DC20200113

Monday, January 13, 2020 1:03 PM

Partial Differential Equations (PDEs) classification groups

PDEs can be classified from different perspectives:

1. **Order of PDE**: The highest order of PDE

$u_t = u_{xx}$ (second order)

$u_t = u_x$ (first order)

$u_t = uu_{xxx} + \sin x$ (third order)

- $u_t + u_{xxx} + uu_x = 0$ (KdV Eqn., third order)
- $u_x^2 + u_y^2 = c^2$ (Eikonal Eqn. of Geometric Optics, first order)

2. **Number of variables**: The number of independent variables for all the involved functions:

$u_t = u_{xx}$ (two variables:

$u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$ (three variables: r, θ, and t)

3. **Homogeneity**: If the source term (right hand side) of the equation is zero the PDE is called homogeneous. The same concept applies to initial (IC) and boundary (BC) conditions of a PDE (RHS of the IC/BC differential operator is zero)

homogeneity

Zero "source"

- $u_t + u_x = 0$ is homogeneous
- $u_{xx} + u_{yy} = x^2 + y^2$ is inhomogeneous

4. **Type of coefficient**:
   - Constant coefficient (function & its derivative terms have constant coefficients)

     numbers

     $$3\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + 5\frac{\partial^2 u}{\partial y^2}x + 2\frac{\partial u}{\partial y} = 0$$

   - Variable coefficient
     - Coefficients only function of independent variables (e.g. x, t)

       more difficult

       $$e^{2x}\frac{\partial^2 u}{\partial x^2} + 2e^{x+y}\frac{\partial^2 u}{\partial x \partial y} + e^{2y}\frac{\partial^2 u}{\partial y^2} = 0$$

     - Coefficients function of independent variables AND the function (e.g. x, t, u)

$$\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} + u^2 = 0$$

factor

derivative term

Source: [Farlow, 2012]

9

5. Hyperbolic / parabolic / elliptic PDEs:

- The classification becomes more clear in the next few slides. Below is the brief description of some of their characteristics and sample applications.

- **Hyperbolic PDEs** correspond to the propagation of waves and there is a finite speed of propagation of waves. They tend to preserve or generate discontinuities (in the absence of damping). Hyperbolic PDEs are often transient although some steady-state limits of transient PDEs can be hyperbolic as well (e.g. steady advection problems).
  Examples: Elastodynamics, Transient electromagnetics; Acoustic equation.
- **Parabolic PDEs**: Unlike hyperbolic PDEs the speed of propagation of information is infinite for parabolic PDEs. They also tend to dissipate sharp solution features and have a "diffusive" behavior. Many transient diffusion problems are modeled (or idealized) by parabolic PDEs. Some examples are
  Examples: Fourier heat equation; Viscous flow (Navier-Stokes equations)
- **Elliptic PDEs**: Elliptic problems are characterized by the global coupling of the solution. They often correspond to steady-state limit of hyperbolic and parabolic PDEs.

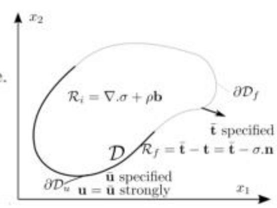
Source: [Farlow, 2012]

1. Elliptic equations:

- Sample:

$$\nabla.\sigma + \rho \mathbf{b} = 0 \quad \text{Elastostatics equation}$$

$$\Delta u = f, \Delta u = \Sigma_{i=1}^{d} u_{,ii} \text{ (Laplacian)}, \quad \text{Poisson equation}$$



$\nabla \cdot K \nabla u = f$...

- The entire domain is physically coupled and often numerical methods involve a global solve.
- They are often steady state limit of parabolic/hyperbolic systems.

2. Parabolic equations(dynamic):

- Sample:

$$C\frac{dT}{dt} - \kappa \Delta T = Q, \text{(constant } \kappa), \quad \text{Parabolic(Fickian) heat equation}$$

- Imply an infinite speed of propagation of information.
- The entire spatial domain is coupled.
- Numerical methods may involve the solution of the global spatial domain or local domains.
- The diffusive operator **smoothens the solution**.

