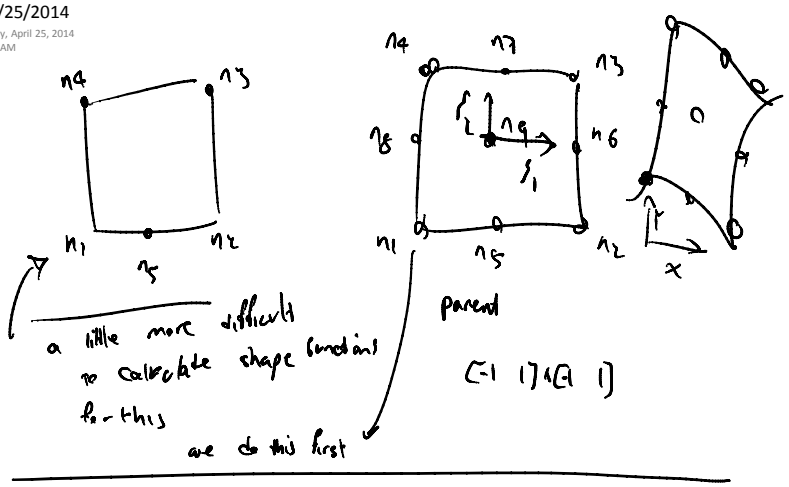
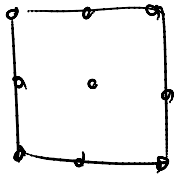


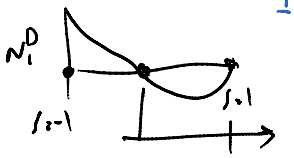

simple product of 1D shape functions



Formula

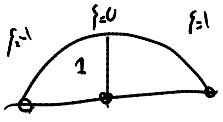
there are poly nomials that are 1 @ one point zero at other points. Use Lagrange polynomials

1D



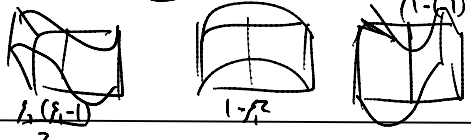

$$N_1^D = \frac{(\xi-(-1))(\xi-0)(\xi-1)}{(-1-(-1))(-1-0)(-1-1)} = \frac{\xi(\xi-1)}{2}$$

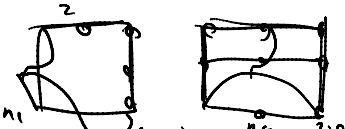
1D ← $N_2^D$




$$N_2^D = \frac{(\xi-(-1))(\xi-1)}{(0-(-1))(0-1)} = 1-\xi^2$$

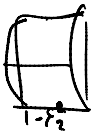
$$\frac{(\xi+1)(\xi-0)}{(1-(-1))(1-0)} = \frac{\xi(\xi+1)}{2}$$



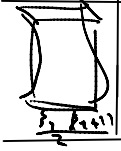




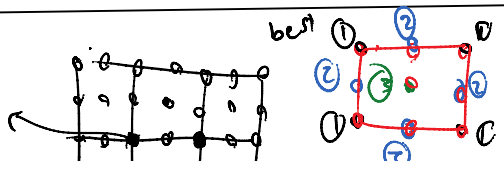


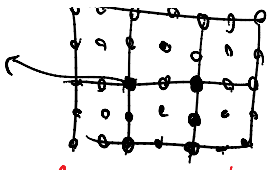


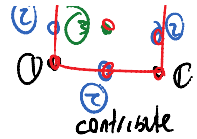
bubble shape function zero around the element




best







contribute less to computational costs

why are dof at element corners better than those on the edges and why those on the edges better than the center one

corner nodes bad

$$\text{average dof / element} = \underbrace{4 \times \frac{1}{4}}_{1} + \underbrace{4 \times \frac{1}{2}}_{2} + 1 \times \frac{1}{1} = 4$$

