Note that the midterm will be graded out of 250 with 50 extra points on the question sheet.

1. (**90 Points**). A cylindrical pressure vessel with closed ends has a radius $R = 1$ m and thickness $t = 40$ mm and is subjected to internal pressure p. The vessel must be designed safely against failure by yielding (according to the von Mises yield criterion) and fracture. The vessel is made of steel with yield stress $\sigma_y = 860$ MPa and fracture toughness $K_{Ic} = 100$ MPa$\sqrt{\text{m}}$.

   (a) For von Mises yield stress, yielding occurs when,

   $$\sigma_v = \sigma_y \quad \text{for} \quad \sigma_v = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}} \tag{1}$$

   where $\sigma_1, \sigma_2, \sigma_3$ are principal stresses.
   By using the values of $\sigma_{zz}, \sigma_{\theta\theta}$, and $\sigma_{rr} = 0$ (exterior surface of the vessel, producing largest σ_v), obtain p_p the maximum allowable p from plastic yielding perspective.

   (b) What direction of the crack between axial to circumferential direction experiences the highest stress intensity factor?

   (c) Plot the maximum permissible pressure p_c versus crack depth a_c considering both plastic yielding and fracture. Employ LEFM model for fracture analysis.

   (d) What is the crack length a_{tran} corresponding to the transition between plastic and fracture failure mechanisms?

   (e) Calculate the maximum permissible crack length a_c for an operating pressure $p = 12$ MPa.

   (f) Calculate the failure pressure p_c for a minimum detectable crack depth $a = 1$ cm.

   (g) Calculate the failure pressure p_c for a minimum detectable crack depth $a = 1$ mm.

2. (**60 Points**). For the notch problem shown in (1) we obtain the power of singularity for strain and strain ($\lambda_1 - 1 = -\frac{1}{2}$) from the equation $\sin(2\pi\lambda_n) = 0 \quad \Rightarrow \quad \lambda_n = \frac{n}{2}, n > 1$. Using the equation,

   $$\sin(2\lambda\alpha) + \lambda \sin(2\alpha) = 0 \quad \text{mode I} \tag{2a}$$

   $$\sin(2\lambda\alpha) - \lambda \sin(2\alpha) = 0 \quad \text{mode II} \tag{2b}$$

   obtain the power of singularity of stress and strain ($\lambda - 1$) for modes I and II. To ensure that internal energy is finite around the crack tip $\lambda - 1 \geq -\frac{1}{2}$ ($U\mathrm{d}A = \sigma\epsilon r \mathrm{d}r \mathrm{d}\theta$ bounded for $r \to 0$). Also, for the singular response $\lambda - 1 < 0$. So the acceptable range for the first term λ is $\frac{1}{2} \leq \lambda < 1$ for a singular response. For more information refer to the course presentation pages 135-138.

   - Find the stress and strain singularity power of mode I and II for 90° notch ($\alpha = \frac{3}{4}\pi$. You need to obtain the λ_1 the minimum root of equations (2) for $\lambda \in [\frac{1}{2}1)$.
   - Noting that $\sigma = K_I r^{\lambda^I - 1} + K_{II} r^{\lambda^{II} - 1} + \cdots$ discuss which mode will dominate the stress field near the crack tip. How is this compared to sharp crack, $\alpha = \pi$, where $\lambda^I = \lambda^{II} = \frac{1}{2}$.
   - **For your interest, no need to submit**. Plot radius of singularity (when applicable) for modes I and II for $\alpha = \pi/2$ to π.

