Discontinuous versus Continuous Galerkin Finite Element Methods for Dynamic Problems

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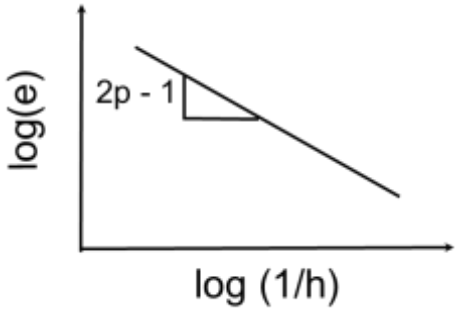
Comparison Criteria

- Convergence Rates: *error* (**e**) vs. *element size* (**h**)



- Solution Scaling: *cost* (CPU & wall **time**, FLOP count) vs *no. of elements* (**N**)

- **Efficiency: cost vs. error**
- Memory usage
- Software Engineering aspects: e.g. ease of implementation and extensibility

Numerical Method and Problem Types

- Underlying mathematical model:
  - Linear versus Non-Linear
  - PDE type: Elliptic, Parabolic and Hyperbolic
- **Solution features**: Sharp discontinuities, Shocks, Singularities, etc.
- Specifics of the numerical method:
  - Formulation type (single-field and multi-field formulations)
  - Element type
  - Integration scheme
- **Adaptive implementations**: *h*- and *hp*-adaptivity
- **Parallel computing**

Problem statement

- PDE: Linearized elastodynamics for 1D-3D (hyperbolic equation)
- Exact solution: Infinitely smooth trigonometric solutions; *e.g.* for 3D:

$$
\begin{aligned}
u_0 &= A_0 \sin(m_0 x_0) \sin(m_1 x_1) \sin(m_2 x_2) \sin(\alpha t), \\
u_1 &= A_1 \cos(m_0 x_0) \cos(m_1 x_1) \sin(m_2 x_2) \sin(\alpha t), \\
u_2 &= A_2 \cos(m_0 x_0) \sin(m_1 x_1) \cos(m_2 x_2) \sin(\alpha t),
\end{aligned}
$$

- Numerical methods:
  - Continuous Galerkin Finite Method with various time integration schemes: Implementations are in deal.ii:

    http://www.dealii.org/
  - Spacetime discontinuous Galerkin Finite Element Method

Time Marching Schemes
Structural Dynamics: $\mathrm{M}\ddot{u} + Ku = f(t,u)$

- Backward Euler: Implicit, first order accurate

$$\dot{U}^{n+1} = \frac{U^{n+1} - U^n}{\Delta t}, \quad \ddot{U} = \frac{U^{n+1}}{(\Delta t)^2} - \frac{U^n}{(\Delta t)^2} - \frac{\dot{U}^n}{\Delta t}$$

- Bathe's Method (2007): Implicit, second order accurate
  - Combines trapezoidal rule step with a backward difference formula

$$\dot{U}^{n+1} = \frac{1}{\Delta t}\left(3U^{n+1} - 4U^{n+1/2} + U^n\right), \quad \ddot{U}^{n+1} = \frac{1}{\Delta t}\left(3\dot{U}^{n+1} - 4\dot{U}^{n+1/2} + \dot{U}^n\right)$$

- Newmark Methods:

$$\begin{aligned} U^{n+1} &= U^n + (\Delta t)\dot{U}^n + \frac{(\Delta t)^2}{2}\left\{(1-2\beta)\ddot{U}^n + 2\beta\ddot{U}^{n+1}\right\} \\ \dot{U}^{n+1} &= \dot{U}^n + (\Delta t)\left\{(1-\gamma)\ddot{U}^n + \gamma\ddot{U}^{n+1}\right\} \end{aligned}$$

  - Average Acceleration : Unconditionally stable, second order accurate
    - $\beta = 1/4,\ \gamma = 1/2$
  - Linear Acceleration: Unconditionally stable, second order accurate
    - $\beta = 1/6,\ \gamma = 1/2$

Time Marching Schemes
Structural Dynamics: $\mathrm{M}\ddot{u} + Ku = f(t,u)$

- Generalized α-Methods: Unconditionally stable, second order accurate

$$M\left[(1-\alpha_m)\ddot{U}^{n+1}+\alpha_m\ddot{U}^n\right]+K\left[(1-\alpha_f)U^{n+1}+\alpha_f U^n\right]=(1-\alpha_f)F^{n+1}+\alpha_f F^n$$

$$\alpha_m<\alpha_f\le\frac{1}{2},\quad \gamma=\frac{1}{2}-\alpha_m+\alpha_f,\quad \beta_n\ge\frac{1}{4}+\frac{1}{2}(\alpha_f-\alpha_m)$$

  - Hilber-Hughes-Taylor (HHT) α-Method $\{\alpha_m = 0\}$

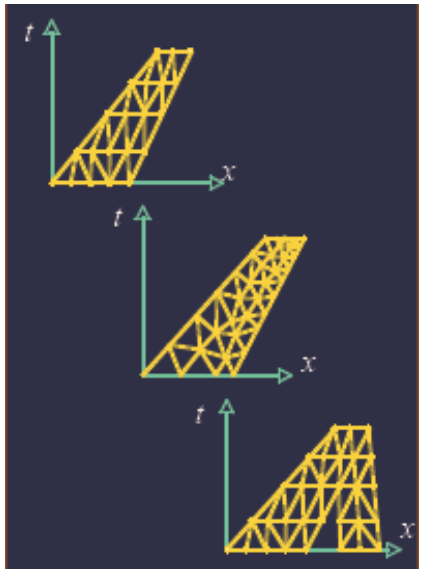
$$\alpha_f\in\left[0,\frac{1}{3}\right],\quad \gamma=\frac{1+2\alpha_f}{2},\quad \beta=\frac{(1+\alpha_f)^2}{4}$$