4 nodes ≡ 1 dof  4 nodes 2 dof  1 node 1 dof

---

How to get rid of "internal" dofs

like $N_9$ here

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \end{bmatrix} = \begin{bmatrix} \underbrace{K_{ee}}_{8\times 8} & \underbrace{K_{ei}}_{8\times 1} \\ \underbrace{K_{ie}}_{1\times 8} & \underbrace{K_{ee}}_{1\times 1} \end{bmatrix} \begin{bmatrix} \vdots \\ \vdots \\ T_9 \end{bmatrix} \begin{matrix} T_e \\ T_i \end{matrix}$$

external

internal

$$F_e = K_{ee} a_e + K_{ei} a_i$$

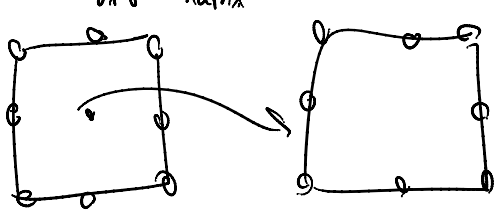
$$F_i = K_{ie} a_e + K_{ii} a_i$$

if it's zero $\quad a_i = -K_{ii}^{-1} K_{ia} a_e$

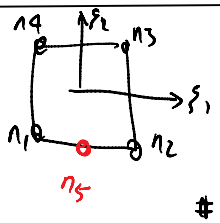
$$\boxed{F_e = (K_{ee} - K_{ei} K_{ii}^{-1} K_{ia}) a_e}$$

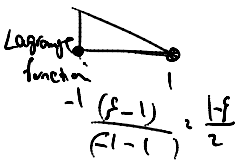
8x8 matrix

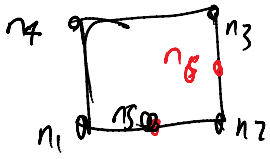
Condensation



$\frac{\text{average dof}}{\text{element}} = 4$  $\frac{\text{average dof}}{\text{element}} = 3$