$T(x, t = 0) = \delta(x)$

$G(x,t) = \sqrt{\frac{C}{4\pi\kappa t}} \exp\left(-\frac{Cx^2}{4\kappa t}\right)$ Green's function

3. **Hyperbolic equations**(dynamic):

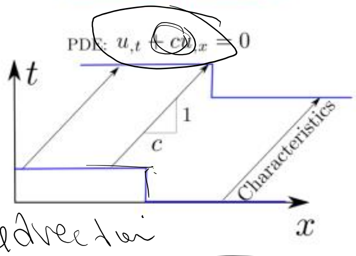
- Sample:

$$\rho\frac{d^2\mathbf{u}}{dt^2} - \nabla.\sigma = \rho\mathbf{b} \qquad \text{Elastostatics equation}$$

$$\frac{d^2u}{dt^2} - k\Delta u = f$$

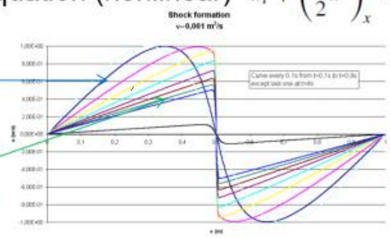
- There is a maximum speed for the propagation of waves (information).
- Due to finiteness of the wave speed the spatial domain is NOT globally coupled.
- Numerical methods may employ the locality of hyperbolic systems to devise local solution schemes.
- Unlike parabolic equations, hyperbolic equations preserve discontinuities or even generate them (nonlinear equations).

$d_t + (u) u_{,x} = 0$ speed



advection eqn moves IC with speed c

Burger's equation (nonlinear) $u_t + \left(\frac{1}{2}u^2\right)_x = 0$



t = 0, smooth solution

t > 0, shock has formed

---

6. Linearity:

- The PDE is linear if the dependent variable and all its derivatives appear linearly in the PDE. The nonlinear PDEs are classified into several groups as their solution characteristics can be quite distinct:

Linear

nonlinear { semi-linear, quasi-linear, fully-nonlinear }

Notations:

- Multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$

  order of the multi-index is $|\alpha| = \alpha_1 + \ldots + \alpha_n$

- Multi-index partial derivative

$$D^\alpha u = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}u.$$

example: $\alpha = (1,2)$ ⟹ $D^\alpha u = \partial_x \partial_y^2 u = u_{xyy}$

- Collection of all partial derivatives of order *k*: $D^k u = \{D^\alpha u \,:\, |\alpha| = k\}$

example: If $u = u(x_1, \ldots, x_n)$, then $D^1 u = \{u_{x_i} \,:\, i = 1, \ldots, n\}$.

Source: [Levandosky, 2002]

$\alpha = (1,2)$ — $x_1(1)$, $x_2(2)$ $\quad D^\alpha u = \partial_x \partial_y \partial_y u = \partial^1_{x_1}\partial^2_{x_2}u$

$\alpha = (4,3)$ $\quad D^\alpha u = \partial^3_{x_1}\partial^4_{x_2}u$ $\quad |\alpha| = 4+3 = 7$ — total # of derivatives

integer

$$D^k u = \{ D^\alpha u \mid |\alpha| = k \}$$

x_1, x_2 independent parameters

$$D^3 u = \{ D^{(3,0)}u, D^{(2,1)}u, D^{(1,2)}u, D^{(0,3)}u \} = \{ \partial^3_{x_1}u, \partial^2_{x_1}\partial_{x_2}u, \partial_{x_1}\partial^2_{x_2}u, \partial^3_{x_2}u \}$$

all 3 derivative terms of u

3 — total #

General form of PDE:

$$F(\vec{x}, u, Du, D^2u, \ldots, D^k u) = 0$$

k^{th} order PDE

x_1, x_2, x_3 indep.

$\{ u_{x_1}, u_{x_2}, u_{x_3} \}$

$$F = u^2 \frac{\partial^7 u}{\partial x_2^7} - 5 = 0 \qquad k = 7$$

$$F(x) = 7x - 3 = 0$$

A. Linear: If u and its derivatives appear in a linear fashion. That is F can be written as,

$$\sum_{|\alpha| \le k} a_\alpha(\vec{x}) D^\alpha u = f(\vec{x}).$$

- summation of $D^\alpha u$'s
- factors are only functions of x

Examples:

- $1 u_t + 1 u_x = 0$ is homogeneous linear
- $1 u_{xx} + 1 u_{yy} = 0$ is homogeneous linear.
- $1 u_{xx} + 1 u_{yy} = x^2 + y^2$ is inhomogeneous linear.
- $1 u_t + x^2 u_x = 0$ is homogeneous linear.
- $1 u_t + 1 u_{xxx} + u u_x = 0$ is not linear.
- $u_x^2 + u_y^2 = 1$ is not linear.

factor is not function of x only

$u_x u_x$

Source: [Levandosky, 2002]

if homog. & linear u_1 & u_2 are sln so is $\alpha u_1 + \beta u_2$

- $u_{xx} + u_{xxx} = 0$ ☆

$u_{xx} + u_{yyy} = 0$ ☆

u_1 & u_2 sln of ☆ so is $\alpha u_1 + \beta u_2$ is

$u_{xx} + u_{yyy} = x^7 + y$ inhomog. (☆☆)

