2014/10/14a

Tuesday, October 14, 2014
10:11 AM

How to calculate K_I & K_{II} using J integral?

Since we are talking about K_I & K_{II}

It means LEFM theory must be valid

"That is SSY condition holds"

$$J_1 = \frac{K_I^2 + K_{II}^2}{E'}$$

$$J_2 = \frac{-2K_I K_{II}}{E'}$$

→ G energy release rate

crack advances with unit length forward

→ Energy release rate when the crack propagates 90° to it's original direction

$G = -\frac{1}{B}\frac{\partial \Pi}{\partial a}$

Calchal

Calculating energy release rate shows $G_2 = -\frac{2K_I K_{II}}{E'}$

$J_2 = \frac{-2 K_I K_{II}}{E'}$ → we can show that the energy release rate for 90° crack extension is

$$J_1 = \int_\Gamma \left(w dy - \mathbf{t} \frac{\partial \mathbf{u}}{\partial x} d\Gamma \right)$$

$$J_2 = \int_\Gamma \left(-w dx - \mathbf{t} \frac{\partial \mathbf{u}}{\partial y} d\Gamma \right)$$

$G_2 = \frac{-2 K_I K_{II}}{E'}$

last time we proved

$J = G$

$J_2 = G_2$ LEFM theory

How do we do it?

→ displacements & stresses are known

→ calculate energy of crack extension

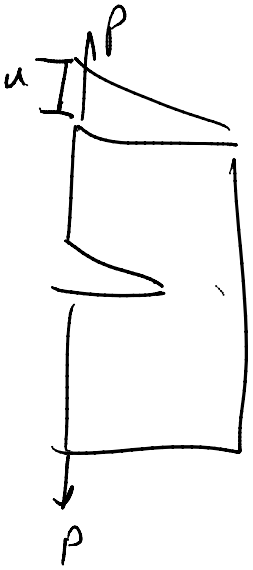
$G_2 = \frac{-2 K_I K_{II}}{E'}$ can be proved by local stress & displacement fields

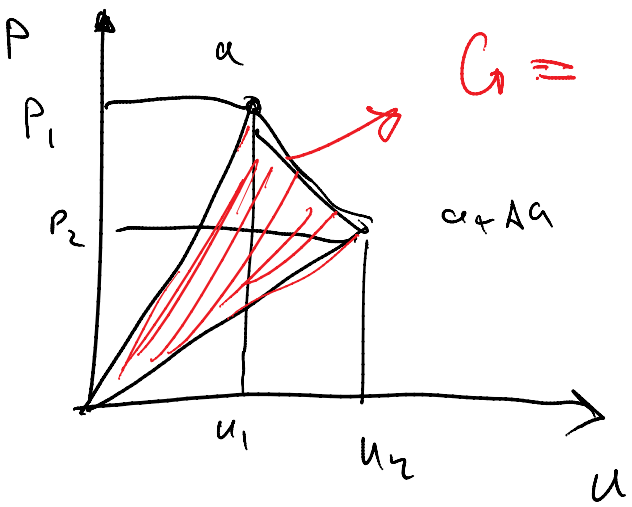
(B=1) unit thickness

$J_2 = G_2$ can be proved similar to the proof for J_2 G

J_1 G_1

Calculating J from $P-u$ curves





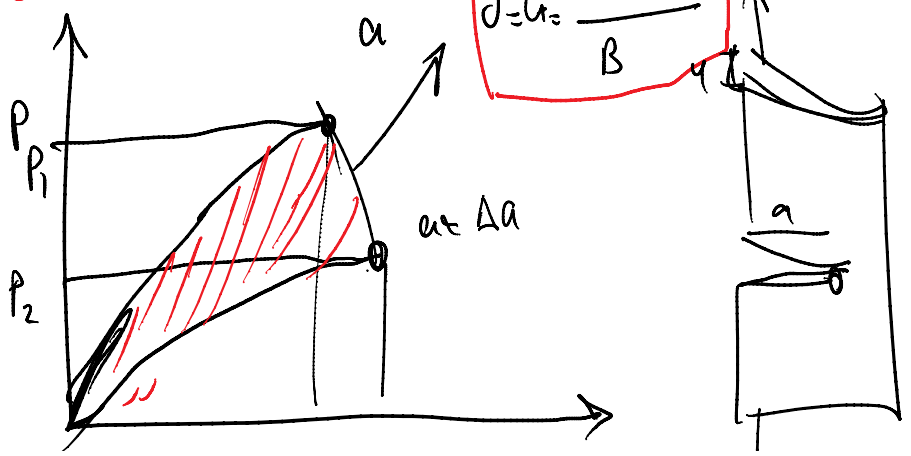
$G = \dfrac{\text{shaded area}}{B}$ for linear case