  - Wood-Bossak-Zienkiewicz (WBZ) α-Method $\{\alpha_f = 0\}$

$$\alpha_m\in\left[0,\frac{1}{3}\right],\quad \gamma=\frac{1+2\alpha_m}{2},\quad \beta=\frac{(1+\alpha_m)^2}{4}$$

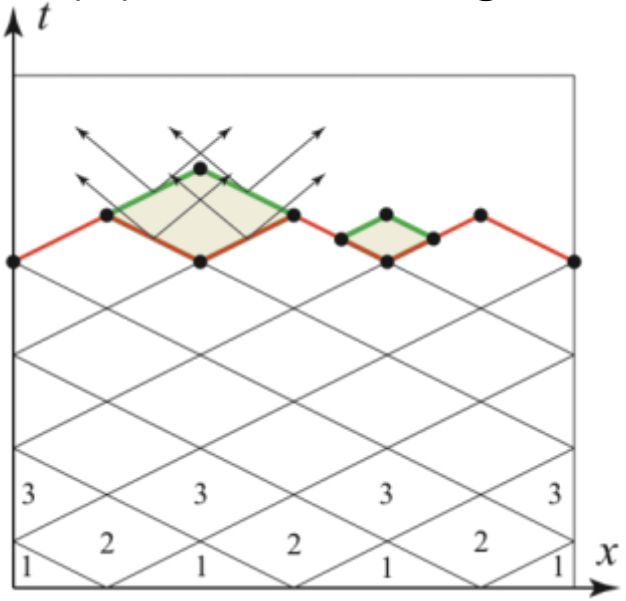
Spacetime Discontinuous Galerkin Method

- Discontinuous basis functions
- Direct discretization of spacetime



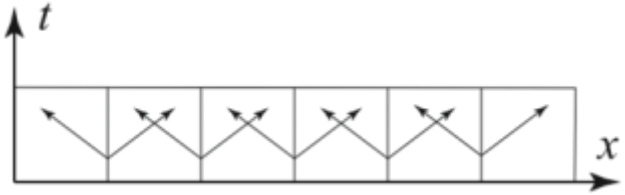
DG + spacetime meshing +
causal meshes for hyperbolic problems:

