phi}]^{\mathrm{T}}[\tilde{\mathbf{a}}] + \boldsymbol{\phi}_p$$



Least Square Method

- The minimum condition for the solution $[\mathbf{a}](\mathbf{u}^h = [\boldsymbol{\phi}]^{\mathrm{T}}[\mathbf{a}] + \phi_p)$ in (155) can be expressed as,

$$[\mathbf{a}] \text{ is minimizer } \Rightarrow \frac{\partial R^2}{\partial a_i} = 0 \Rightarrow$$

$$\int_{\mathcal{D}} 2\frac{\partial L_M(\mathbf{u}^h)}{\partial a_i}(L_M(\mathbf{u}^h) - \mathbf{r})\,\mathrm{dv} + \int_{\partial\mathcal{D}_f}(-2)\frac{\partial L_f(\tilde{\mathbf{u}}^h)}{\partial a_i}(\bar{\mathbf{f}} - L_f(\tilde{\mathbf{u}}^h))\,\mathrm{ds} = 0 \tag{156}$$

- Noting the linearity of L_M and L_f and $[\mathbf{a}](\mathbf{u}^h = [\boldsymbol{\phi}]^{\mathrm{T}}[\mathbf{a}] + \phi_p)$ we observe,

$$L_M(\mathbf{u}^h) = a_i L_M(\boldsymbol{\phi}_i) + \boldsymbol{\phi}_p \Rightarrow \qquad \frac{\partial L_M(\mathbf{u}^h)}{\partial a_i} = L_M(\boldsymbol{\phi}_i) \tag{157a}$$

$$L_f(\mathbf{u}^h) = a_i L_f(\boldsymbol{\phi}_i) + \boldsymbol{\phi}_p \Rightarrow \qquad \frac{\partial L_f(\mathbf{u}^h)}{\partial a_i} = L_f(\boldsymbol{\phi}_i) \tag{157b}$$

Least Square Method

- Equations (156) and (157) yield,

$$\forall \boldsymbol{\phi}_i \in \mathcal{V}^h : \int_{\mathcal{D}} L_M(\boldsymbol{\phi}_i) \underbrace{\left(L_M(\mathbf{u}^h) - \mathbf{r}\right)}_{\mathcal{R}_i} \,\mathrm{dv} + \int_{\partial \mathcal{D}_f} \left(-L_f(\boldsymbol{\phi}_i)\right) \underbrace{\left(\bar{\mathbf{f}} - L_f(\tilde{\mathbf{u}}^h)\right)}_{\mathcal{R}_f} \,\mathrm{ds} = \mathbf{0} \tag{158}$$

- In comparison to (126) for the general statement of weighted residual methods we observe,

$$\forall \mathbf{w} \in \mathcal{W}^h : \int_{\mathcal{D}} \mathbf{w}.\mathcal{R}_i(\mathbf{u}^h) \,\mathrm{dv} + \int_{\partial \mathcal{D}_f} \mathbf{w}^f.\mathcal{R}_f(\mathbf{u}^h) \,\mathrm{ds} = \mathbf{0}$$

- Discrete Least Square problem for linear differential operators L_M and L_f is equivalent to a discrete weighted residual statement with the weight functions:

Weight functions corresponding to Least Square Method

$$\mathbf{w} = L_M(\boldsymbol{\phi}) \tag{159a}$$

$$\mathbf{w}^f = \left(-L_f(\boldsymbol{\phi})\right) \tag{159b}$$

Least Square Method

- According to (159), discrete least square method corresponds to a weighted residual method with particular weight functions given therein. Accordingly, (132) and (133) take the form:

$$\mathbf{K} = \int_{\mathcal{D}} [L_M(\boldsymbol{\phi})].[L_M(\boldsymbol{\phi})]^{\mathrm{T}} \, \mathrm{dv} + \int_{\partial\mathcal{D}_f} [L_f(\boldsymbol{\phi})].[L_f(\boldsymbol{\phi})]^{\mathrm{T}} \, \mathrm{ds} \tag{160a}$$

$$\mathbf{F} = \int_{\mathcal{D}} [L_M(\boldsymbol{\phi})].(\mathbf{r} - L_M(\boldsymbol{\phi}_p)) \, \mathrm{dv} - \int_{\partial\mathcal{D}_f} [L_f(\boldsymbol{\phi})].(L_f(\boldsymbol{\phi}_p) - \bar{\mathbf{f}}) \, \mathrm{ds} \tag{160b}$$

