Continuum Mechanics: Equation Sheet

Tensor Algebra

Indicial Notation

Kronecker’s Delta

\[ \delta_{ij} = \delta_{ji} \]
\[ \delta_{ii} = 3 \]
\[ \delta_{ij}a_{p\ldots j\ldots q} = a_{p\ldots i\ldots q} \]
\[ \delta_{ij} = 3 \]
\[ \delta_{ij}a_j = a_i \]

Alternating symbol

Kronecker’s Delta

\[ \epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = -\epsilon_{jik} = -\epsilon_{kji} = -\epsilon_{ikj} \]
\[ \epsilon_{ijk} \epsilon_{ipq} = \delta_{jp} \delta_{kq} - \delta_{jq} \delta_{kp} = \left| \begin{array}{cc} \delta_{jp} & \delta_{jq} \\ \delta_{kp} & \delta_{kq} \end{array} \right| \]
\[ \epsilon_{ijk} \epsilon_{ijq} = 2\delta_{kq} \]
\[ \epsilon_{ijk} \epsilon_{ijk} = 6 \]

0.1 Determinant and Matrix Inverse

\[ \det A = \epsilon_{ijk} A_{i1} A_{j2} A_{k3} = \epsilon_{ijk} A_{i1} A_{j2} A_{k3} \]
\[ \epsilon_{pqr} \det A = \epsilon_{ijk} A_{ip} A_{jq} A_{kr} = \epsilon_{ijk} A_{pi} A_{qj} A_{rk} \]
\[ \det A = \frac{1}{6} \epsilon_{ijk} \epsilon_{pqr} A_{ip} A_{jq} A_{kr} = \frac{1}{6} \epsilon_{ijk} \epsilon_{pqr} A_{ip} A_{qj} A_{rk} \]
\[ \epsilon_{ijk} \epsilon_{pqr} \det A = \left| \begin{array}{ccc} A_{ip} & A_{iq} & A_{ir} \\ A_{jp} & A_{jq} & A_{jr} \\ A_{kp} & A_{kq} & A_{kr} \end{array} \right| \]
\[ A_{rk}^{-1} = \frac{1}{2 \det A} \epsilon_{ijk} \epsilon_{pqr} A_{ip} A_{jq} \]
\[ A x = b \quad (x_r = A_{rk}^{-1} b_k) \quad \Rightarrow \quad x_r = \frac{1}{2 \det A} \epsilon_{ijk} \epsilon_{pqr} A_{ip} A_{jq} b_k \]
\[ \epsilon_{ijk} A_{im} A_{jn} = \epsilon_{mnp} \det(A) A_{pk}^{-1} \]
\[ \frac{d(\det A)}{d\alpha} = \text{trace} \left( \frac{dA}{d\alpha} A^{-1} \right) \det A \quad \alpha \text{ any argument (dependency) of } A \text{ such as time } t \]

Definition of Tensor Product

\[ (a \otimes b) v = (b \cdot v) a \]
Properties of the Tensor Product

\[(a \otimes b)^T = (b \otimes a)\]
\[(a \otimes b)(c \otimes d) = (b \cdot c) a \otimes d\]

Let \(\{e_1, e_2, e_3\}\) be an orthonormal set, then

\[(e_i \otimes e_i)(e_j \otimes e_j) = \begin{cases} 0, & \text{for } i \neq j, \\ e_i \otimes e_i, & \text{for } i = j, \end{cases}\]
\[\sum_{i=1}^{3} e_i \otimes e_i = I\]

Definition of Trace of a Tensor Product

\[\text{tr}(a \otimes b) = a \cdot b\]
\[\text{tr}A = \sum_{i=1}^{3} A_{ii}\]
\[\text{tr}S^T = \text{tr}S\]
\[\text{tr}(ST) = \text{tr}(TS)\]
\[S \cdot T = \text{tr}(S^T T)\]

Properties of the Inner Product

\[R \cdot (ST) = (S^T R) \cdot T = (RT^T) \cdot S\]
\[u \cdot S v = S \cdot (u \otimes v)\]
\[(a \otimes b) \cdot (u \otimes v) = (a \cdot u)(b \cdot v)\]
\[S(a \otimes b) = (Sa) \otimes b\]
\[(a \otimes b) S = a \otimes (S^T b)\]
\[\sum_{i=1}^{3} (Se_i) \otimes e_i = S\]