- Local solution property
- O(N) solution scaling



SDG

Extruded meshes impose a
global artificial coupling

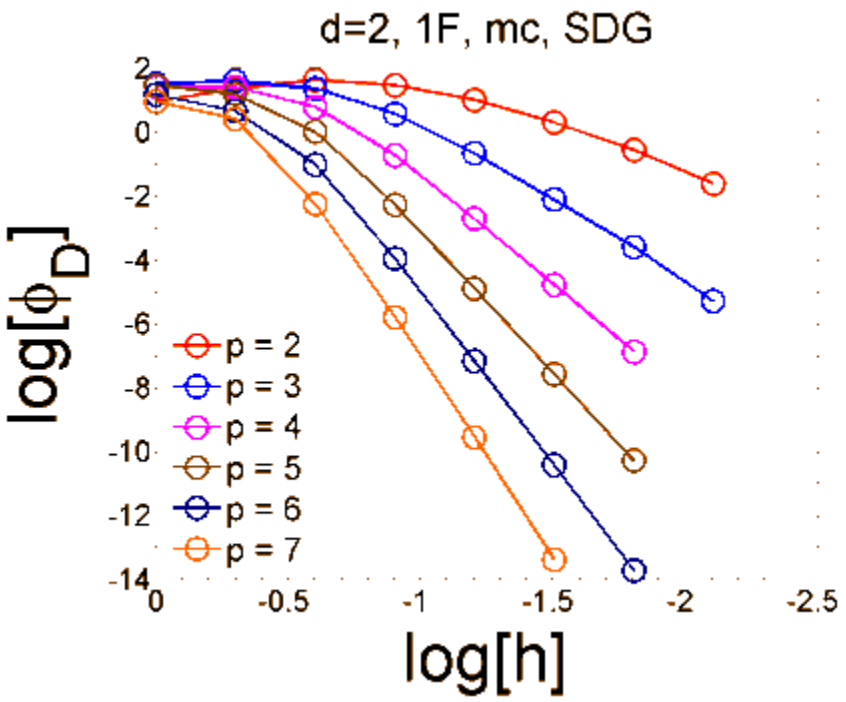


Time marching methods

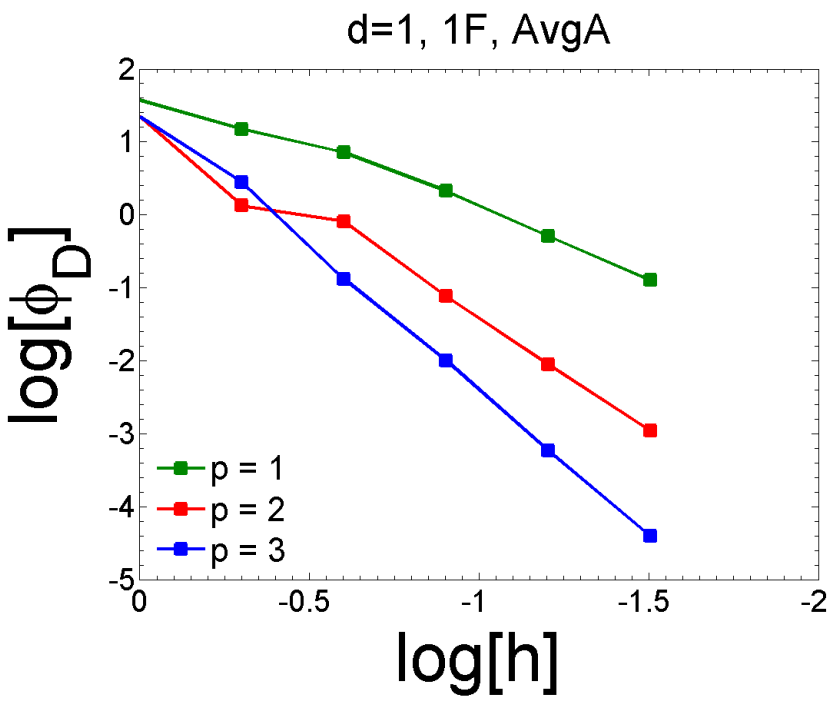
FEM comparisons

Convergence Rate

SDG



CFEM, Average α

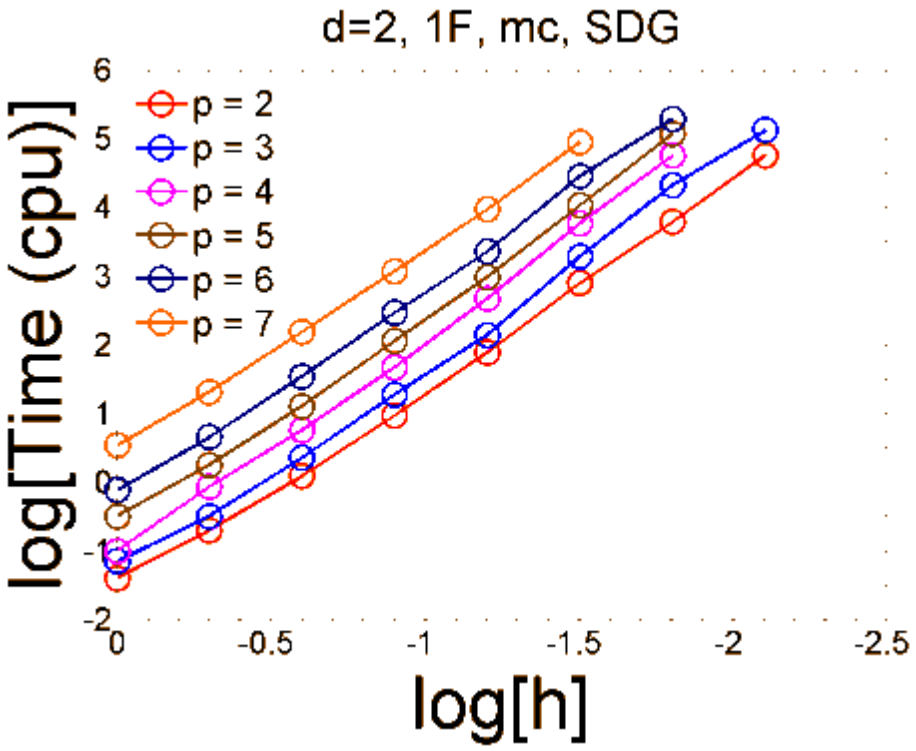


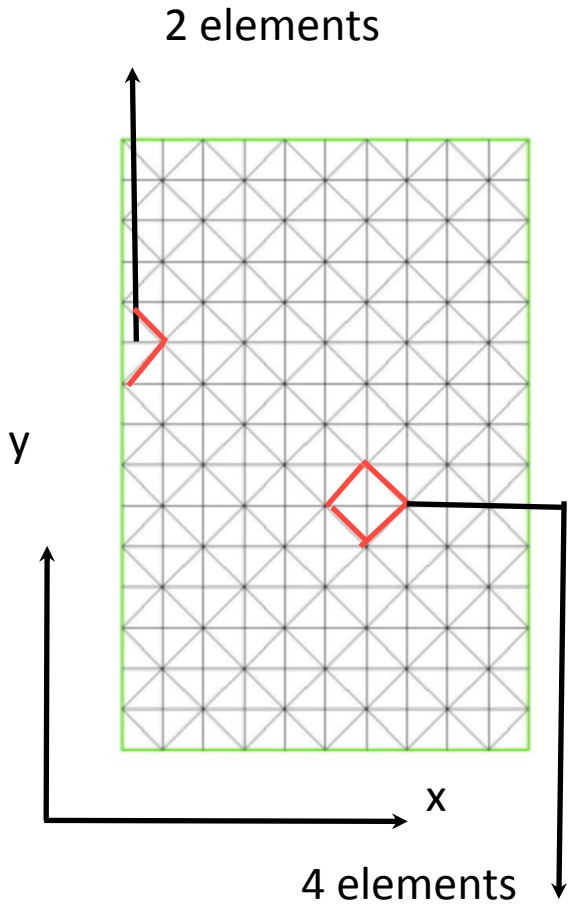
$$e = Ch^{ap+b} \quad \Rightarrow \log(e) = log(C) - (ap+b)\log(1/h)$$

a and b depend on both spatial and temporal discretization and integration schemes