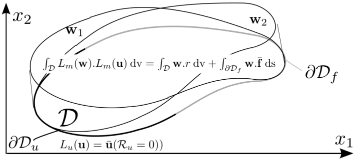
or alternatively the individual components are given,

$$K_{ij} = \int_{\mathcal{D}} L_M(\boldsymbol{\phi}_i) L_M(\boldsymbol{\phi}_j) \, \mathrm{dv} + \int_{\partial\mathcal{D}_f} L_f(\boldsymbol{\phi}_i) L_f(\boldsymbol{\phi}_j) \, \mathrm{ds} \tag{161a}$$

$$F_i = \int_{\mathcal{D}} L_M(\boldsymbol{\phi}_i)(\mathbf{r} - L_M(\boldsymbol{\phi}_p)) \, \mathrm{dv} - \int_{\partial\mathcal{D}_f} L_f(\boldsymbol{\phi}_i)(L_f(\boldsymbol{\phi}_p) - \bar{\mathbf{f}}) \, \mathrm{ds} \tag{161b}$$

- The matrix $\mathbf{K}$ is always symmetric for least square method
- **Optional:** While all three variations of the continuum least square method (50), (51), and (52) can be used to form the discrete least square method, (52) (PDE residual only) is often used for presentation purposes. For other cases, care should be taken to ensure that all integrals have the same physical dimension.

Discrete Weak Statement

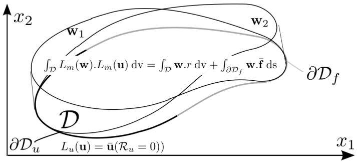


- Weak statement is basically a weighted residual statement where the derivative orders on the solution and weight function has been distributed to obtain a (more) balanced derivative order between the two.
- The advantage of the weak statement is the enlargement of the allowable solution functions.
- As will be shown shortly, this function space enlargement has immense practical importance in the design of finite element method.
- While the approach from balance law $\rightarrow$ strong form $\rightarrow$ weak form is quite cumbersome, we observed that the energy method provides a very direct way to obtain the weak statement.
- According to (74) the continuum weak statement is,

$$\text{Find } \mathbf{u} \in \mathcal{V} \text{ such that } \forall \mathbf{w} \in \mathcal{W} \int_{\mathcal{D}} L_m^w(\mathbf{w}) L_m(\mathbf{u})\,\mathrm{dv} = \int_{\mathcal{D}} \mathbf{w}.\mathbf{r}\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}.\bar{\mathbf{f}}\,\mathrm{ds}$$

where L_m^w and L_m are differential operators acting on weight and solution functions.

Discrete Weak Statement



- The discrete weak statement is obtained in the same fashion that we derived the discrete weighted residual statement from continuum statement (as mentioned weak statement can be considered as a manipulated weighted residual statement).
- The same trial function and weight functions spaces (122) and (123) are used for the discrete statement. The discrete weak form reads as:

$$\text{Find } \mathbf{u}^h \in \mathcal{V}^h \text{ such that } \forall \mathbf{w} \in \mathcal{W}^h \int_{\mathcal{D}} L_m^w(\mathbf{w}) L_m(\mathbf{u}^h)\,\mathrm{dv} = \int_{\mathcal{D}} \mathbf{w}.\mathbf{r}\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}.\bar{\mathbf{f}}\,\mathrm{ds} \tag{162}$$