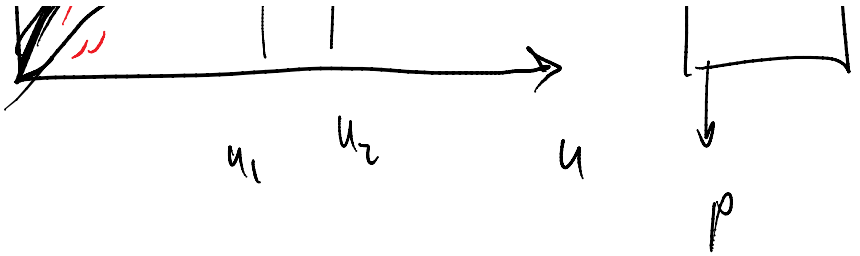
$$G = \frac{-1}{B}\frac{\partial \Pi}{\partial a}$$

What is the generalization of this concept for nonlinear case?

we want to evaluate J

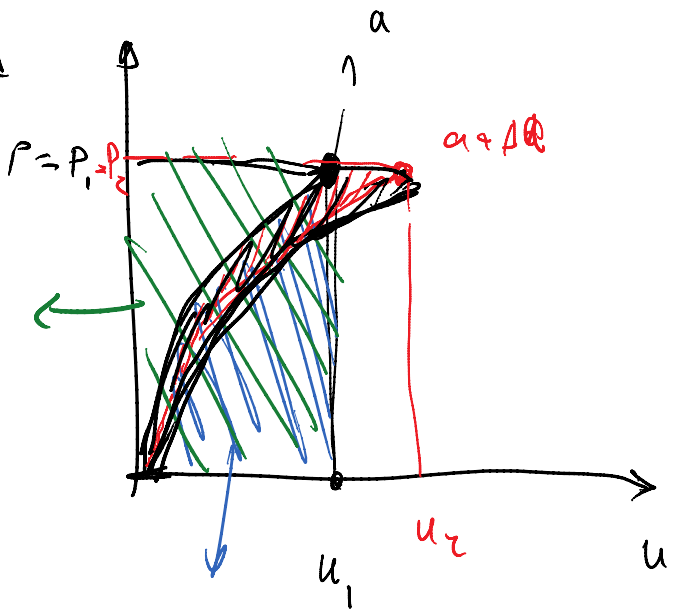
$$J = G = -\frac{1}{B}\frac{\partial \Pi}{\partial a}$$