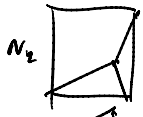



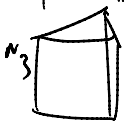

# nodes

n=4 n=5 n=6

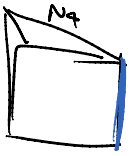
$\frac{(1-\xi_1)(1-\xi_2)}{4}$



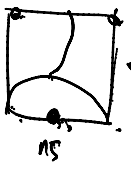

$\frac{(1+\xi_1)(1-\xi_2)}{4}$




$\frac{(1+\xi_1)(1+\xi_2)}{4}$

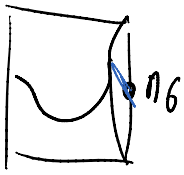



$\frac{(1-\xi_1)(1+\xi_2)}{4}$




$\frac{(\xi_2-0)(\xi_2-1)}{(-1-0)(-1-1)}$

$\frac{(\xi_1-\xi_1)(\xi_1-1)}{(0-(-1))(0-1)}$



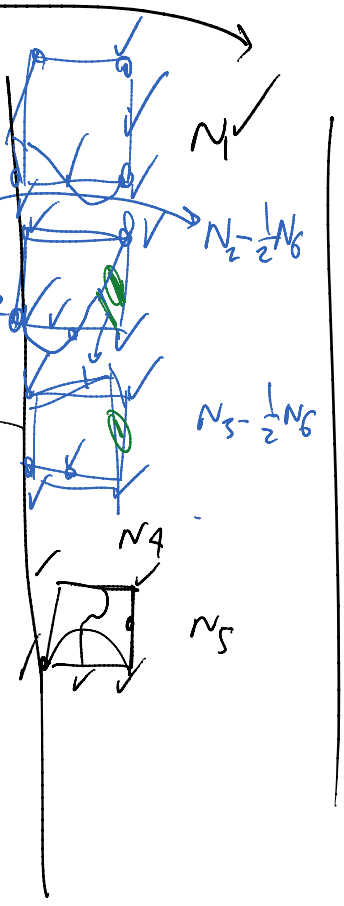
$\frac{(1+\xi_1)^2 \xi_2(\xi_2-1)}{2}$

$\frac{\xi_1(\xi_1+1)(1-\xi_2)}{2}$

$N_1 \to N_1 - \frac{1}{2}N_5$

$N_2 \to N_2 - \frac{1}{2}N_5$ new $N_2$

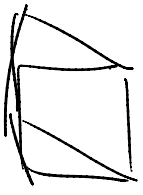
$N_3$

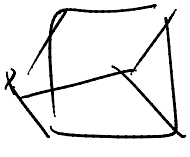



$N_1$

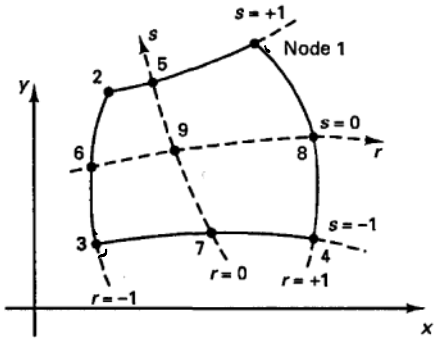
$N_2 - \frac{1}{2}N_6$

$N_3 - \frac{1}{2}N_6$





$N_1 = \frac{(1-\xi_1)}{2}$ $\frac{(1-\xi_2)}{2}$




(a) 4 to 9 variable-number-nodes two-dimensional element

Include only if node $i$ is defined

| | | $i=5$ | $i=6$ | $i=7$ | $i=8$ | $i=9$ |
|---|---|---|---|---|---|---|
| $h_1 =$ | $\frac{1}{4}(1+r)(1+s)$ | $-\frac{1}{2}h_5$ | | | $-\frac{1}{2}h_8$ | $-\frac{1}{4}h_9$ |
| $h_2 =$ | $\frac{1}{4}(1-r)(1+s)$ | $-\frac{1}{2}h_5$ | $-\frac{1}{2}h_6$ | | | $-\frac{1}{4}h_9$ |
| $h_3 =$ | $\frac{1}{4}(1-r)(1-s)$ | | $-\frac{1}{2}h_6$ | $-\frac{1}{2}h_7$ | | $-\frac{1}{4}h_9$ |
| $h_4 =$ | $\frac{1}{4}(1+r)(1-s)$ | | | $-\frac{1}{2}h_7$ | $-\frac{1}{2}h_8$ | $-\frac{1}{4}h_9$ |
| $h_5 =$ | $\frac{1}{2}(1-r^2)(1+s)$ | | | | | $-\frac{1}{2}h_9$ |
| $h_6 =$ | $\frac{1}{2}(1-s^2)(1-r)$ | | | | | $-\frac{1}{2}h_9$ |
| $h_7 =$ | $\frac{1}{2}(1-r^2)(1-s)$ | | | | | $-\frac{1}{2}h_9$ |
| $h_8 =$ | $\frac{1}{2}(1-s^2)(1+r)$ | | | | | $-\frac{1}{2}h_9$ |
| $h_9 =$ | $(1-r^2)(1-s^2)$ | | | | | |

(b) Interpolation functions

From Bathe's book slightly different numbering scheme (start from $\xi_1 = \xi_2 = 1$ instead of $\xi_1 = \xi_2 = -1$)

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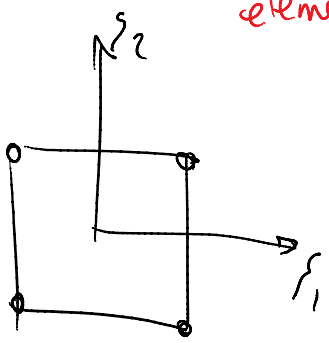
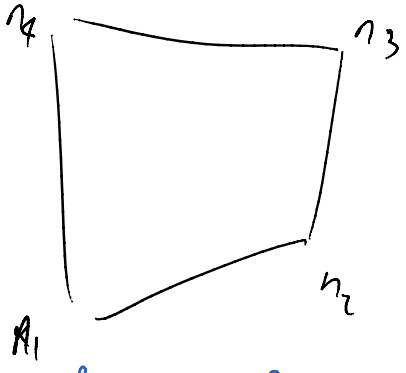
# Subparametric, Isoparametric, Superparametric elements




what where (2) uses of shape functions

1. interpolating the solution

$$T^h = [N_1(\vec{\xi}) \; N_2(\vec{\xi}) \; N_3(\vec{\xi}) \; N_4(\vec{\xi})] \begin{bmatrix} a_1 \\ a_4 \\ a_5 \\ a_9 \end{bmatrix}$$

9

2.