3. (**150 Points**). Figure2 shows a point force displacement system with crack length A, force P, and beam width and height B and 2H, respectively. The moment at the end of the crack due to the

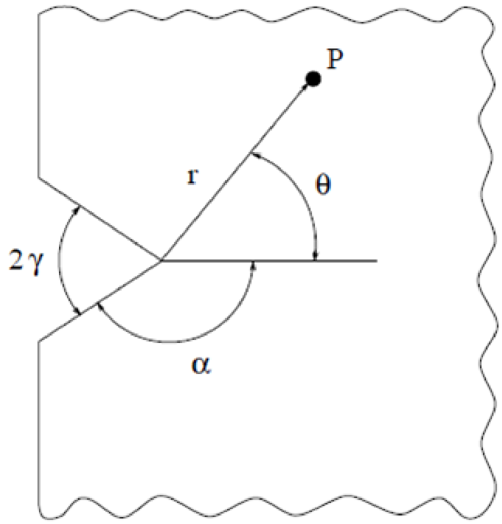


Figure 1: Schematic of notch geometry

force is $M = PA$. To distinguish A from area of the crack surface we use $\mathcal{A} = AB$ for the latter. We employ the following nondimensional parameters to facilitate the analysis of this problem,

$$a = \frac{A}{H} \qquad \text{normalized crack length} \tag{3a}$$

$$p = \frac{P}{\sigma_y BH} \qquad \text{normalized force} \tag{3b}$$

$$m = pa = \frac{PA}{\sigma_y BH^2} \qquad \text{normalized moment} \tag{3c}$$

$$\delta = \frac{\Delta}{H}\frac{E}{\sigma_y} \qquad \text{normalized displacement (crack opening)} \tag{3d}$$

where σ_y is the yield stress.

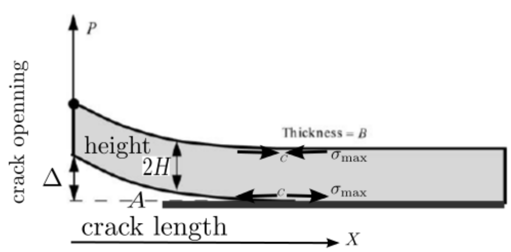


Figure 2: Force displacement relation for a point force system.

The purpose of this problem is plastic fracture mechanics analysis of this crack and comparison with LEFM. We adopt an elastic-perfectly plastic material behavior. From linear analysis we know that the maximum moment M that this beam can withstand without plastic deformation is when $\sigma_{\max}$ at points C in the figure reach σ_y. If M further increases (through increasing P or crack length A) we will have plastic yielding in points C and the plastic region further penetrates inside the domain, until M at crack tip eventually reaches maximum possible moment that the section can withstand. The limit for initiation of plastic deformation and maximum value moments are,

$$M_{\text{lmax}} = \frac{I}{\sigma_y} z_{\max} = \frac{2}{3} B H^2 \sigma_y \qquad \text{maximum moment for linear response} \tag{4a}$$

$$M_{\max} = B H^2 \sigma_y \text{ (all interface is yielded)} \qquad \text{maximum moment of the interface} \tag{4b}$$

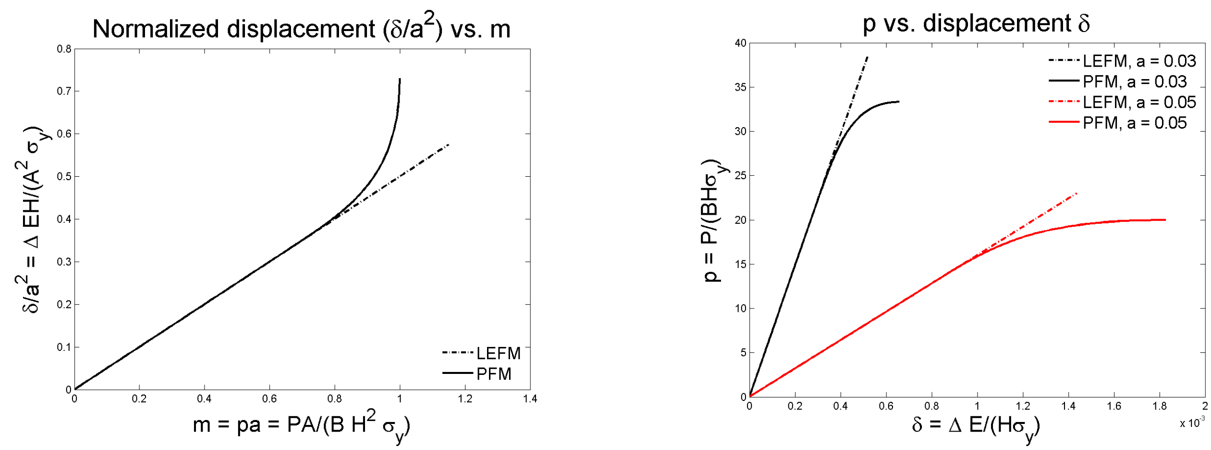
To determine the deflection Δ at the tip of the crack we employ relations between $M(x)$ and $\frac{\mathrm{d}^2\Delta(x)}{\mathrm{d}X^2}$ as follows:

