A Riemann-Solver Free Discontinuous Galerkin Method for Hyperbolic Systems

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Spacetime
Discontinuous
Galerkin Method
Discontinuous Galerkin (DG) Finite Element Methods

- Weakly enforce conservation jump conditions (e.g., Rankine–Hugoniot)
- Can recover balance properties at the element level (vs global domain)
- Support for nonconforming meshes and
- Arbitrary changes in element polynomial order
- Superior performance for resolving discontinuities (discrete solution space better resembles the continuum solution space)

Sample DG solutions with no evident numerical artifacts

Numerical artifacts generally spoil continuous FE solutions in the presence of shocks
Direct discretization of spacetime

- Replaces a separate time integration; no global time step constraint
- Unstructured meshes in spacetime
- No tangling in moving boundaries
- Arbitrarily high and local order of accuracy in time
- Unambiguous numerical framework for boundary conditions

Shock capturing more expensive, less accurate

Shock tracking in spacetime: more accurate and efficient
courtesy of Scott Miller
Spacetime Discontinuous Galerkin (SDG) 
Finite Element Method

DG + spacetime meshing + causal meshes for hyperbolic problems:

- Local solution property
- $O(N)$ complexity (solution cost scales linearly vs. number of elements $N$)
- Asynchronous patch-by-patch solver
  - incoming characteristics on red boundaries
  - outgoing characteristics on green boundaries
  - The element can be solved as soon as inflow data on red boundary is obtained $\Rightarrow$
    - partial ordering & local solution property
    - elements of the same level can be solved in parallel

Time marching

Time marching or the use of extruded meshes imposes a global coupling that is not intrinsic to a hyperbolic problem

Elements labeled 1 can be solved in parallel from initial conditions; elements 2 can be solved from their inflow element 1 solutions and so forth.
Tent Pitcher:
Patch–by–patch meshing

- meshing and solution are interleaved
  - patches (‘tents’) of tetrahedra are solve immediately \( \Rightarrow O(N) \) property
  - rich parallel structure: patches can be created and solved in parallel

![Tent Pokémon images]

**tent–pitching sequence**
Riemann solution
Riemann solutions in SDG method

• The weighted residual statement includes jumps relative to a target fluxes:

\[
\int_{Q} \left[ \mathbf{i} \hat{\epsilon} \wedge (dM - \rho b) + (du - v) \wedge \hat{u} \wedge dt + d\epsilon \wedge \nabla \hat{M} \right] \\
+ \int_{\partial Q} \left[ \mathbf{i} \hat{\epsilon} \wedge (M^* - M) + (u^* - u) \wedge \hat{u} \wedge dt + (\epsilon^* - \epsilon) \wedge \nabla \hat{M} \right] = 0
\]

• Similar to Finite Volume and other Discontinuous Galerkin methods: Riemann solutions (or approximate numerical fluxes) needed at noncausal interfaces.
Why use Riemann fluxes?

• Specially for problems with shocks and discontinuities, Riemann fluxes yield smaller errors.
• Choice of error is important in presence of discontinuities:

![Graphs showing total error and energy dissipation error](image-url)
Why use Riemann fluxes?

- Higher stability due to preservation of characteristics:

Benchmark problem [Hughes (76); Laursen, Chawla (97); Czekanski, Meguid (01); Cirak, West (05); etc.]

Riemann solutions incorporated in *Spacetime Discontinuous Galerkin* finite element method

Unlike other solutions, SDG results are not overly damped and are free of numerical oscillations and overshoot / undershoot
Challenges in obtaining/using Riemann fluxes: Linear Problems

- **Anisotropic materials**: Complicated equations for wave speeds and characteristic values.

- **Multiphysics problems** may result in large system of equations; e.g. thermoelasticity:

  \[
  c_d = \left[ \frac{1}{2} \left( c_D^2 + c_T^2 + \beta - \sqrt{-4c_D^2 c_T^2 + (c_D^2 + c_T^2 + \beta)^2} \right) \right]^{1/2} \\
  c_T = \left[ \frac{1}{2} \left( c_D^2 + c_T^2 + \beta + \sqrt{-4c_D^2 c_T^2 + (c_D^2 + c_T^2 + \beta)^2} \right) \right]^{1/2}
  \]

  \[
  \gamma_1 = \{0, 0, 0, \rho c_D^2 (c_t - c_T)(c_t + c_T), 0, c_t (c_D^2 - c_T^2), 0, 0, -c_D^2 c_t C \theta, c_D^2 \theta, 0\}, \\
  \gamma_2 = \{0, 0, \rho c_D^2 (c_d - c_T)(c_d + c_T), c_d (c_D^2 - c_T^2), 0, 0, -c_D^2 c_d C \theta, c_D^2 \theta, 0\}, \\
  \gamma_3 = \{0, 0, 0, 2 \rho c_5, 0, 1, 0, 0, 0, 0\}, \\
  \gamma_4 = \{0, 1, 0, 0, 0, 0, 0, 0, 0\}, \\
  \gamma_5 = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 1\}, \\
  \gamma_6 = \{0, 0, 0, 0, -\lambda, 0, \rho c_D^2, 0, 0, 0\}, \\
  \gamma_7 = \{1, 0, 0, 0, 0, 0, 0, 0, 0\}, \\
  \gamma_8 = \{0, 0, 0, -2 \rho c_5, 0, 1, 0, 0, 0, 0\}, \\
  \gamma_9 = \{0, 0, \rho c_D^2 (c_d - c_T)(c_d + c_T), c_d (c_D^2 - c_T^2), 0, 0, C c_d c_D^2 \theta, c_D^2 \theta, 0\}, \\
  \gamma_{10} = \{0, 0, \rho c_D^2 (c_t - c_T)(c_t + c_T), c_t (c_D^2 - c_T^2), 0, 0, C c_D^2 c_t \theta, c_D^2 \theta, 0\}
  \]