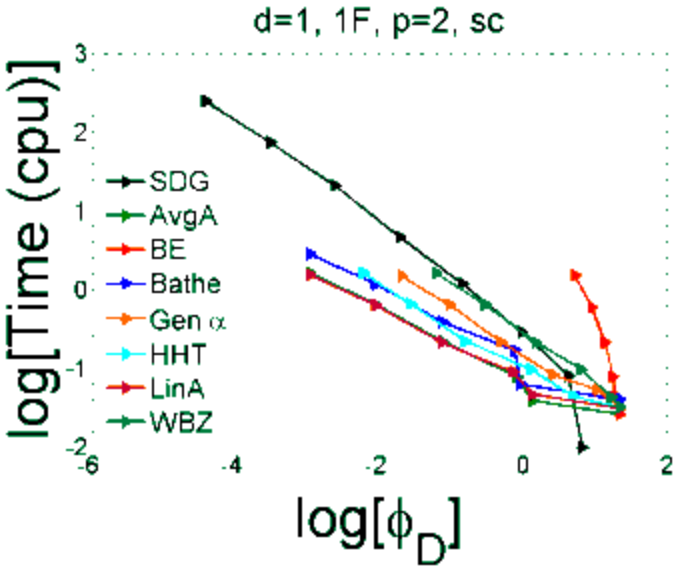
Solution Scaling

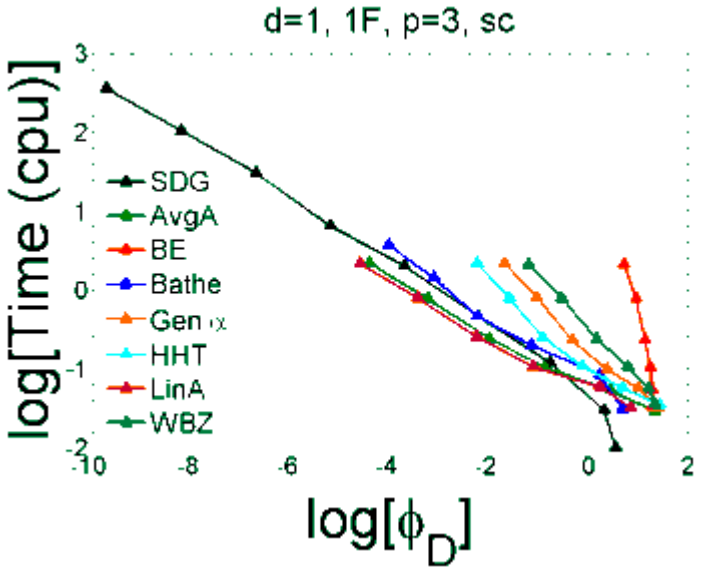
SDG: Solution scales versus number of elements