$$K = \int_A B^T D B \, dA \qquad dA = |J| \, d\Delta_\xi$$

$B = L_x N$

$\xi \rightarrow x$

e.g. $\nabla_x N(\xi)$

$$\vec{x} = N^i(\xi) \vec{\xi}^i$$

$$x = N^1(\xi_1,\xi_2) x^1 + N^2(\xi_1,\xi_2) x^2 + N^3(\xi_1,\xi_2) x^3 + N^4(\xi_1,\xi_2) x^4$$

$$y = N^1(\xi_1,\xi_2) y^1 + N^2(\xi_1,\xi_2) y^2 + N^3(\xi_1,\xi_2) y^3 + N^4(\xi_1,\xi_2) y^4$$

(2) $\xi \rightarrow x$ we use shape functions

---

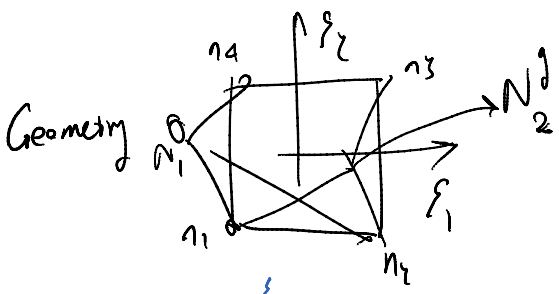
(1) SUPERPARAMETRIC

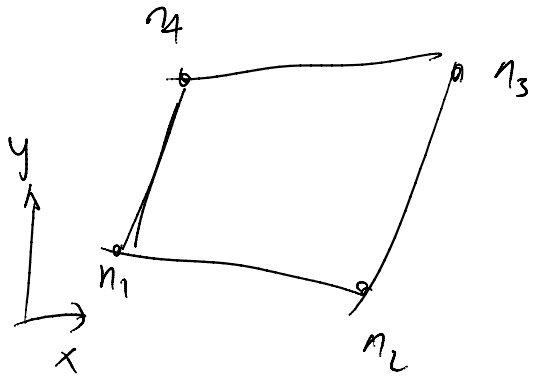
solution interpolated with higher order $N$'s

solution order > geom order

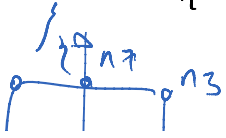
Example

Geometry

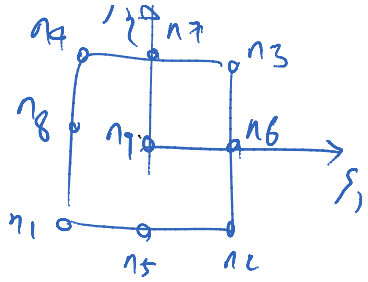



solution



Solution



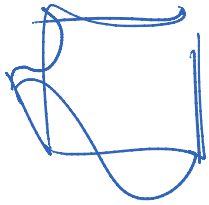

$$T^h = [N_1(\xi_1,\xi_2) \quad N_2 \quad N_3 \quad \cdots \quad N_9] \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_9 \end{bmatrix}$$

9 dof's per element

$$N_1(\xi_1,\xi_2) = \frac{\xi_1(\xi_1-1)\,\xi_2(\xi_2-1)}{4}$$

2nd order in $\xi_1$

" " " $\xi_2$



$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_9}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots & \frac{\partial N_9}{\partial y} \end{bmatrix}$$

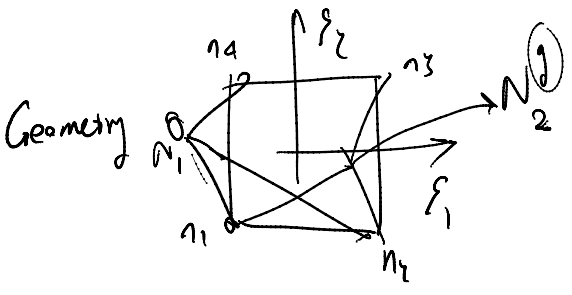
2 × 9 matrix

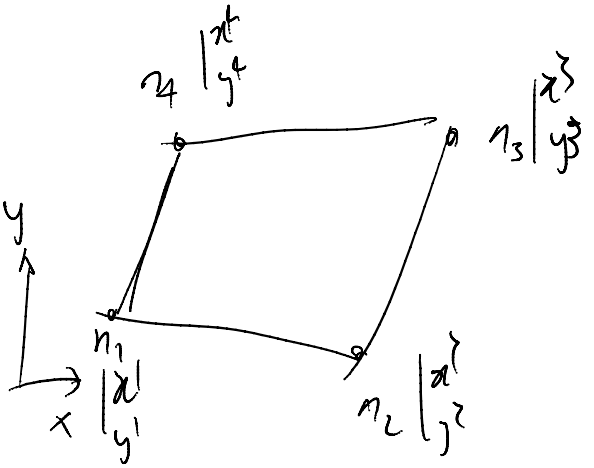
N's are functions of $\xi_1, \xi_2$

we need a map between $\xi$ & x.