- For linear L_m (L_m^w is always linear) by using the expansion $\mathbf{u}^h = \sum_{j=1}^n a_j \boldsymbol{\phi}_j + \boldsymbol{\phi}_p$ (117a) and $\mathbf{w} = [\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n]^{\mathrm{T}}$ we can derive the following linear system:

$$\int_{\mathcal{D}} L_m^w([\mathbf{w}]) L_m([\boldsymbol{\phi}]^{\mathrm{T}}[\mathbf{a}] + \boldsymbol{\phi}_p)\,\mathrm{dv} = \int_{\mathcal{D}} [\mathbf{w}].\mathbf{r}\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} [\mathbf{w}].\bar{\mathbf{f}}\,\mathrm{ds} \;\Rightarrow \tag{163}$$

$$\left\{\int_{\mathcal{D}} L_m^w([\mathbf{w}]) L_m([\boldsymbol{\phi}])^{\mathrm{T}}\,\mathrm{dv}\right\}[\mathbf{a}] = \int_{\mathcal{D}} \{[\mathbf{w}].\mathbf{r} - L_m^w([\mathbf{w}]).L_m(\boldsymbol{\phi}_p)\}\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} [\mathbf{w}].\bar{\mathbf{f}}\,\mathrm{ds}$$

Discrete Weak Statement

- From (163) the matrix equations for $\mathbf{K}\mathbf{a} = \mathbf{F}$ (125) are (*cf.* (132) and (133) for comparison):

$$\mathbf{K} = \int_{\mathcal{D}} L_m^w([\mathbf{w}]).[L_m(\boldsymbol{\phi})]^{\mathrm{T}} \, \mathrm{dv} \tag{164a}$$

$$\mathbf{F} = \int_{\mathcal{D}} \{[\mathbf{w}].\mathbf{r} - L_m^w([\mathbf{w}]).L_m(\boldsymbol{\phi}_p\} \, \mathrm{dv} + \int_{\partial\mathcal{D}_f} [\mathbf{w}].\bar{\mathbf{f}} \, \mathrm{ds} \tag{164b}$$

or alternatively the individual components are given,

$$K_{ij} = \int_{\mathcal{D}} L_m^w(\mathbf{w}_i).L_m(\boldsymbol{\phi}_j) \, \mathrm{dv} \tag{165a}$$

$$F_i = \int_{\mathcal{D}} \{\mathbf{w}_i.\mathbf{r} - L_m^w(\mathbf{w}_i).L_m(\boldsymbol{\phi}_p\} \, \mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}_i.\bar{\mathbf{f}} \, \mathrm{ds} \tag{165b}$$

Discrete Weak Statement: Self-adjoint property

- Self-adjoint operators are prevalent in physics and mathematics. They convey commutative property of a binary operator. For example, for the weak statement the self-adjoint property of the problem translates to:

$$\forall \mathbf{w}, \mathbf{v} \in \mathcal{V}^h : \int_{\mathcal{D}} L_m^w(\mathbf{w}).L_m(\mathbf{v}) \, \mathrm{dv} = \int_{\mathcal{D}} L_m^w(\mathbf{v}).L_m(\mathbf{w}) \, \mathrm{dv} \tag{166}$$

- We want to investigate what self-adjointness translates to in discrete setting.
- We consider a Galerkin weak statement, that is weight functions are equal to trial functions: $\mathbf{w}_i = \boldsymbol{\phi}_i$ (so $\mathcal{W}^h = \mathcal{V}^h$). Then from (164) and (165) linear matrix equation components become,

$$\mathbf{K} = \int_{\mathcal{D}} L_m^w([\boldsymbol{\phi}]).[L_m(\boldsymbol{\phi})]^{\mathrm{T}} \, \mathrm{dv} \tag{167a}$$

$$\mathbf{F} = \int_{\mathcal{D}} \{[\boldsymbol{\phi}].\mathbf{r} - L_m^w([\boldsymbol{\phi}]).L_m(\boldsymbol{\phi}_p\} \, \mathrm{dv} + \int_{\partial\mathcal{D}_f} [\boldsymbol{\phi}].\bar{\mathbf{f}} \, \mathrm{ds} \tag{167b}$$

or alternatively the individual components are given,

$$K_{ij} = \int_{\mathcal{D}} L_m^w(\boldsymbol{\phi}_i).L_m(\boldsymbol{\phi}_j) \, \mathrm{dv} \tag{168a}$$

$$F_i = \int_{\mathcal{D}} \{\boldsymbol{\phi}_i.\mathbf{r} - L_m^w(\boldsymbol{\phi}_i).L_m(\boldsymbol{\phi}_p\} \, \mathrm{dv} + \int_{\partial\mathcal{D}_f} \boldsymbol{\phi}_i.\bar{\mathbf{f}} \, \mathrm{ds} \tag{168b}$$