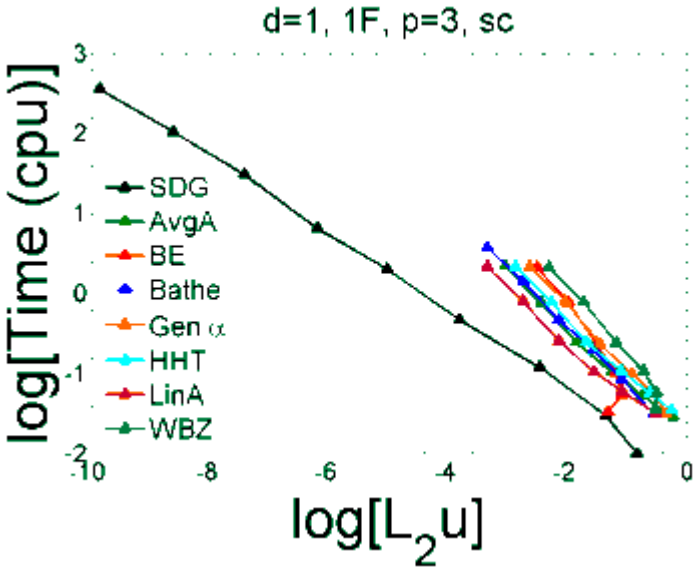




Efficiency



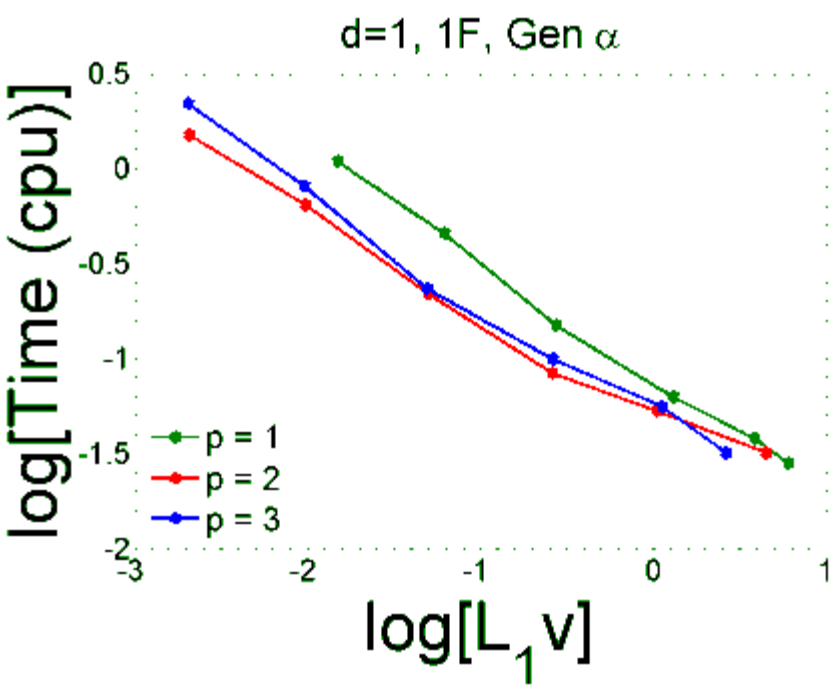


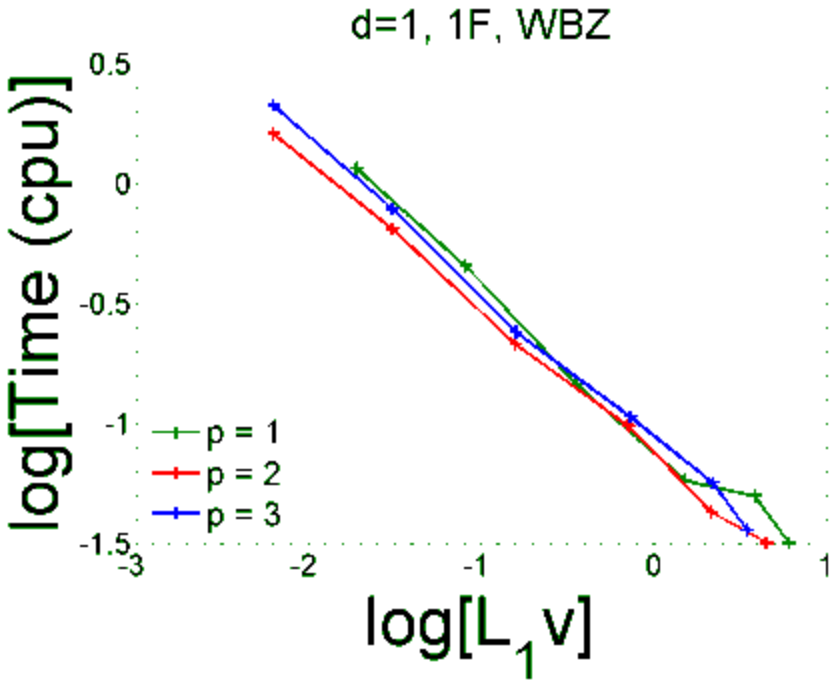


- Average and Linear a are the most efficient in time marching schemes.
- SDG is less efficient that CFEM time marching, but as p increases SDG becomes more efficient.
- SDG method can exhibit higher convergence rate as p increases.

Other criteria

1. Order of accuracy in time

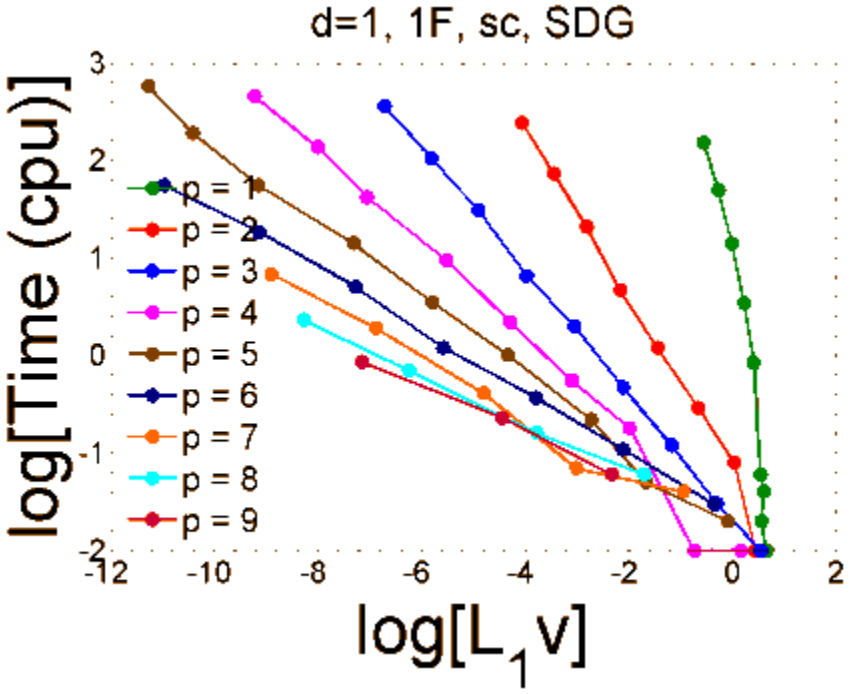


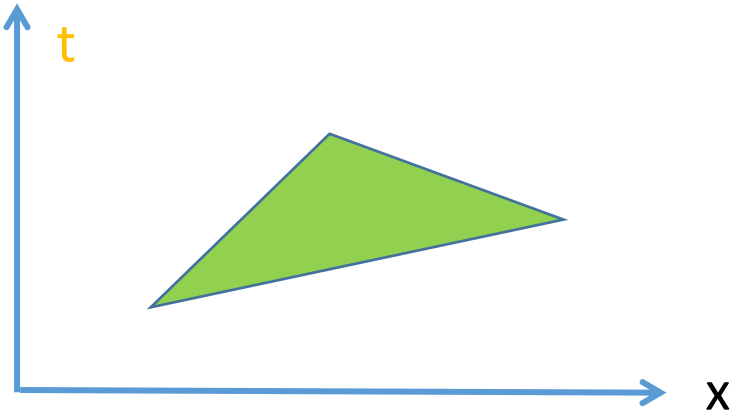


$$e = O\left(h^{ap+b}\right), \quad \text{spatial}$$
$$e = O(\Delta t^n) \quad \text{termporal}$$

- Eventually order of integration in time can limit the convergence rate.
- The efficiency of the method may decrease or not change by increasing *p*.

1. Order of accuracy in time (SDG)



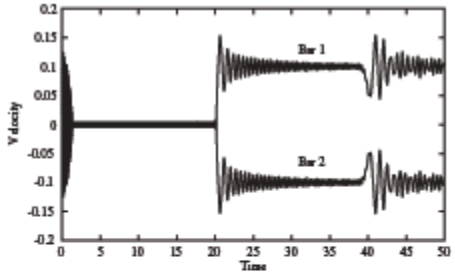


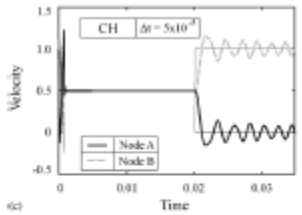
$$u = a_0 1 + a_1 x + a_2 t + a_3 x^2 + a_4 x t + a_5 t^2 + \ldots$$

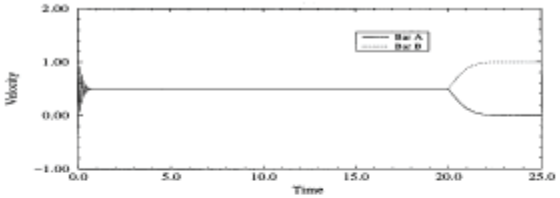
- The solution is simultaneously interpolated by finite elements in space & time.
- Order of convergence and efficiency increase as p increases.