$$\frac{\mathrm{d}^2\Delta(x)}{\mathrm{d}X^2} = \begin{cases} \frac{M(x)}{EI} & M(x) < M_{\text{lmax}} \\ \frac{\sigma_y}{HE} \frac{1}{\sqrt{3}\sqrt{1 - M(x)/M_{\max}}} & M_{\text{lmax}} < M(x) < M_{\max} \end{cases} \tag{5}$$

Note that $\Delta(x), M(x)$ denote displacement and moment values along the beam while undecorated Δ and M denote their maximum values at the two end points of the crack. By locating the initiation position of plastic deformation in the beam and integrating (5) we obtain,

$$\delta = \frac{\Delta}{H}\frac{E}{\sigma_y} = a^2 f(m), \quad f(m) = \begin{cases} \frac{1}{2} m & m < \frac{2}{3} \\ \frac{20}{27m^2} - \frac{2}{3\sqrt{3}m^2}\sqrt{1-m}(2+m) & \frac{2}{3} < m < 1 \end{cases} \tag{6}$$

Equation (6) implies that when the applied moment $m = pa$ is small ($< \frac{2}{3}$ corresponding to M_{lmax}) the linear response holds between load and displacement. However, as m increases either through increasing load P or crack length A, the $P - \Delta$ relation is no longer linear.



(a) Relation between normalized displacement and moment, equation (6).

(b) $P - \Delta$ plots for sample crack lengths $A = aH$ based on (6).

Figure 3: Linear and nonlinear $P - \Delta$ relations for the crack problem in figure 2. The dash line LEFM curves show that for small "loading" (m, P), the actual $P - \Delta$ relation is linear.

(a) **Energy release rate J**: To characterize plastic fracture response of this crack, we need to evaluate energy release rate $J = G$. Since equation (6) is the $P - \Delta$ relation (in normalized form), we should be able to evaluate internal (strain) energy $U(\Delta, A) = \int_0^{\Delta} P(\bar{\Delta})\mathrm{d}\bar{\Delta}|_{\text{fixed A}}$ or complimentary internal energy $U^*(P, A) = \int_0^{P} \Delta(\bar{P})\mathrm{d}\bar{P}|_{\text{fixed A}}$. Subsequently, using one of the following equations $J = G = -\frac{1}{B}\frac{\mathrm{d}U(\Delta,A)}{\mathrm{d}A}|_{\text{fixed }\Delta}$ or $J = G = \frac{1}{B}\frac{\mathrm{d}U^*(P,A)}{\mathrm{d}A}|_{\text{fixed P}}$ we can evaluate J. Note that J is taken as the energy release rate per unit area of crack advance $\mathcal{A} = AB$ rather than crack length A.