Relation between Skew-symmetric Tensors and Cross-Product

There is a one-to-one correspondence between vectors and skew-symmetric tensors: given any skew-symmetric tensor \(W\) there exists a unique vector \(w\) such that

\[Wv = w \times v, \quad \forall v \in \mathcal{V}.\]

Furthermore, if \(\{\alpha, \beta, \gamma\}\) are the components of the vector \(w\) with respect to an orthonormal basis, then

\[\begin{bmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{bmatrix}.\]
Orthogonal tensors - Rotations

Any orthogonal tensor $Q$ corresponds to a rotation (potentially with zero angle, i.e., identity map), and a reflection if $Q$ is improper, that is $\det Q = -1$. If we have the axis $a$ and rotation angle $\theta$, we can express $Q$ as,

$$Q = \cos \theta I + (1 - \cos \theta) a \otimes a + \sin \theta a x(a)$$

Other useful identities are,

$$Qa = \pm 1a \quad \text{for proper and - for improper orthogonal } Q,$$

that is $a$ is an eigenvector of $Q$.

Inversely, if we have the rotation $Q$ and want to obtain the axis $a$ and angle $\theta$ of rotation, we use the above equation to derive them as follows (see “Brannon_Rebecca_rotation.pdf” pages 43-44):

1. Obtain $c = \frac{\text{trace}(Q)-1}{2}$.
2. If $c = 1$, $Q = I$. That is, $Q$ is an identity tensor, and any direction would be an axis of rotation with zero angle of rotation.
3. If $c = -1$, angle of rotation is $\theta = \pi$(radians) = $180^\circ$. That is, $Q$ is reflection with respect to the axis of $Q$, $a$. The axis of $Q$ is obtained by normalizing any nonzero column of $Q+I$.
4. If $|c| \neq 1$ we need to compute the angle and axis of $Q$ as follows,

   (a) $s = \pm \sqrt{1-c^2}$.

   (b) Compute the axis of rotation:

$$a = ax(Q) = \frac{1}{\sin \theta} ax[\text{skew}(Q)] \quad \Rightarrow \quad a_1 = \frac{1}{2s}(Q_{32} - Q_{23}) \quad a_2 = \frac{1}{2s}(Q_{13} - Q_{31}) \quad a_3 = \frac{1}{2s}(Q_{21} - Q_{12})$$

Spectral Decomposition Theorem

Let $S$ be a symmetric tensor. Then there is an orthonormal basis for $V$ consisting entirely of eigenvectors of $S$. Moreover, for any such basis \{\textbf{e}_1, \textbf{e}_2, \textbf{e}_3\}, the corresponding eigenvalues $\omega_1$, $\omega_2$, and $\omega_3$, when ordered, form the entire spectrum of $S$ and

$$S = \sum_{i=1}^{3} \omega_i \textbf{e}_i \otimes \textbf{e}_i.$$ 

Conversely, if $S$ has the form

$$S = \sum_{i=1}^{3} \omega_i \textbf{e}_i \otimes \textbf{e}_i,$$

with \{\textbf{e}_1, \textbf{e}_2, \textbf{e}_3\} orthonormal, then $\omega_1$, $\omega_2$, and $\omega_3$ are eigenvalues of $S$ with $\textbf{e}_1$, $\textbf{e}_2$, and $\textbf{e}_3$ being the corresponding eigenvectors. Furthermore,

(a) $S$ has exactly three distinct eigenvalues if and only if the characteristic spaces of $S$ are three mutually perpendicular lines through $0$. 

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(b) $S$ has exactly two distinct eigenvalues if and only if $S$ admits the representation

$$S = \omega_1 e \otimes e + \omega_2 (I - e \otimes e), \quad \|e\| = 1 \text{ and } \omega_1 \neq \omega_2.$$ 

In this case $\omega_1$ and $\omega_2$ are two distinct eigenvalues and the corresponding characteristic spaces are $\text{span}\{e\}$ and $\{e\}^\perp$, respectively. Conversely, if $\text{span}\{e\}$ and $\{e\}^\perp$ (with $\|e\| = 1$) are the characteristic spaces of $S$ then $S$ must have the form

$$S = \omega_1 e \otimes e + \omega_2 (I - e \otimes e), \quad \omega_1 \neq \omega_2.$$ 