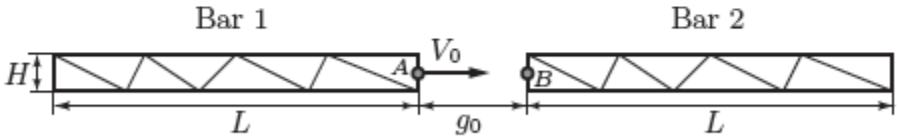
2. Discontinuities & high gradients in solution

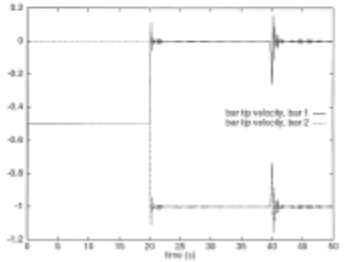
Benchmark problem [Hughes (76);Laursen, Chawla (97); Czekanski, Meguid (01); Cirak, West (05); *etc.*]



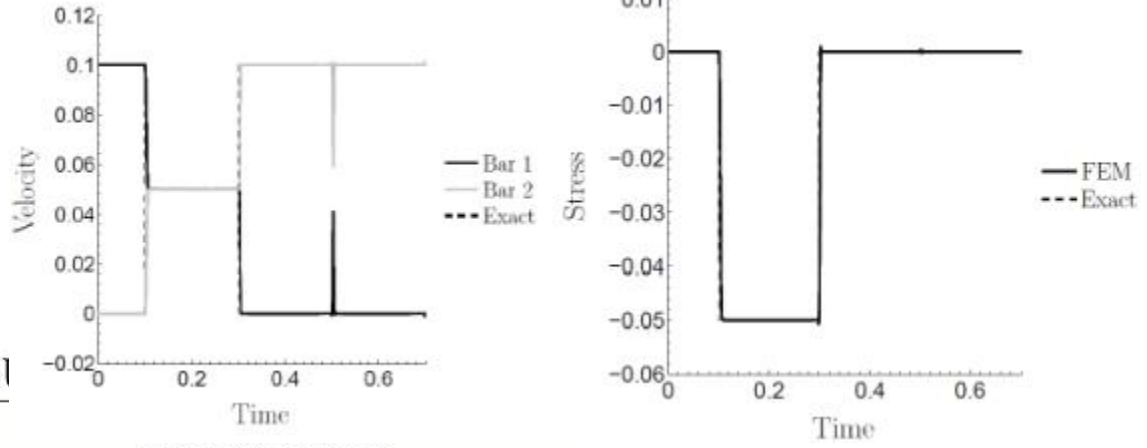








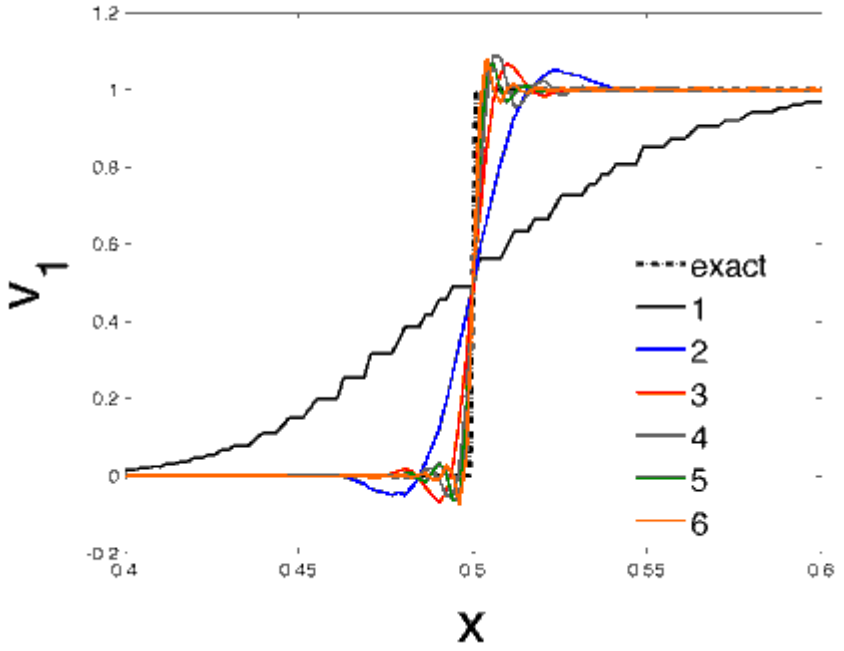
Riemann solutions incorporated in *Spacetime Discontinuous Galerkin* finite element method

