04/08/2014-2

Tuesday, April 08, 2014
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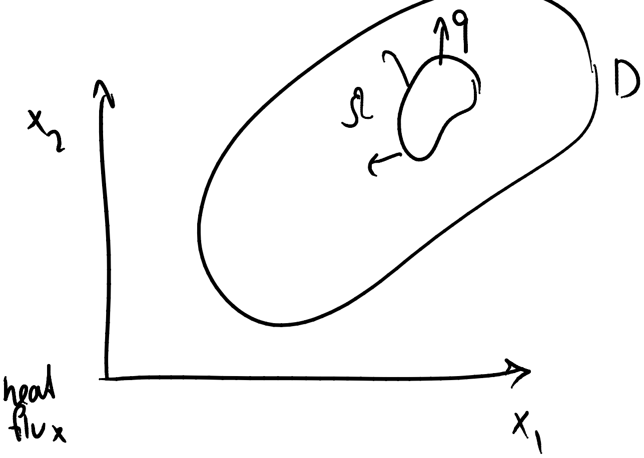
Heat Problem:



outward heat flux

$$-\int_{\partial\Omega} q\cdot n\, ds + \int_{\Omega} Q\, dv = 0$$

heat source

$$\int_{\partial\Omega} q\cdot n\, ds - \int_{\Omega} Q\, dv = 0$$ Balance law

$$\int_{\partial\Omega} q\cdot n\, ds = \int_{\partial\Omega} q_i n_i\, ds = \int_{\Omega} \overbrace{q_{i,i}}^{\nabla\cdot q}\, dv$$ Divergence theorem

$$\int_{\Omega} (\nabla\cdot q - Q)\, dv = 0$$ localization (Ω is arbitrary) $\Longrightarrow$

$$\boxed{\nabla\cdot q - Q = 0}$$ Strong form

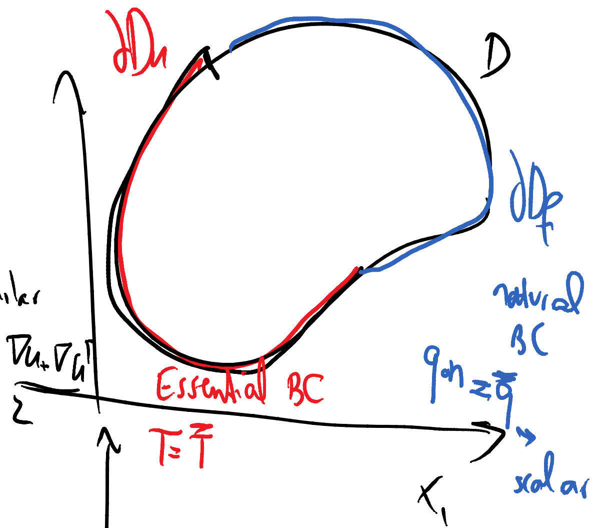
$[q]\cdot n = 0$ jump condition

static problem
we don't deal

$[q]\cdot n = 0$ ...

we don't deal with physical jump

$$\boxed{\nabla\cdot q - Q = 0}$$ strong form



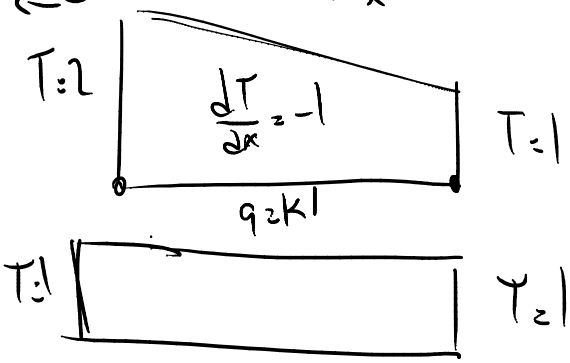
1 material properties k

constitutive relation

vector ← $q = -K \nabla T$ → vector

compare with ← $\sigma = C\,\varepsilon$ → matrix

∇T similar to $\varepsilon = \nabla u + \nabla u^T$



q vector $n_{dim} = 2$ $= -K$ ∇T vector $n_{dim} = 2$

K: $n_{dim} \times n_{dim}$ matrix

$K = k\,\mathbb{1}$ isotropic conductivity

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② weighted Residual

$R_i = \nabla\cdot q - Q$

$R_f = \bar{q} - q\cdot n$ on ∂D_f Natural B

$R_u = \bar{T} - T$ on ∂D_u

↳ typically statisfied strongly

strong satisfaction of Essential BC

Find $T \in V^{WR} = \{ u \mid \forall x \in \partial D_u \; u(x) = \bar{T}(x) \}$

$\ni \forall \omega$

$$\int_D \omega R_i + \int_{\partial D_f} \omega R_f = 0$$

$$\int_D \omega (\nabla \cdot q - Q) \, dV + \int_{\partial D_f} \omega (\bar{q} - n \cdot q) = 0$$