(c) $S$ has exactly one eigenvalue if and only if

$$S = \omega I.$$ 

In this case $\omega$ is the eigenvalue and $\mathcal{V}$ is the corresponding characteristic space. Conversely, if $\mathcal{V}$ is a characteristic space for $S$ then $S$ has the form

$$S = \omega I.$$ 

Polar Decomposition Theorem

Let $F \in \text{Lin}^+$, i.e., let $F$ be a second order tensor with positive determinant. Then there exists positive definite symmetric tensor $U$ and $V$ along with a proper rotation $R$ such that

$$F = RU = VR.$$ 

Moreover, each of these decompositions is unique; in fact

$$U = \sqrt{FF^T} \quad \text{and} \quad V = \sqrt{FF^T}.$$ 

We call the representations $F = UR$ and $F = RV$ the right and left polar decompositions of $F$, respectively.

Principal Invariants of a Tensor

Given a tensor $S$, the determinant of the tensor $S - \omega I$ admits the representation

$$\det (S - \omega I) = -\omega^3 + \mathcal{I}_1(S) \omega^2 - \mathcal{I}_2(S) \omega + \mathcal{I}_3(S) \quad \forall \omega \in \mathbb{R},$$

where

$$\mathcal{I}_1(S) = \text{tr} (S),$$

$$\mathcal{I}_2(S) = \frac{1}{2} \left( (\text{tr}(S))^2 - \text{tr} (S^2) \right),$$

$$\mathcal{I}_3(S) = \det (S).$$

The elements of the list $\{\mathcal{I}_1(S), \mathcal{I}_2(S), \mathcal{I}_3(S)\}$ are called the principal invariants of $S$. Finally, if $S$ is symmetric, its principal invariants are completely characterized by the spectrum of $S$. In fact, we have

$$\mathcal{I}_1(S) = \omega_1 + \omega_2 + \omega_3,$$

$$\mathcal{I}_2(S) = \omega_1 \omega_2 + \omega_2 \omega_3 + \omega_1 \omega_3,$$

$$\mathcal{I}_3(S) = \omega_1 \omega_2 \omega_3,$$

$\omega_1, \omega_2, \text{ and } \omega_3$ being the eigenvalues of $S$. 

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**Cayley-Hamilton Theorem**

Recall that the equation

\[
\det (S - \omega I) = 0
\]

is called the characteristic equation of \( S \).

Every second order tensor \( S \) satisfies its own characteristic equation, that is,

\[
S^3 - I_1(S) S^2 + I_2(S) S - I_3(S) I = 0.
\]
Tensor Analysis

Useful Differentiation Formulas

Let $F$, $U$, and $S$ be smooth tensor fields over. Let $u$ and $v$ be smooth vector fields. Finally, let $\phi$ be a smooth scalar field. Then,

\[
D \det (F) [U] = \det (F) \tr (UF^{-1}), \\
\div v = \tr (\grad v), \\
\grad (\phi v) = \phi \grad v + v \otimes \grad \phi, \\
\div (\phi v) = \phi \div v + v \cdot \grad \phi, \\
\grad (v \cdot u) = (\grad u)^T v + (\grad v)^T u, \\
\div (v \otimes u) = v \div u + (\grad v) u \\
\div (S^T v) = S \cdot \grad v + v \cdot \div S \\
\div (\phi S) = \phi \div S + S \grad \phi
\]

Divergence Theorem

Let $B$ be a regular region of a Euclidean space $\mathcal{E}$. Let $\partial B$ be the boundary of $B$ with outer unit normal $n$. Let $S$ be a smooth tensor fields over $B$. Let $v$ and $w$ be smooth vector fields over $B$. Finally, let $\phi$ be a smooth scalar field over $B$. Then,

\[
\int_{\partial B} \phi n ds = \int_B \grad \phi dv \\
\int_{\partial B} v \cdot n ds = \int_B \div v dv \\
\int_{\partial B} Sn ds = \int_B \div S dv \\
\int_{\partial B} v \otimes n ds = \int_B \grad v dv \\
\int_{\partial B} (Sn) \otimes v ds = \int_B [(\div S) \otimes v + S (\grad v)^T] dv \\
\int_{\partial B} v \cdot Sn ds = \int_B (v \div S + S \cdot \grad v) dv \\
\int_{\partial B} v(w \cdot n) ds = \int_B [v \div w + (\grad v)w] dv
\]