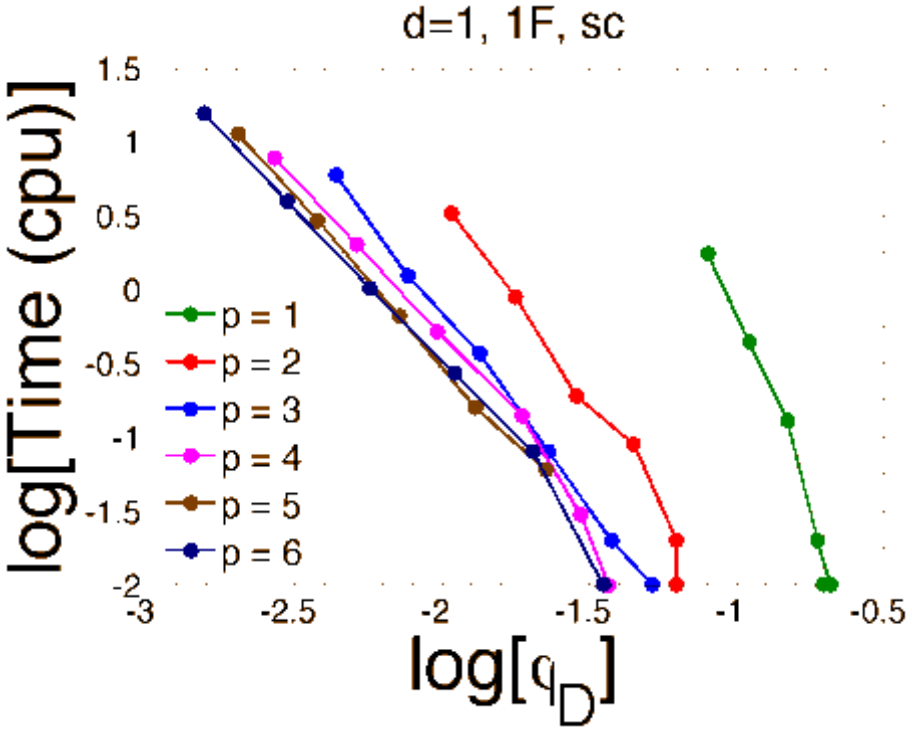


Unlike other solutions, SDG results are not overly damped and are free of numerical oscillations and overshoot / undershoot

2. Discontinuities & high gradients in solution

Step function captured by SDG method for 1/h = 128, p = 1 to p = 6





- Solution is free from global scale numerical artifacts.
- Although the order of convergence is limited by discontinuity, higher order polynomials are still more efficient!

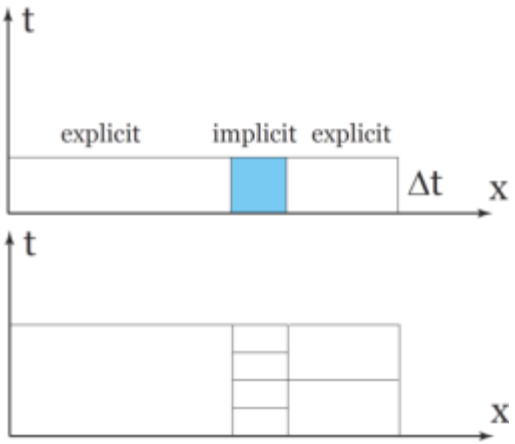
3. Multiscale features

Time-marching methods

- **Time step is limited by smallest elements**

→ Explicit: stability
Implicit: Accuracy

- **Improvements:**
  - **Implicit-Explicit (IMEX)** methods: increase the time step by using implicit integration for small elements
  - **Local time-stepping**: subcycling for smaller elements enables using larger global time steps



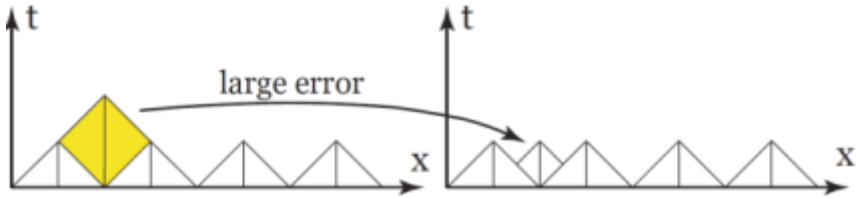
SDGFEM

- Small elements locally have smaller progress in time (no global time step constrains)
- None of the complicated "improvements" of time marching methods needed

SDGFEM graciously and efficiently handles highly multiscale domains

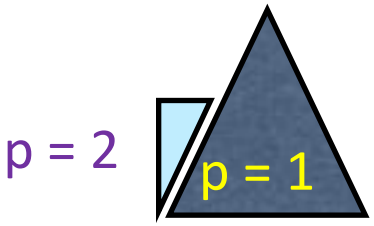
4. Adaptivity

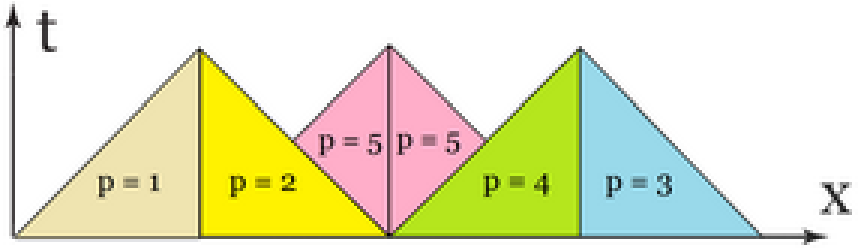
- **Local adaptive operations**: no need for reanalysis of the entire domain



- **Arbitrary h-refinement**
- **Arbitrary p-enrichment**

Similar to other DG methods, it does not require transition elements to change *h* or *p*