Divergence theorem

$$\int_D f_{i,i} \, dV = \int_{\partial D} f_i n_i \, dS$$

$$\int_D \omega q_{i,i} \, dV$$

$AB' = (AB)' - A'B$

$$\omega q_{i,i} = \underline{(\omega q_i)_{,i}} - \omega_{,i} q_i$$

∂D

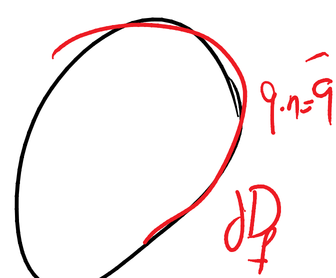
$$\int_D w(\nabla\cdot q - Q)\,dv + \int_{\partial D_f} w(\bar{q} - q\cdot n)\,ds = 0$$

$$\int_D w\, q_{i,i}\, dv = \int_D \left\{ (w q_i)_{,i} - w_{,i}\, q_i \right\} dv$$

$$= \int_D (w q_i)_{,i}\, dv - \int_D w_{,i}\, q_i\, dv$$

$$\int_{\partial D} w q_i \underline{n_i}\, dS - \int_D w_{,i}\, q_i\, dv$$

$$\int_D w(\nabla\cdot q - Q)\,dv + \int_{\partial D_f} w(\bar{q} - q\cdot n)\,ds$$



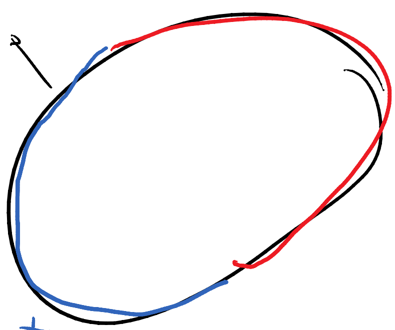
$$\int_D (-w_{,i}\, q_i - wQ)\,dv + \int_{\partial D} w q_i n_i\, ds + \int_{\partial D_f} w(\bar{q} - \underline{q_i n_i})\, ds = 0$$

$$+ \cancel{\int_{\partial D_f} w q_i n_i \, ds} + \int_{\partial D_u} w q_i n_i \, ds + \int_{\partial D_f} w \bar{q} \, ds$$

$$- \cancel{\int_{\partial D_f} w q_i n_i \, ds} = 0$$

$$\boxed{\int_{\Omega} \left(-w_{,i} q_i - w Q\right) dV + \int_{\partial D_u} w\, q_i n_i \, ds + \int_{\partial D_f} w \bar{q} \, ds = 0}$$

$q = ?$



we choose weight function to be zero @ ∂D_u

$q = -k \nabla T$ constitutive relation

Find $T \in V = \left\{ u \in \cancel{C}\, H^1 \;\Big|_{\partial D_u} u(x) = \bar{T}(x) \right\}$ (*)

$$\exists \; \forall w \in V_0 = \left\{ u \in \cancel{C^1} H^1 \;\middle|\; \forall x \in \partial D_u \;\; \underline{u(x) = 0} \right\}$$

in weak statement

$$\boxed{\int_D \nabla w \, k \, \nabla T = \int_D w Q - \int_{\partial D_f} w \bar{q}}$$

weight function satisfies homogeneous essential BC

3) Path to FE formulation

a) Discretization

$$T \rightarrow T^h = \sum_{i=1}^{n} a_i \phi_i(x) + \phi_p$$

discretized T

$w_i \; \ldots \; w_n$ the weight function

$$T^h \in V \qquad \phi_i \in V_0$$

$$\phi_p \in V$$

$$w_i \in V_0$$

$w_i \in V_0$

b) Galerkin method

$$V = \{ u \in H^1(D) \mid \forall x \in \partial D_u \quad u(x) = \bar{T}(x) \}$$

$$V_0 = \{ u \in H^1(D) \mid \forall x \in \partial D_u \quad u(x) = 0 \}$$

For $i = 1, \dots, n$
$$\left. \begin{array}{l} \phi_i \in V_0 \\ \phi_p \in V \end{array} \right\}$$

Find $a_1, \dots, a_n$ such that

$$T^h = \sum_{i=1}^{n} a_i \phi_i + \phi_p$$

satisfies

$$\int_D \nabla w \, k \, \nabla T^h = \int_D w Q - \int_{\partial D_f} w \bar{q} \, ds$$

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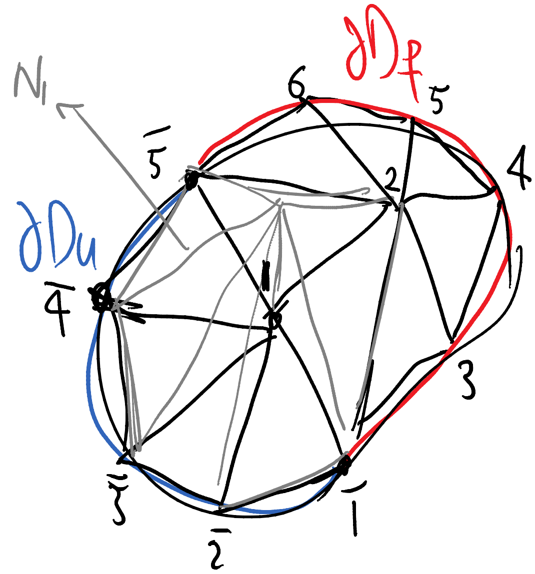
in FE we use N (shape function) instead