Discrete Weak Statement: Self-adjoint property

- If the problem is self adjoint from (166) and (168a) we observe:

$$K_{ij} = \int_{\mathcal{D}} L_m^w(\boldsymbol{\phi}_i).L_m(\boldsymbol{\phi}_j)\,\mathrm{dv} = \int_{\mathcal{D}} L_m^w(\boldsymbol{\phi}_j).L_m(\boldsymbol{\phi}_i)\,\mathrm{dv} = K_{ji} \quad \Rightarrow$$

Self adjoint property implications in discrete setting

For a self-adjoint problem, the discrete matrix $\mathbf{K}$ is symmetric

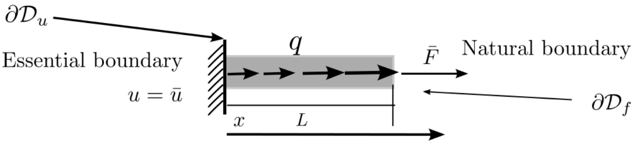
- Recalling the matrix $\mathbf{K}$ components for Least Square method (161a), how does this results compare to least square method:

$$K_{ij} = \int_{\mathcal{D}} L_M(\boldsymbol{\phi}_i) L_M(\boldsymbol{\phi}_j)\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} L_f(\boldsymbol{\phi}_i) L_f(\boldsymbol{\phi}_j)\,\mathrm{ds} \quad \text{for least square method}$$

While least square method **always** generates symmetric matrices, symmetry only occurs for self adjoint problems in weak statement with Galerkin weight option

- The self adjoint property can also be checked on the weighted residual statement, but it is easier to check this property in the weak statement because the weight and trial functions take a similar form.
- What are the benefits of a symmetric matrix in discrete setting?

Example: Self-adjoint problems



- Consider the 1D solid bar example shown in the figure. The residuals for this problem are (essential boundary conditions are strongly satisfied.

$$\mathcal{R}_i(x) = L_M(u(x)) - r(x) = \frac{\mathrm{d}\ \overbrace{F(x)}^{A\sigma(u(x))}}{\mathrm{d}x} + q(x) \quad \Rightarrow L_M(u(x)) = \frac{\mathrm{d}}{\mathrm{d}x}\left(EA\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)$$

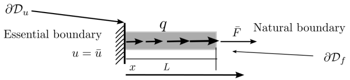
$$\mathcal{R}_f(x) = \bar{F} - F(x) = \bar{F} - A\sigma(u(x)) = \bar{F} - AE\frac{\mathrm{d}u(x)}{\mathrm{d}x}$$

- The weighted residual statement for this problem is,

$$\text{Find } u \in \mathcal{V}, \text{ such that } \forall w \in \mathcal{W} : \int_0^L \mathcal{R}_i(x)\,\mathrm{d}x + w(x).(\bar{F} - F(x))|_{x=2} = 0$$

Example: Self-adjoint problems

- We obtain the weak statement for this problem (by one integration by part) to assess if the problem is self adjoint or not (this is equation (73) we directly derived from energy method):



$$\text{Find } u\in\mathcal{V}=\{v\in C^1([0,L])\mid v(0)=\bar{u}\},\ \text{such that,}$$

$$\forall w\in\mathcal{W}=\{v\in C^1([0,L])\mid v(0)=0\}$$

$$\int_0^L \underbrace{w'(x)}_{\epsilon(w(x))}\ \underbrace{AEu'(x)}_{F(u(x))=A\sigma(u(x))}\,\mathrm{d}x=\int_0^L w(x)q(x)\,\mathrm{d}x+w(L)\bar{F}$$

- That is,

$$L_m^w(w)=w'(x)=\epsilon(w(x)) \tag{169a}$$

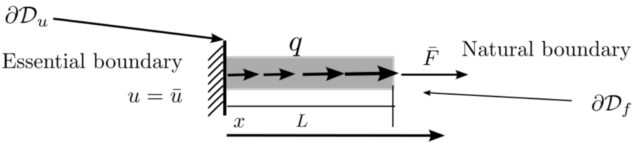
$$L_m(u)=AEu'(x)=A\sigma(u(x))=F(u(x)) \tag{169b}$$