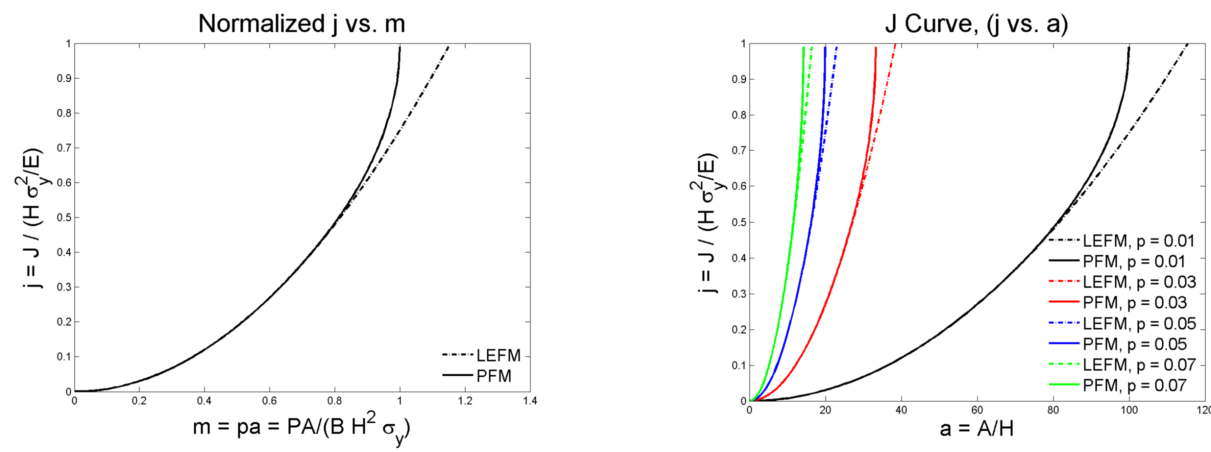
Choosing the appropriate form of J in terms of U or U^* for this problem show that,

$$J(m) = \frac{\sigma_y^2 H}{E} \left\{ \int_0^m f(\bar{m}) \mathrm{d}\bar{m} + m f(m) \right\} \tag{7}$$

(b) **LEFM vs. PFM, Small Scale Yielding (SSY)**: After evaluating (7) we can show (no need to prove (7) yields (8)) that normalized energy release rate j is equal to,

$$j = \frac{J}{\frac{\sigma_y^2 H}{E}} = \begin{cases} \frac{3}{4} m^2 & m < \frac{2}{3} \\ 1 - \frac{2}{\sqrt{3}}\sqrt{1-m} & \frac{2}{3} < m < 1 \end{cases} \tag{8}$$

This $J - m$ relation and its realization as J curve for specific load samples p are shown in (3b).



(a) Energy release rate as a function of normalized moment $m = pa$, (8)

(b) J curve ($J vs. A$) plots for sample applied loads $P = p\sigma_y HB$ based on (8).

Figure 4: Energy release rate J as a function of normalized moment $m = pm$ and its realization for specific load values p. The LEFM solution does not take material yielding into account.

i. What is the limiting m value, m_{tran}, below which LEFM solution can be used? For the geometry shown in 2, what is the transition load $P_{\text{tran}}(A)$ for a given crack length A for which LEFM solution can be employed?

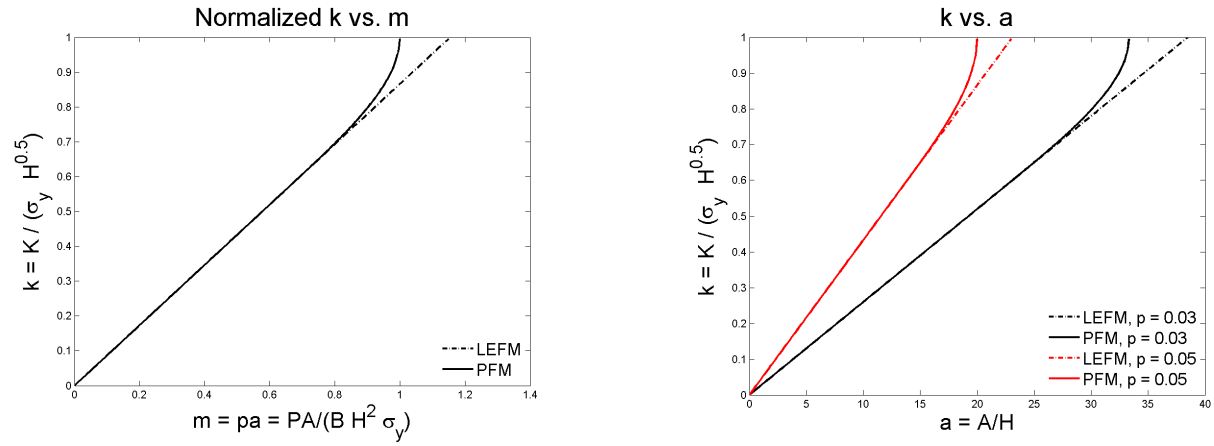
ii. Briefly (less than 2-4 sentences) explain why for $j > j_{\text{tran}}$ ($P > P_{\text{tran}}(A)$) plastic solution has a larger energy release rate?