How do form the map between x & ξ?

Geometry is interpolated with lower order shape functions



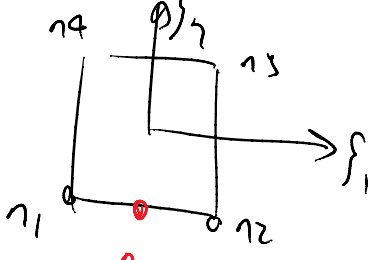



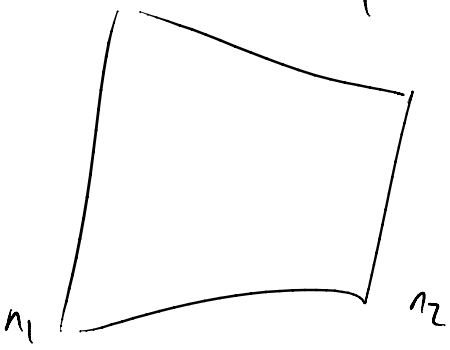

$$x = x(\xi,\eta) = x^1 N^1(\xi,\eta) + x^2 N^2(\xi,\eta) + x^3 N^3(\xi,\eta) + x^4 N^4(\xi,\eta)$$

$$x = X(\xi_1, \xi_2) = x^1 N^{1g}(\xi_1, \xi_2) + x^2 N^{2g}(\xi_1, \xi_2) + x^3 N^{3g}(\xi_1, \xi_2) + x^4 N^{4g}(\xi_1, \xi_2)$$

$$= x^1 \frac{(1-\xi_1)(1-\xi_2)}{4} + x^2 \frac{(1+\xi_1)(1-\xi_2)}{4} + x^3 \frac{(1+\xi_1)(1+\xi_2)}{4} + x^4 \frac{(1-\xi_1)(1+\xi_2)}{4}$$

What are $x, y$ for $\eta_5$



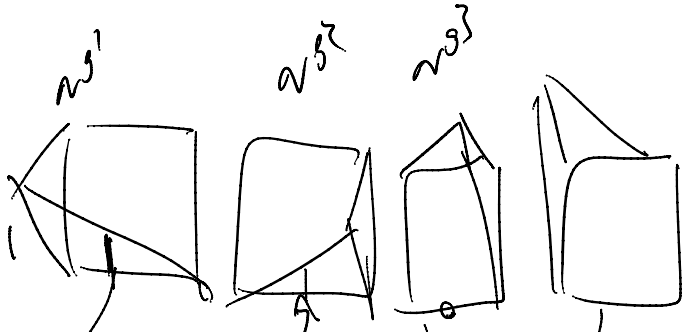




$\eta_5$

↓

not used for geometry interpolation that we don't have $N^{g5}$

$\eta_5$ | 0, -1



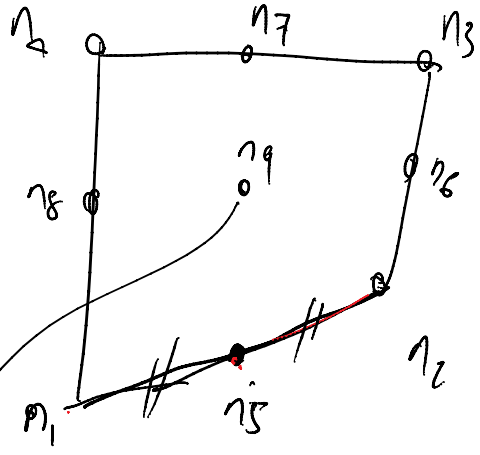

$\xi_1 = 0 \quad \xi_2 = -1$

$$x^5 = x^1 N^{g1}(\eta_5) + x^2 N^{g2}(\eta_5) + x^3 N^{g3}(\eta_5) + x^4 N^{g4}(\eta_5)$$