- It is easy to verify (166):

$$\int_{\mathcal{D}} L_m^w(w).L_m(\mathbf{u})\,\mathrm{dv}=\int_0^L w'(x).AEu'(x)\,\mathrm{d}x$$

$$=\int_0^L u'(x).AEw'(x)\,\mathrm{d}x=\int_{\mathcal{D}} L_m^w(u).L_m(\mathbf{w})\,\mathrm{dv}$$

Example: Self-adjoint problems



- So, 1D bar example is self adjoint.
- It is easy to verify (168a) by using (169)

$$K_{ij} = \int_0^L \frac{\mathrm{d}u_i}{\mathrm{d}x} . AE \frac{\mathrm{d}u_j}{\mathrm{d}x} \,\mathrm{d}x = \int_0^L \frac{\mathrm{d}u_j}{\mathrm{d}x} . AE \frac{\mathrm{d}u_i}{\mathrm{d}x} \,\mathrm{d}x = K_{ji}$$

- For elastostatics (linear or nonlinear) the self adjoint property well stemming from a more general property:

$$\epsilon(\mathbf{w}) : \sigma(\mathbf{u}) = \epsilon(\mathbf{u}) : \sigma(\mathbf{w})$$

which is equivalent to the 1D case where we had:

$$\epsilon(w)F(u) = \frac{\mathrm{d}w}{\mathrm{d}x} AE \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} AE \frac{\mathrm{d}w}{\mathrm{d}x} = \epsilon(\mathbf{u})F(\mathbf{w})$$

Example: Self-adjoint problems

- As an example of a problem that is not self-adjoint consider the following boundary value problem for $u \in \mathcal{D} = [0\ 1]$:

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}x} - r(x) = 0 & \text{Differential equation} \\ u(0) = u(1) = 0 & \text{Essential boundary condition} \end{cases} \tag{170}$$

- The weighted residual statement for this problem is,

$$\text{Find } u \in \mathcal{V} = \{v \in C^1([0,1]) \mid v(0) = v(1) = 0\}, \text{ such that,}$$
$$\forall w \in \mathcal{W} = \{v \in C^0([0,1]) \mid v(0) = v(1) = 0\}$$
$$\int_0^1 w(x).\underbrace{\left(\frac{\mathrm{d}u(x)}{\mathrm{d}x} - r(x)\right)}_{\mathcal{R}_i(x)} \mathrm{d}x = 0$$

- For the problem to be self adjoint the bilinear form (a linear function in each of its arguments):

$$a(u,w) = \int_0^1 w(x).\frac{\mathrm{d}u(x)}{\mathrm{d}x}\,\mathrm{d}x$$

should be commutative. However, It is clear that $a(u,w) \neq a(w,u)$ (for example consider $w(x) = 1, u(x) = x$).

- **Optional** Self adjoint property not only has many computational advantages (half the storage and faster solution algorithm for symmetric matrices) but also has several physical interpretations and facilitates many existence and uniqueness proofs.

Energy Method, approach A, Ritz method

We want to obtain the discrete solution corresponding to the continuum energy method (80):

$$\text{Find } \mathbf{u} \in \mathcal{V} = \{\mathbf{v} \,|\mathbf{v} \in C^m(\mathcal{D}), L_u(\mathbf{u}) = \bar{\mathbf{u}}\} \text{ such that}$$
$$\forall \tilde{\mathbf{u}} \in \mathcal{V} : \boxed{\Pi(\mathbf{u}) \leq \Pi(\tilde{\mathbf{u}})} \quad \text{or alternatively} \tag{171}$$
$$\forall\, \delta\mathbf{u} \in \mathcal{V}_0 = \{\mathbf{v} \,|\mathbf{v} \in C^m(\mathcal{D}, L_u(\mathbf{u}) = 0\} : \quad \Pi(\mathbf{u}) \leq \Pi(\mathbf{u} + \delta\mathbf{u})$$

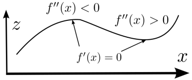
There are two paths to discretize the energy method.