Challenges in obtaining/using Riemann fluxes: Nonlinear Problems

- Requires a nonlinear solution at quadrature points on element boundaries:

\[
\begin{bmatrix}
\rho \\
u \\
p \\
\end{bmatrix}_t + \begin{bmatrix}
u & \rho & 0 \\
0 & u & 1/\rho \\
0 & \gamma p & u \\
\end{bmatrix} \begin{bmatrix}
\rho \\
u \\
p \\
\end{bmatrix}_x = 0.
\]

- Typical solution for 1D Euler equation:
Riemann-solver free method
Multi-cell elements

• Inflow and outflow facets of the patch are causal ⇒
  • Riemann solution is trivial upstream value
• If interior facets are eliminated we obtain a Riemann-solver free method ⇒

Interior cells are integration cells eliminating all nontrivial Riemann solutions.

• Typical solution for 1D Euler equation:
Advantages of Riemann-solver-free method

- Riemann fluxes do not exist or are computationally difficult for:
  - Nonlinear Riemann solutions
  - Anisotropic materials
  - Multiphysics problem.

- Lower assembly cost by elimination of interior facet integration.
- Lower system solution cost:

<table>
<thead>
<tr>
<th>d</th>
<th>dof reduction</th>
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<tbody>
<tr>
<td>1</td>
<td>2</td>
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<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>24</td>
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</tbody>
</table>

Caveat: Multi-cell elements have an equivalent $h$ more than two times of counterpart single cell elements.
Multi-cell elements: Are they more efficient?

- **Cheaper**: Substantially reduce number of dof & assembly cost
- **Less accurate**: Larger elements

For almost all the polynomial tested single element patches were more efficient

How about problems with larger dofs?
Mixed finite element formulation

• Mixed finite element formulations can increase order of convergence
• 3-field elastodynamics: \( u \) displacement, \( v \) velocity, and \( E \) strain interpolated:

\[
\int_\Omega [i \dot{\varepsilon} \land (dM - \rho b) + d\varepsilon \land i\hat{M} + (du - v) \land \hat{f}] \\
+ \int_{\partial\Omega} [i \dot{\varepsilon} \land (M^* - M) + (\varepsilon^* - \varepsilon) \land i\hat{M} + (u^* - u) \land \hat{f}] = 0 \\
\forall (\hat{u}, \hat{v}, \hat{E}) \in \mathcal{U}^2 \times \mathcal{V}^2 \times \mathcal{E}^2.
\]

• 1-field elastodynamics: \( u \) displacement interpolated. Systematically derived from 3-field formulation:

\[
\int_\Omega [i \dot{\varepsilon} \land (dM - \rho b) + \int_{\partial\Omega} [i \dot{\varepsilon} \land (M^* - M) + (\varepsilon^* - \varepsilon) \land i\hat{M} \\
+ (u^* - u) \land \hat{f}]_I = 0 \quad \forall \hat{u} \in \mathcal{U}^2
\]

Mixed formulations

<table>
<thead>
<tr>
<th>Formulation</th>
<th>Interpolated fields</th>
<th>Numerical dissipation convergence rate</th>
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</thead>
<tbody>
<tr>
<td>1F</td>
<td>U</td>
<td>$h^{2p-1}$</td>
</tr>
<tr>
<td>3F</td>
<td>U, v, E</td>
<td>$h^{2p+1}$</td>
</tr>
</tbody>
</table>

- For moderate systems higher convergence rate of 3F formulation results in higher efficiency although it has more dofs than 1F formulation.
- As dof increase (e.g. when d and p increase) 1F starts to become more efficient albeit its lower energy error convergence rate.
Mixed formulations

• For large systems:
  • High spatial dimension (d)    High polynomial order (p)    Single-cell elements

1F formulation is more efficient due to substantial portion of LU factorization:
Multi-cell elements: More efficient for 3F?

- As dof increase (e.g. when d and p increase) multicell elements starts to become more efficient albeit their larger effective element size.
Multi-cell elements

- For large systems:
  - High spatial dimension \( (d) \)
  - High polynomial order \( (p) \)
  - 3F (three field formulation)

- Multi-cell elements are more efficient for such configurations with substantial reduction in the portion of LU factorization:

\[
\text{Time LU \% (cpu)}
\]

\[
\log[h]
\]

**Graphs:**
- \( d=2, 3F, \text{sc, SDG} \)
- \( d=2, 3F, \text{mc, SDG} \)
Riemann-solver free method for anisotropic elastodynamics
Riemann-solver free method for anisotropic elastodynamics

[Click to play movie]
Conclusions

- Riemann fluxes preserve the characteristics structure of the wave equation for hyperbolic problems.
- For problems with discontinuities they provide additional stability and reduce computational errors.
- Direct discretization of spacetime and using causal patch boundary facets enables derivation of Riemann-solver-free by using simplex cells as integration cells.
- The new solution scheme eliminates the need to obtain Riemann fluxes or appropriate substitutes for nonlinear, anisotropic, and multiphysics problems.
- A efficiency analysis for linear elastodynamics with exact solution shows that the Riemann free method is more efficient than single element patches for high dofs (mixed formulation, high d and p).
- The gains are deemed to be substantially more for nonlinear problems such as Euler’s equations for inviscid fluid.