$$x^1 \tfrac{1}{2} + x^2 \tfrac{1}{2}$$

$$x^5 = \frac{x^1 + x^2}{2}$$

$$y^5 = \frac{y^1 + y^2}{2}$$




$$x^9 = \frac{x^1 + x^2 + x^3 + x^4}{4}$$

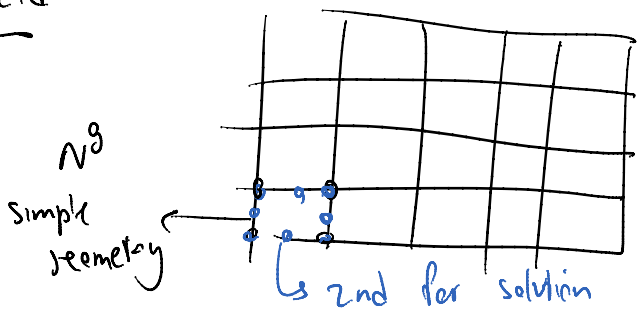
Where do we use these element.

here $N^h$ Solution is 2nd order in $\xi_1, \xi_2$

$N^g$ Geometry is 1st order "

"Geometry is less accurately interpolated."

If you have Simple geometry but rich solution field

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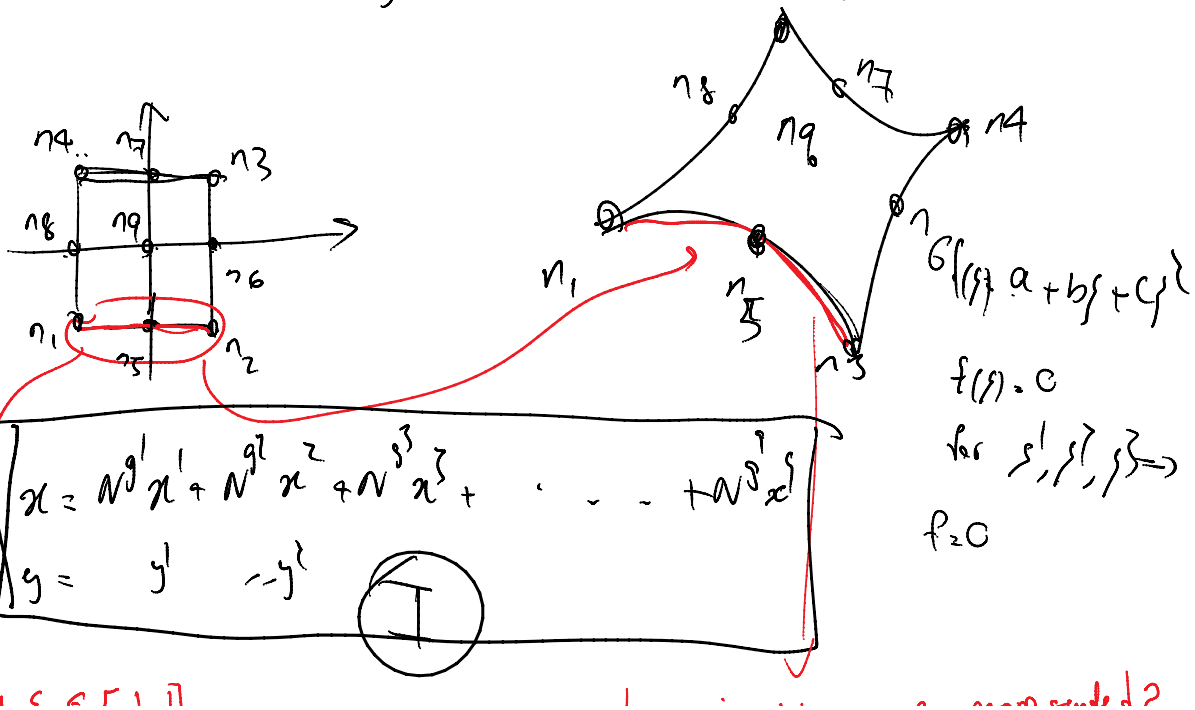
## 2. SUBPARAMETRIC

geometry order < solution order

Geometry is more accurately represented



$\xi_1 \in [-1, 1]$

$\xi_2 = -1$

how is this curve represented?