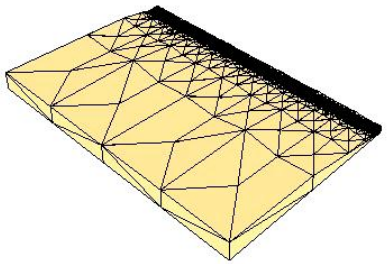


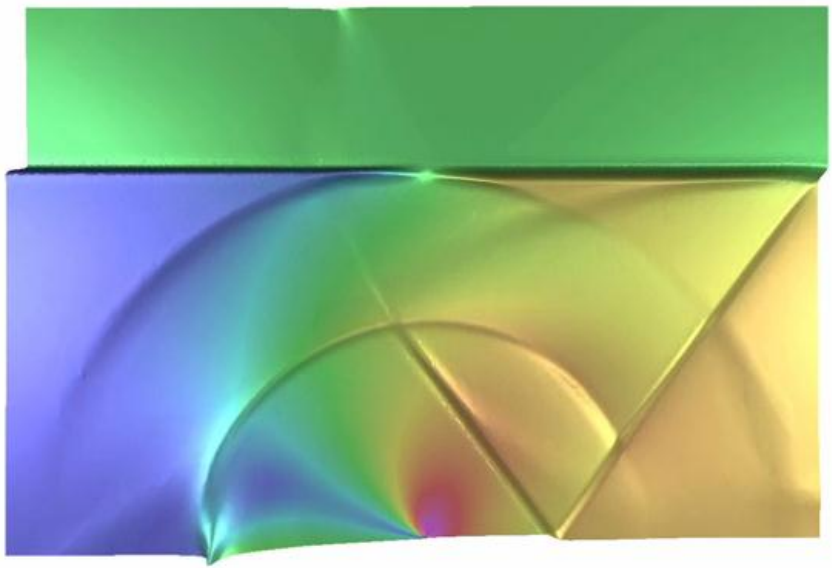


- **Arbitrarily high & local resolution in time**
- **SDGFEM ideal for multiscale meshes:** meshes generated by adaptive operations are highly multiscale, with the latter being a major concern in time marching schemes (previous slide)

Crack-tip wave scattering

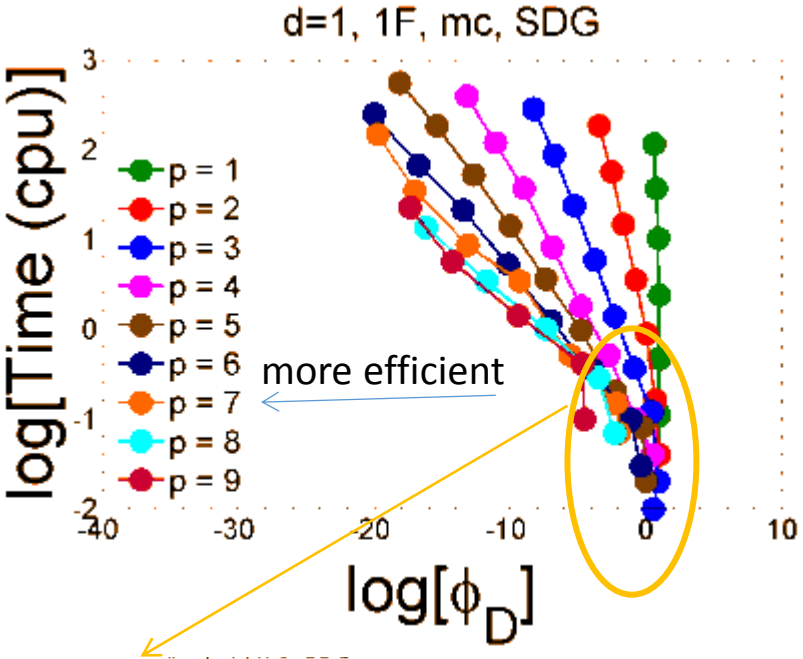
- These meshes for a crack-tip wave scattering problem are generated by adaptive operations. Refinement ratio smaller than 10-4.

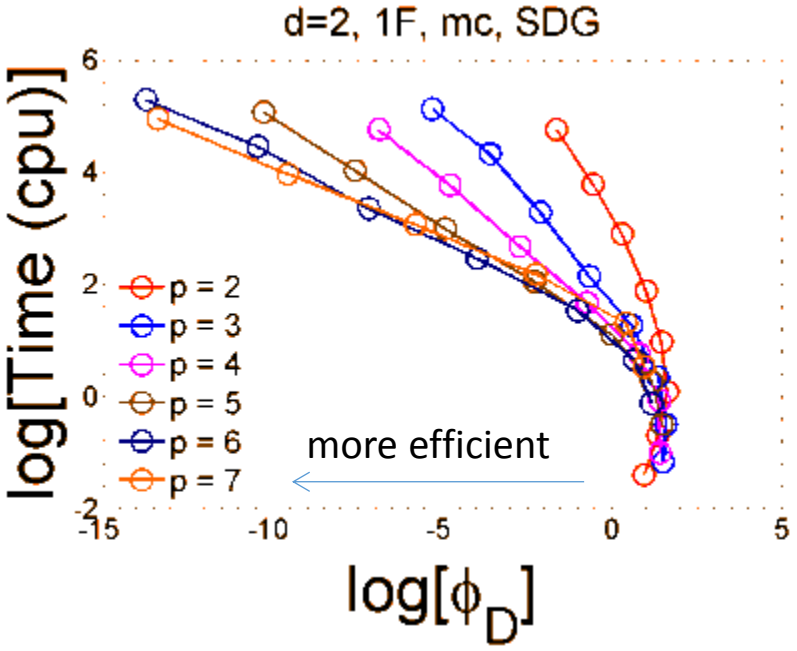


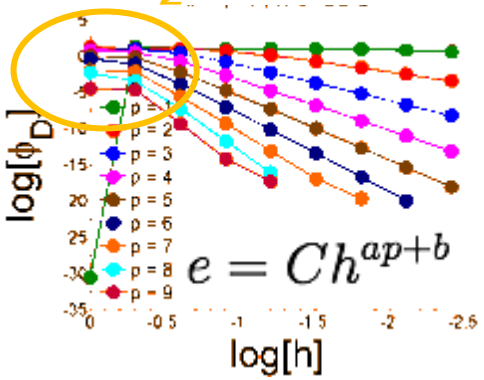


5. Higher order methods

- Is p-enrichment always more efficient for smooth problems?







- For larger systems (3F vs 1F), higher order polynomials may not be more efficient!
- How about continuous Galerkin methods?

Conclusion

- For more thorough study to better capture SDGFEM benefits there needs to be:
  - Higher order dimension
  - Higher polynomial order
- SDGFEM are more accurate and efficient for high order dimension problems with high order polynomial
- SDGFEM better capture discontinuous features
- Local solution property of the method implies that both assembly and solution stages of the FEM simulation scale linearly versus number of elements

THANK YOU FOR YOUR TIME AND ATTENTION!