iii. Since for LEFM $K = \sqrt{GE}$ (plane stress), the "effective" normalized K for this problem is,

$$k = \frac{K}{\sigma_y \sqrt{H}} = \begin{cases} \frac{\sqrt{3}}{2} m & m < \frac{2}{3} \\ \sqrt{1 - \frac{2}{\sqrt{3}}\sqrt{1-m}} & \frac{2}{3} < m < 1 \end{cases} \tag{9}$$

Consider a loading $P_1 = 0.05\sigma_y BH$, $A = 10H$. What is the stress intensity factor K_1 corresponding to this load? What is the stress intensity factor K_2 for $P_2 = 2P_1$ and same A? What is the relation between $2K_1$ and K_2? Using figure 5(b) explain why the superposition principle (*e.g.,* K of $2P$ is $2K$) does not hold here.

iv. Principle of SSY states that LEFM solutions are acceptable when *fracture process zone size* is much smaller than *other relevant length scales of the problem* such as domain



(a) Normalized stress intensity factor $k = \frac{K}{\sigma_y \sqrt{H}}$, (9)

(b) Stress intensity factor plots for sample applied loads $P = p\sigma_y HB$ based on (9).

Figure 5: "Effective" stress intensity factor computed from J.

dimensions and radius of singularity. Show that plastic zone size $r_p \propto \left(\frac{K}{\sigma_y}\right)^2$ as the representative of fracture process zone size satisfies $r_p \propto jH$. By comparing r_p with relevant length scales of the problem show that SSY condition is directly related to the condition $j < j_{\text{tran}}$ we obtained for the validity of LEFM solution.

(c) **Critical load P_{cr} and displacement Δ_{cr}** correspond to load and displacement values that the crack can start propagating for a given fracture resistance J_c. To determine P_{cr}, as done before, in R plot (*e.g.,* 4(b)) we find the smallest load (for load control) or displacement (displacement control) value whose J curve intersect R curve for the initial crack length A_0. If $R(A)$ is constant $R(A) = J_c$, for linear regime ($j_c = \frac{J_c}{\frac{\sigma_y^2 H}{E}} < \frac{1}{3}$, *cf.* (8), we obtain,

$$J = J_c \Rightarrow j = \frac{3}{4}m^2 = j_c \; (m < \frac{2}{3} \text{ linear branch of } j) \quad \Rightarrow \quad m_{cr} = p_{cr}a_0 = \frac{2}{\sqrt{3}}\sqrt{j_c} \Rightarrow$$

$$P_{cr} = BH\sigma_y p_{cr} = BH\sigma_y \frac{m_{cr}}{a_0} = BH\sigma_y \frac{2}{\sqrt{3}}\sqrt{j_c}\frac{H}{A_0} = \frac{2}{\sqrt{3}}\sqrt{j_c}\frac{BH^2}{A_0}\sigma_y \tag{10}$$