$x = X(\xi_1)$

$y = Y(\xi_1)$

↓

parametric representation of the line

$$x = x^1 N^1(\xi_1, \xi_2) + x^2 N^2(\xi_1, \xi_2) + x^3 N^3(\xi_1, \xi_2) + \cdots$$

$\xi_2 = -1$

$N_3, N_4, N_6, N_7, N_8, N_9$ are zero on $L$

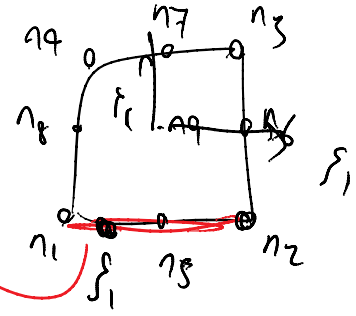
$N^1(\xi_1, -1)$

$N^2(\xi_1, -1)$

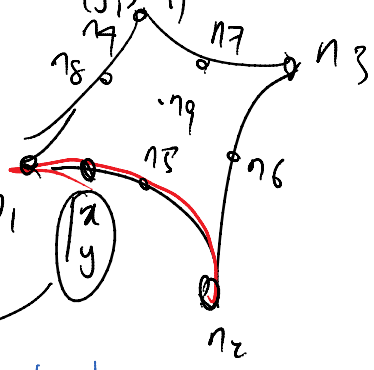
$$x = \frac{\xi_1(\xi_1 - 1)}{2} x^1 + \frac{\xi_1(1+\xi_1)}{2} x^2 + (1 - \xi_1^2) x^5$$

$$y = \frac{\xi_1(\xi_1 - 1)}{2} y^1 + \frac{\xi_1(1+\xi_1)}{2} y^2 + (1 - \xi_1^2) y^5$$

on the line




$N^5(\xi_1, -1)$

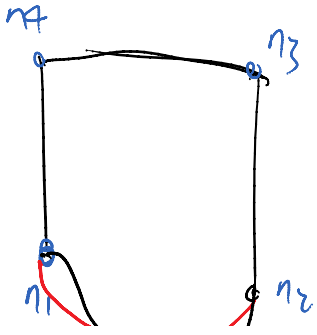



parametric representation of the boundary

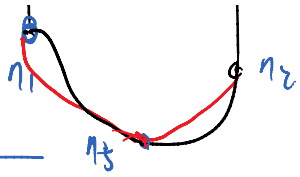
1. would the element boundary pass through $\eta_1, \eta_5, \eta_2$?

obviously yes!

2. would the element geometry perfectly match domain boundary?



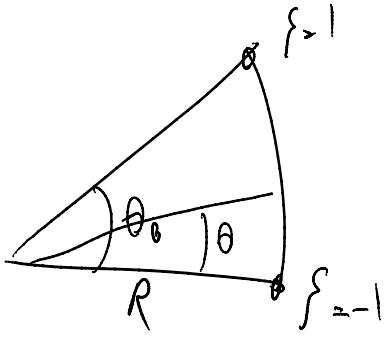
match domain boundary?




we choose $\eta_3$ on the actual domain boundary

No if the boundary doesn't have second order parametric representation of $\xi_1, \xi_2$ we cannot represent it exactly (we can show that this second order parametric representation can EXACTLY express parabolic boundaries)

with these elements can we exactly capture circular arcs? No.