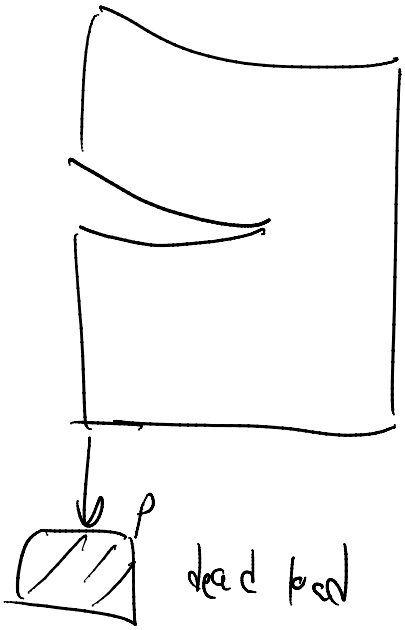




Simple Cases : A) load control : with constant load

P is fixed



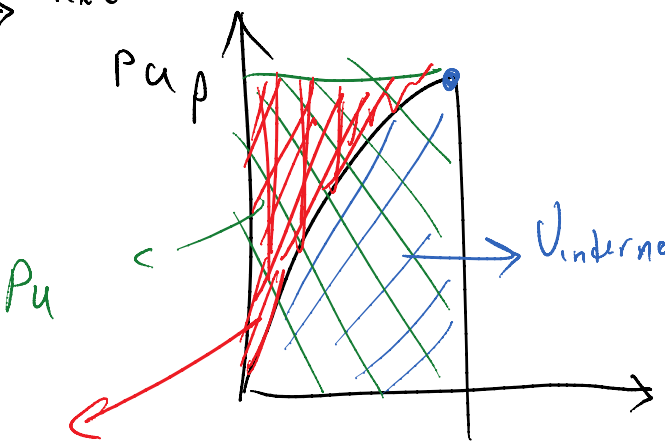


$$J = G = -\frac{\partial \Pi}{\partial a}$$

$$\Pi = U_i - \overbrace{U_{external}}^{\text{work}}$$

$$= \int_0^u P\,du = \text{fixed}$$

$$U_i = P_u$$



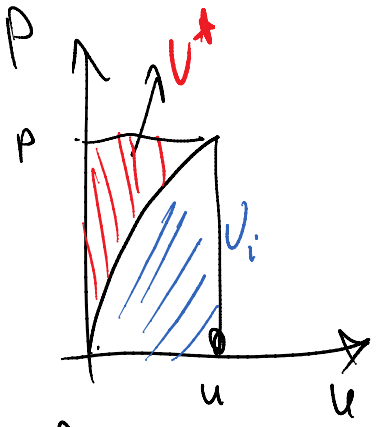
Complimentary energy = PU_ area under the curve

complimentary energy

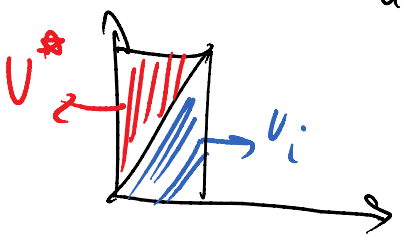
$\Pi = -U^*$ for dead load (constant load)

$$G = \frac{-1}{B}\frac{d\Pi}{da} = \frac{-1}{B}\frac{d-U^*}{da} \Rightarrow$$

$$\boxed{\mathcal{J} = G = \frac{1}{B}\frac{dU^*}{da}\bigg|_{P\ \text{constant}}}$$



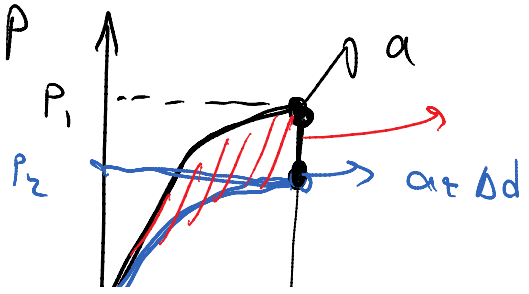
linear case $\leftarrow$ $G = \frac{1}{B}\frac{dU_i}{da}\bigg|_{P\ \text{constant}}$