In FE we use N (shape function) instead of ϕ (trial function)



6x6 system

Shape functions take the value of 1 at one d.o.f and zero elsewhere.
(Not node)

For this problem there is 1 dof per node

So it happens that shape functions take the value of 1 at 1 node and zero elsewhere

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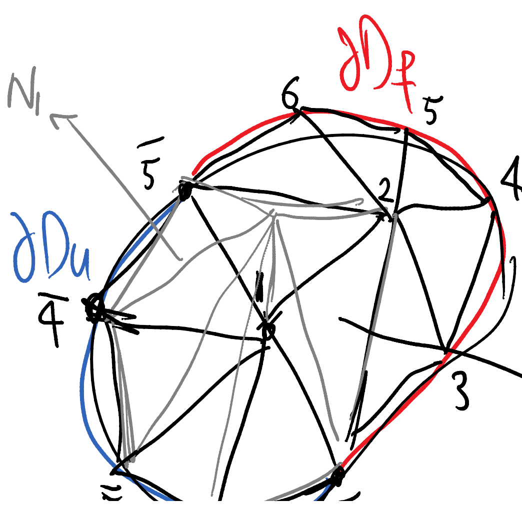
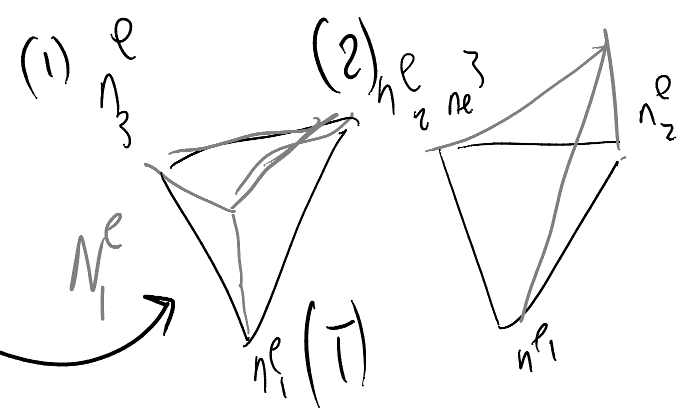
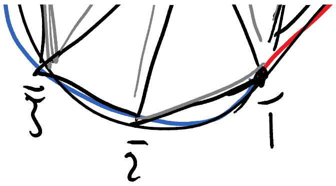
Globally

local @ element level



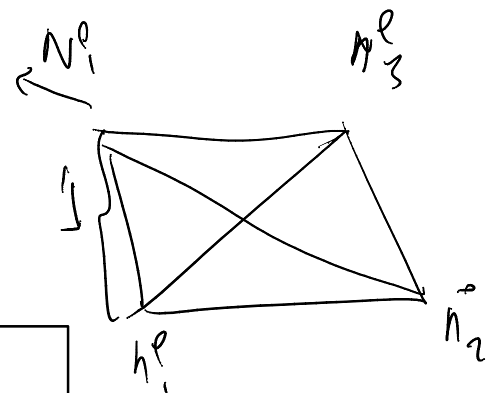




For mulas for element stiffness & forces:

$$f_r^e = \int N^T Q \qquad f_N^e = \int_{\partial D f} N^T \bar{q}\, ds \qquad f_D^e = k a^e$$

$$\int_D \nabla w\, k\, \nabla T^h = \int_D w Q - \int_{\partial D f} w \bar{q}\, ds \qquad f^e = f_r^e + f_N^e + f_D^e$$



$$N = [N_1 \quad N_2 \quad N_3]$$

$$T^h = N^e a^e$$

$$\boxed{\begin{gathered} k^e = \int_e (\nabla N)^T k_{3\times 3} \nabla N_{3\times 1}\, dv \\ k^e = \int_e B^T D B\, dv \\ B = \nabla N \qquad D = k \end{gathered}}$$

$$N = [N_1 \quad N_2 \quad N_3] \implies B = \nabla N = \begin{bmatrix} N_{1,x} & N_{2,x} & N_{3,x} \\ N_{1,y} & N_{2,y} & N_{3,y} \end{bmatrix}$$

$\quad {}_{1,y} \quad {}_{2,y} \quad {}_{3,y}]$

$$
K^e = \int \begin{bmatrix} N_{1,x} & N_{1,y} \\ N_{2,x} & N_{2,y} \\ N_{3,x} & N_{3,y} \end{bmatrix} k \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} N_{1,x} & N_{2,x} & N_{3,x} \\ N_{1,y} & N_{2,y} & N_{3,y} \end{bmatrix} dV
$$