1. **Ritz Method**: Direct discretization of the continuum energy minimization principle: By changing $\mathbf{u}$ to $\mathbf{u}^h \in \mathcal{V}^h$ and minimizing Π with respect to solution coefficients $\mathbf{a}$ instead of continuum condition $\Pi(\mathbf{u}) \leq \Pi(\tilde{\mathbf{u}})$ for $\mathbf{u}, \tilde{\mathbf{u}} \in \mathcal{V}$ (infinite dimensional space) we obtain,

$$\text{Find } \mathbf{u}^h \in \mathcal{V}^h \text{ such that} \tag{172}$$
$$\forall \tilde{\mathbf{u}}^h \in \mathcal{V}^h : \quad \Pi(\mathbf{u}^h) \leq \Pi(\mathbf{u})$$

Noting that $\mathcal{V}^h$ is an n-dimensional space (122) and $\mathbf{u}^h = \sum_{j=1}^{n} a_j \boldsymbol{\phi}_j + \boldsymbol{\phi}_p$ (117a) the minimum condition can be expressed as,

$$\text{Find } [\mathbf{a}] \in \mathbb{R}^n \text{ such that} \tag{173}$$
$$\forall [\tilde{\mathbf{a}}] \in \mathbb{R}^n : \quad \Pi([\mathbf{a}]) \leq \Pi([\tilde{\mathbf{a}}]) \text{ where}$$
$$\Pi([\tilde{\mathbf{a}}]) = \Pi(\tilde{\mathbf{u}}^h) \text{ for } \tilde{\mathbf{u}}^h = [\boldsymbol{\phi}]^{\mathrm{T}}[\tilde{\mathbf{a}}] + \boldsymbol{\phi}_p$$



Necessary optimality condition $\nabla\Pi([\mathbf{a}]) = 0$ (gradient $= 0$) for Π at $\mathbf{a} \Rightarrow$:

$$\boxed{\nabla\Pi(\mathbf{a}) = 0 \;\Rightarrow\; \frac{\partial \Pi}{\partial a_i}(\mathbf{a}) = 0 \quad \text{Ritz method minimizes } \Pi \text{ of the discrete system}} \tag{174}$$

Energy Method, approach B, weak formulation

2. **Weak method**: Minimizing the continuum problem then discretizing: In the second approach we proceed by first minimizing the continuum statement. From (96)

$$\Pi = \Pi(y, y', \dots, y^{(n)}) \quad \Rightarrow \quad \delta\Pi = \frac{\partial \Pi}{\partial y}\delta y + \frac{\partial \Pi}{\partial y'}\delta y' + \cdots + \frac{\partial^n \Pi}{\partial {y^{(n)}}^n}\delta y^{(n)}. \tag{175}$$

the continuum energy method would directly result in a weak statement of the form:

$$\text{Find } \mathbf{u} \in \mathcal{V} : \ \forall \mathbf{w} \in \mathcal{V}_0$$
$$\int_{\mathcal{D}} L_m^w(\mathbf{w}) L_m(\mathbf{u})\, \mathrm{dv} = \int_{\mathcal{D}} \mathbf{w}.\mathbf{r}\, \mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}.\bar{\mathbf{f}}\, \mathrm{ds}$$

where $\mathbf{w} = \delta\mathbf{u}$ stands for the variation of the solution $\mathbf{u}$ and $\mathcal{V}, \mathcal{V}_0$ are given in (171). We already discussed the solution of a discrete weak problem (for example refer to (162) and (163)). The linear system matrix $\mathbf{K}$ and force vectors $\mathbf{F}$ were given in (164):

$$\mathbf{K} = \int_{\mathcal{D}} L_m^w([\mathbf{w}]).[L_m(\boldsymbol{\phi})]^{\mathrm{T}}\, \mathrm{dv} \tag{176}$$
$$\mathbf{F} = \int_{\mathcal{D}} \{[\mathbf{w}].\mathbf{r} - L_m^w([\mathbf{w}]).L_m(\boldsymbol{\phi}_p\}\, \mathrm{dv} + \int_{\partial\mathcal{D}_f} [\mathbf{w}].\bar{\mathbf{f}}\, \mathrm{ds}$$