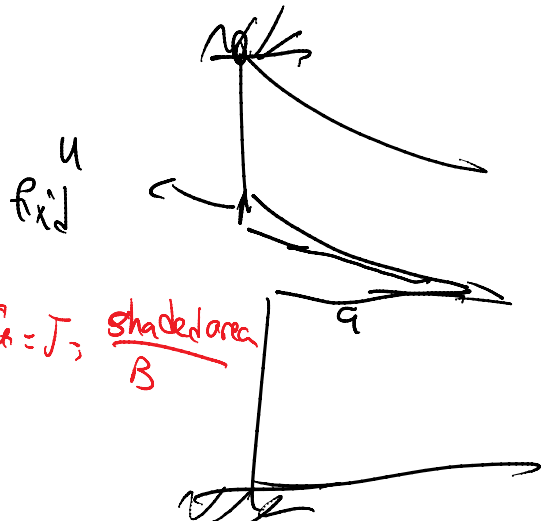


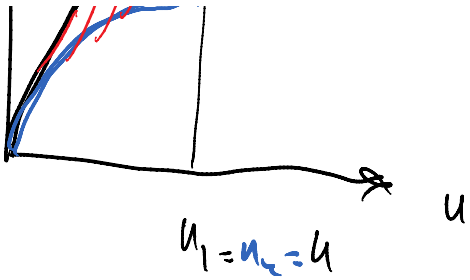
In linear case $U^* = U_i$

2nd case fixed grip

U is fixed







$$\Delta \Pi = U_2 - U_1 - W_{12}$$

$$W_{12} = \int_{u_1}^{u_2} P\,du = 0 \qquad u_1 = u_2 = u$$

$$\Delta \Pi = U_2 - U_1 = \Delta U_i$$

$$G = J = -\frac{1}{B}\frac{\Delta \Pi}{\Delta a} \Rightarrow \qquad \Delta a \to 0$$

$$\boxed{G = -\frac{1}{B}\left.\frac{dU_i}{da}\right|_{\text{fixed } u}}$$

Fixed Grip

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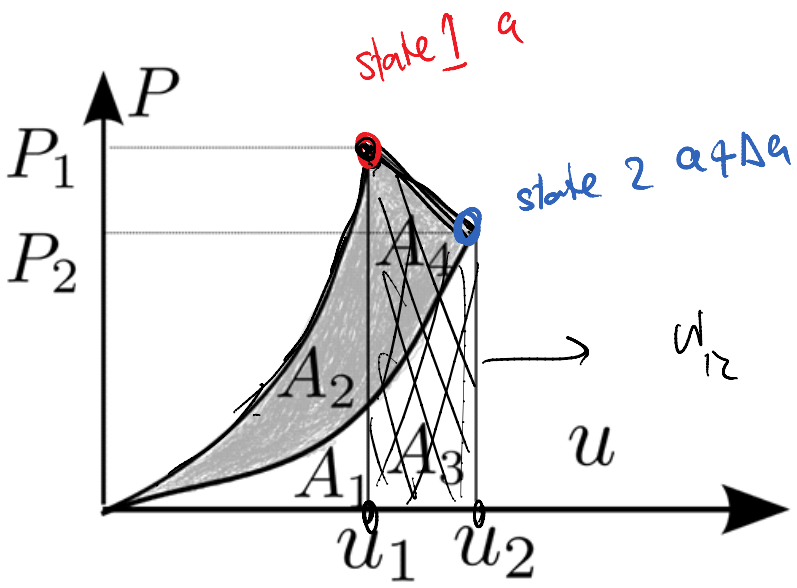
nonlinear $J = \frac{1}{B}\left(\frac{\partial U_e^*}{\partial a}\right)_P$ (load control) $\qquad J = -\frac{1}{B}\left(\frac{\partial U_e}{\partial a}\right)_u$ (displacement control)

$$J = -\frac{1}{B}\left(\frac{\partial U_e}{\partial a}\right)_u$$

linear case $\quad J = \frac{1}{B}\left(\frac{\partial U_e}{\partial a}\right)_P$

because for linear case $U_e^* = U_e$

General case



state 1 a

state 2 a+Δa

$$J = G = \frac{\text{shaded area}}{B\,\Delta a}$$

$$W = \int_{u_1}^{u_2} P\,du$$

Similar to G!

$$G = -\lim_{\Delta a \to 0} \frac{\Delta \Pi}{\Delta a B}$$

$$\Delta \Pi = U_e^2 - U_e^1 - W_{12}$$

$$= (A_1 + A_3) - (A_1 + A_2) - (A_3 + A_4)$$

$$= A_3 - A_2 - A_3 - A_4$$

$$= -(A_2 + A_4)$$

shaded area

$$G = \lim_{\Delta a \to 0} \frac{-(A_2 + A_4)}{\Delta a B}$$

$$G = \frac{\text{shaded area}}{\Delta a\, B}$$

In linear case we obtained

$$G = \frac{P^2}{2B}\frac{dC}{da} = -\frac{u^2}{2B}\frac{dk}{da}$$

(compliance, stiffness)