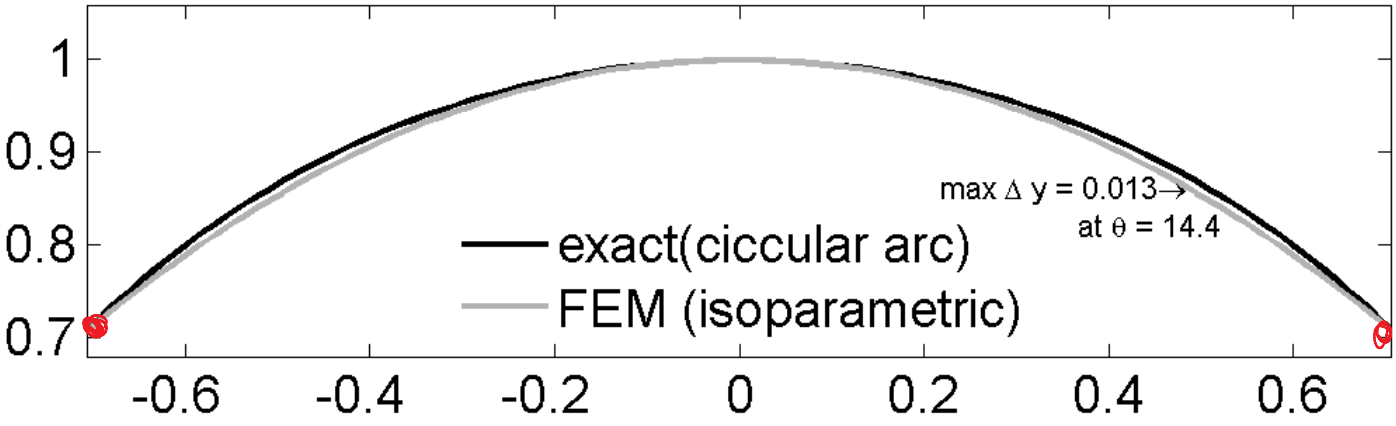

$$\xi = 2\frac{\theta}{\theta_0} - 1 \qquad \xi \in [-1,1] \Rightarrow \theta = \frac{\theta_0}{2}(\xi+1)$$

$$x = R\cos\theta = R\cos\frac{\theta_0}{2}(\xi+1)$$

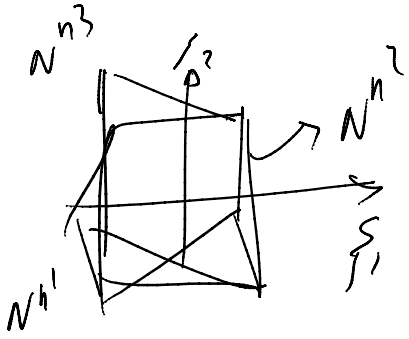
$$y = R\sin\frac{\theta_0}{2}(\xi+1)$$

This figure shows an example were 2nd order Q4 attempts to captures circular arc domain boundary but capturing 3 points. As can be there is very little error between exact (Circular) and FEM boundary.

and FEM boundary




How about the solution $N^h$




$$T^h = \begin{bmatrix} N^{h1} & N^{h2} & N^{h3} & N^{h4} \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

$$B = \begin{bmatrix} \frac{\partial N^{h1}}{\partial x} & \frac{\partial N^{h2}}{\partial x} & \cdots & \frac{\partial N^{h4}}{\partial x} \\ \frac{\partial N^{h2}}{\partial y} & \frac{\partial N^{h2}}{\partial y} & \cdots & \frac{\partial N^{h4}}{\partial y} \end{bmatrix}$$

what is the size of $k^e$

$$k^e = \int B^T K B \, dA$$

↓

4x4 because we have 4 $N^h$'s

— how do we form B

$$B = J^{-1} \begin{bmatrix} \frac{\partial N_1^h}{\partial \xi_1} & \frac{\partial N_2^h}{\partial \xi_1} & | & | \\ \frac{\partial N_1^h}{\partial \xi_2} & \frac{\partial N_2^h}{\partial \xi_2} & | & | \end{bmatrix}$$

$$N = \begin{bmatrix} \frac{(1-\xi_1)(1-\xi_2)}{4} & \frac{(1+\xi_1)(1-\xi_2)}{4} & \frac{(1+\xi_1)(1+\xi_2)}{4} & \frac{(1-\xi_1)(1+\xi_2)}{4} \end{bmatrix}$$

→ where do we get $J$

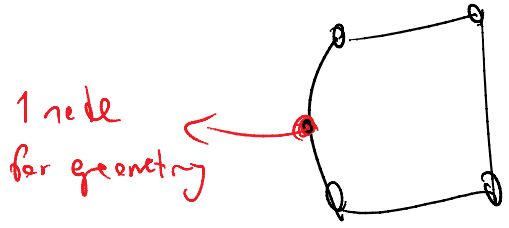
use equation (I) to calculate

$$J = \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} \end{bmatrix}$$

Where do we use subparametric elements?

Higher order geometry than than the solution?

complex geometry, simple solution



Generally we use isoparametric elements

$N^e$, $N^h$