Comparison of two approaches for discretizing the energy method

- In the second approach (weak form obtained from energy functional) as can be seen in (176), we have complete freedom to choose any set of weight functions (for example subdomain, collocation, or Galerkin options).
- While in the first approach (Ritz method) we do not directly deal with weight functions, generally it corresponds to the weak system in (175) with **Galerkin** weight functions, that is $\mathbf{w} = \phi$ and (176) simplifies to

$$\mathbf{K} = \int_{\mathcal{D}} L_m^w([\boldsymbol{\phi}]).[L_m(\boldsymbol{\phi})]^{\mathrm{T}}\,\mathrm{dv}$$

$$\mathbf{F} = \int_{\mathcal{D}} \{[\boldsymbol{\phi}].\mathbf{r} - L_m^w([\boldsymbol{\phi}]).L_m(\boldsymbol{\phi}_p\}\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} [\boldsymbol{\phi}].\bar{\mathbf{f}}\,\mathrm{ds}$$

- This is very similar to least square method which also did not directly include weight functions but we concluded that for linear differential operators L_M and L_f it corresponded to weight functions of the form $\mathbf{w} = L_M(\phi)$ and $\mathbf{w}^f = -L_f(\phi)$.

Summary

Basically in the first approach (Ritz method) we first discretize then optimize while in the second approach (weak statement) we first optimize then discretize. The commutation of these operations are shown in the next slide.

Relation between Energy Method and Weak Statement

Energy method

$\mathit{\Pi}(\mathbf{u}) \le \mathit{\Pi}(\tilde{\mathbf{u}})$

$\mathbf{u} \in \mathcal{V} : \ \forall\, \tilde{\mathbf{u}} \in \mathcal{V}$

Stationary condition

Continuum weak statement

$\delta \mathit{\Pi}(\mathbf{u}) = 0 \quad \mathbf{w} = \delta \mathbf{u}$

$\mathbf{u} \in \mathcal{V} \ : \forall \mathbf{w} \in \mathcal{V}_0$

$$\int_{\mathcal{D}} L_m^w(\mathbf{w}) L_m(\mathbf{u}) \,\mathrm{dv} = \int_{\mathcal{D}} \mathbf{w}.\mathbf{r} \,\mathrm{dv} + \int_{\partial \mathcal{D}_f} \mathbf{w}.\bar{\mathbf{f}} \,\mathrm{ds}$$

example: 1D bar

$$\int_0^L w'(x) E A u'(x) \,\mathrm{d}x = \int_0^L w(x) q(x) \,\mathrm{d}x - w(L)\bar{F}$$

Descritization

Discrete weak statement $\mathbf{A}\mathbf{a} = \mathbf{b}$

$$\left\{ \int_{\mathcal{D}} L_m^w([\mathbf{w}]) L_m([\boldsymbol{\phi}])^{\mathrm{T}} \,\mathrm{dv} \right\} [\mathbf{a}] = \int_{\mathcal{D}} \left\{ [\mathbf{w}].\mathbf{r} - L_m^w([\mathbf{w}]).L_m(\boldsymbol{\phi}_p) \right\} \,\mathrm{dv} + \int_{\partial \mathcal{D}_f} [\mathbf{w}].\bar{\mathbf{f}} \,\mathrm{ds}$$

Descritization

$\mathit{\Pi}(\mathbf{u}^h) \le \mathit{\Pi}(\tilde{\mathbf{u}}^h)$

$\mathit{\Pi}(a_1, \ldots, a_n) \le \mathit{\Pi}(\tilde{a}_1, \ldots, \tilde{a}_n)$ for

$\mathbf{u}^h = a_1 \phi_1 + \cdots + a_n \phi_n + \phi_p$

$\tilde{\mathbf{u}}^h = \tilde{a}_1 \phi_1 + \cdots + \tilde{a}_n \phi_n + \phi_p$

Stationary condition

Ritz method

$\nabla_{\mathbf{a}} \mathit{\Pi} = \mathbf{0}$

$\frac{\partial \mathit{\Pi}}{\partial a_1} = \ldots = \frac{\partial \mathit{\Pi}}{\partial a_n} = 0$

Equivalent for Galerkin DWS *i.e.*, $\mathbf{w} = \phi$