$$C = \frac{u}{P}$$

What is the generalization of this equation for nonlinear case

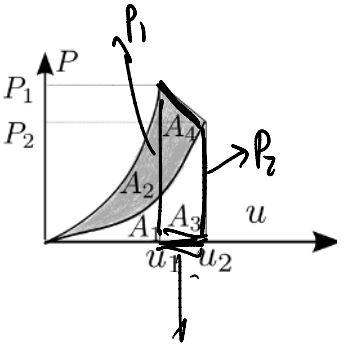
$$J = -\lim_{a\to 0}\frac{\Delta\Pi}{B\Delta a}$$

$$= -\lim_{a\to 0}\frac{U_{e2} - U_{e1} - W_{12}}{B\Delta a}$$

$$= -\lim_{a\to 0}\frac{1}{B}\left\{\frac{U_{e2} - U_{e1}}{a_2 - a_1} - \frac{P_1 + P_2}{2}\frac{u_2 - u_1}{a_2 - a_1}\right\}$$

$$= \frac{1}{B}\left\{-\frac{dU_e}{da} + P\frac{du}{da}\right\}$$

W_{12}, $= P$



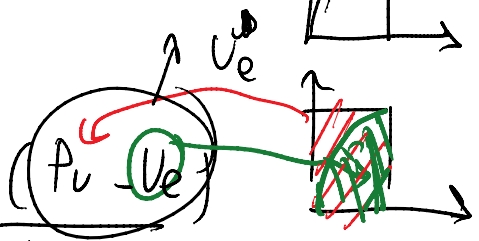
$W_{12} = \left(\frac{P_1+P_2}{2}\right)(u_2 - u_1)$

$$G = \frac{1}{B}\left\{-\frac{dU_e}{da} + P\frac{du}{da}\right\} = \frac{1}{B}\left\{-\frac{dU_e}{du} + P\right\}\frac{du}{da}$$

1) How do we recover fixed load equation?

fixed load $\quad G = \frac{1}{B}\frac{dU_e}{da}\Big|_P$ → complimentary

P is fixed $\quad$ P_u is free



$$G = \frac{1}{B}\left(-\frac{dU_e}{da} + \frac{d(P_u)}{da}\right) = \frac{1}{B}\frac{d(P_u - U_e)}{da}$$

$$G = \frac{1}{B}\frac{dU_e^*}{da}\Big|_P \quad \text{for fixed load}$$

$$G = \frac{1}{B}\left\{-\frac{dU_e}{da} + P\frac{du}{da}\right\} = \frac{1}{B}\left\{-\frac{dU_e}{du} + P\right\}\frac{du}{da}$$

How do we recover fixed grip equation

u is fixed

$$G = \frac{-1}{B}\left.\frac{dU_e}{da}\right|_u$$

u constant $\Rightarrow \frac{du}{da} = 0$

$$G = \frac{1}{B}\left(-\frac{dU_e}{da} + P\frac{du}{da}\right)$$

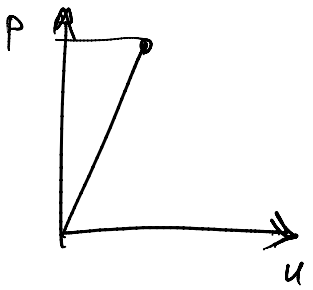
Last question what does this relation reduce to for linear case

$U_e(u)$ very challenging

$$G = \frac{1}{B}\left\{-\frac{dU_e}{da} + P\frac{du}{da}\right\} = \frac{1}{B}\left\{-\frac{dU_e}{du} + P\right\}\frac{du}{da}$$

$$U_e = \frac{1}{2}Pu = \frac{1}{2}ku^2 \quad \leftarrow \text{stiffness}$$

linear case $P = ku$



$$G = \frac{1}{B}\left\{-\frac{d}{da}\left(\frac{ku^2}{2}\right) + ku\frac{du}{da}\right\} =$$

$$\frac{1}{B}\left\{-ku\frac{du}{da} - \frac{dk}{da}\frac{u^2}{2} + ku\frac{du}{da}\right\}$$

$$\boxed{G = \frac{-1}{2B}u^2\frac{dk}{da} = \frac{1}{2B}P^2\frac{dC}{da}}$$