Note that P_{cr} depends on the initial crack length A_0. Similarly, by plugging m_{cr} in (6) we obtain Δ_{cr}, the critical displacement for crack propagation initiation. Δ_{cr} is either directly applied in displacement control loading or is the displacement corresponding to P_{cr} for load control setting. These values are summaries as follows,

$$P_{cr} = \frac{2}{\sqrt{3}}\sqrt{j_c}\frac{BH^2}{A_0}\sigma_y \tag{11a}$$

$$\Delta_{cr} = \sqrt{\frac{j_c}{3}}\frac{\sigma_y}{E}\frac{A_0^2}{H} \tag{11b}$$

for $m < \frac{2}{3}(j_c < \frac{1}{3})$.

i. Evaluate P_{cr}, Δ_{cr} for the nonlinear range $1 > m > \frac{2}{3}(1 > j_c > \frac{1}{3})$ in terms of j_c and m_{cr}.

ii. Combining the solution from (11a) and your solution for $1 > m > \frac{2}{3}$, plot P_{cr} in the form $p_{cr}a_0 = P\frac{A_0}{BH^2\sigma_y}$ versus $j_c = \frac{J_c}{H\sigma_y^2/E}$ for the entire range $j_c = 0$ to 1. In addition to P_{cr} from PFM, add the P_{cr} that you would have obtained from LEFM analysis for the entire $j_c \in [0\ 1]$ using (11a).

iii. For what ranges of j_c, P_{cr} from LEFM and PFM analysis are different and in that range is PFM P_{cr} smaller or larger than that of LEFM analysis. Explain (less than 2-3 sentences) why P_{cr} of PFM is smaller or larger than that of LEFM.

iv. Similarly, plot $\varDelta_{cr}$ in the form $\frac{\delta_{cr}}{a_0} = \varDelta \frac{HE}{A_0^2 \sigma_y}$ versus $j_c = \frac{J_c}{H\sigma_y^2/E}$ for $j_c \in [0\ 1]$ for both PFM and LEFM solutions using (11b) and your solution.

v. Compare $\varDelta_{cr}$ from LEFM and PFM and comment on in which range they are different and briefly explain the cause of difference. You can refer to figure 6 for the explanation of your results.

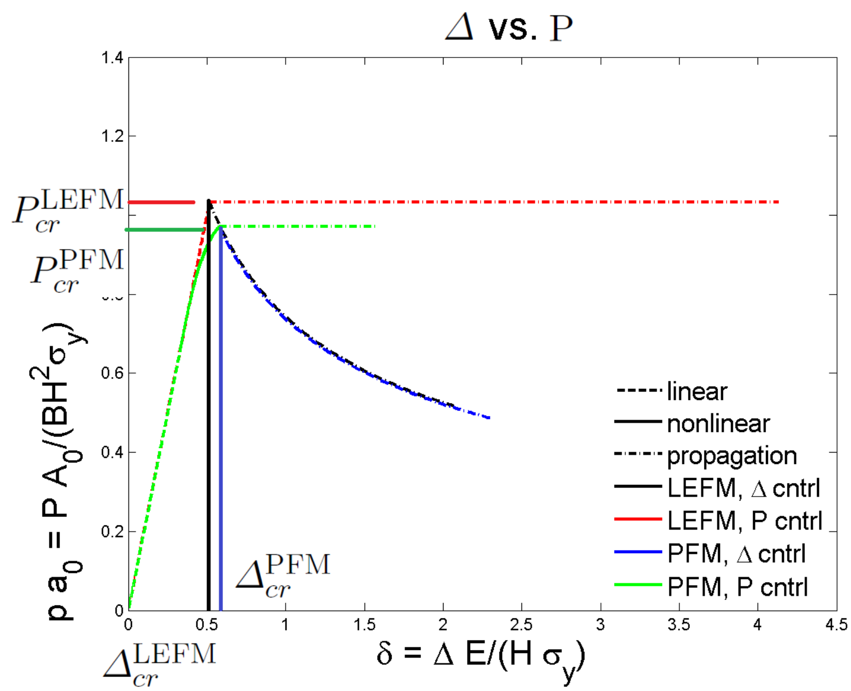


Figure 6: P_{cr} and $\varDelta_{cr}$ from LEFM and PFM analysis of the crack with initial length A_0 for $j_c = 0.8$.