$u = u_p + \sum \alpha_i u_i$ is a sln to (☆☆)

superposition

$uu_{,x}$ → not linear — does not apply

B. Semi-linear: is a ~~nonlinear PDE where the~~ highest order derivatives can be written in a linear fashion of functions of ***x***. That is, such coefficients are only functions of independent coordinate ***x***. The PDE can be written as,

$$\sum_{|\alpha|=k} a_\alpha(\vec{x}) D^\alpha u + a_0(D^{k-1}u, D^{k-2}u, \ldots, Du, u, \vec{x}) = 0.$$

$a(x,y,u)\,u_{,x} + b(x,y,u)\,u_{,y} = c(x,y,u)$ adv. eqn

nonlinear because of ↑ ↑

1st order — 0th order

semi-linear — linear — I don't care

$a(x,y)\,u_{,x} + b(x,y)\,u_{,y} = c(x,y,u)$ — $(x^7+y^7)u^2$

highest derivatives are linear

linear $a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y)$

0th term — source term

- $u_t + u_{xxx} + uu_x = 0$ is semilinear. (3)
- $u_{xx} + u_{yy} = u^3$ is semilinear. (order 2)
- $u_t + xu_x = 0$ is linear.

- $u_{xx} + u_{yy} = u^3$ is semilinear.
- $u_t + x u_x = 0$ is linear.
- $u_t + u u_x = 0$ quasilinear butnot semilinear.

C. Quasi-linear: is a nonlinear PDE, that is not semilinear and its highest derivatives can be written as linear function of functions of x **and lower order derivatives of u**. That is, it can be written as,

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \ldots, Du, u, \vec{x}) D^\alpha u + a_0(D^{k-1}u, \ldots, Du, u, \vec{x}) = 0.$$

That is the coefficients of highest order terms depend on

$$\vec{x}, u, \ldots, D^{k-1}u, \text{ but not on } D^k u.$$

Examples:

$a(x,y,u)\dfrac{\partial u}{\partial x} + b(x,y,u)\dfrac{\partial u}{\partial y} = c(x,y,u)$ Is quasilinear
semilinear if a(x, y), b(x,y)
linear if a(x, y), b(x, y) and c = u d(x, y)

- $u_t + a(u) u_x = 0$ is quasilinear.
- $u_x^2 + u_y^2 = 1$ is not quasilinear.

16

D. Fully-nonlinear: If it's nonlinear and cannot be written in quasi-linear, semi-linear forms.

$$\frac{\partial u}{\partial x_1} + \left(\frac{\partial u}{\partial x_2}\right)^2 = 0$$

$$u_{xx}\, u_{yy} - (u_{xy})^2 = \psi \qquad \text{Monge-Ampère equation}$$

- $u_x^2 + u_y^2 = c^2$ (Eikonal Eqn. of Geometric Optics, first order)

For a list of well-known nonlinear (semi-linear, quasi-linear, and fully nonlinear) PDEs refer to here (https://en.wikipedia.org/wiki/List_of_nonlinear_partial_differential_equations)

17

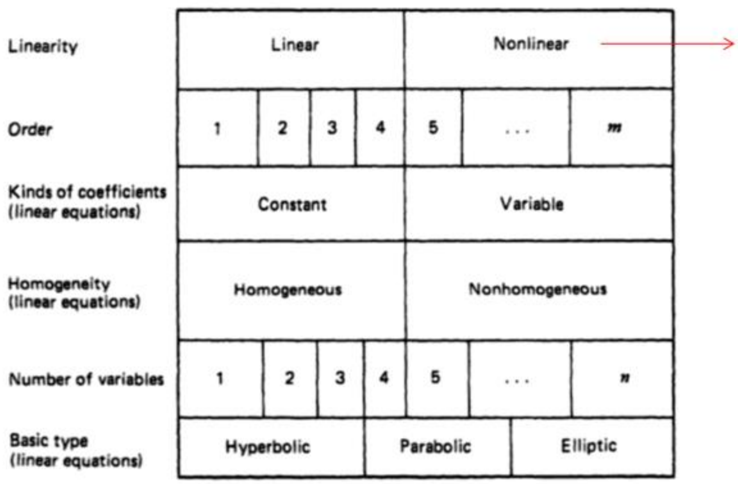
Source: [Levandosky, 2002]

$\sigma = S(\varepsilon) \rightarrow \bar{S}(\nabla u)$

$\nabla \cdot \bar{S}(\nabla u) + \rho b = 0$

$S_{ij}(u_{,k})_{,j}$

$S_{ij,j}(u_{,k})$



- Semi-linear
- Quasi-linear
- Fully-nonlinear

FIGURE 1.1 Classification diagram for partial differential equations.

Source: [Farlow, 2012]

18

Concept of characteristics for hyperbolic PDEs:

- We are interested in solving the **1st order linear PDE in two variables**:

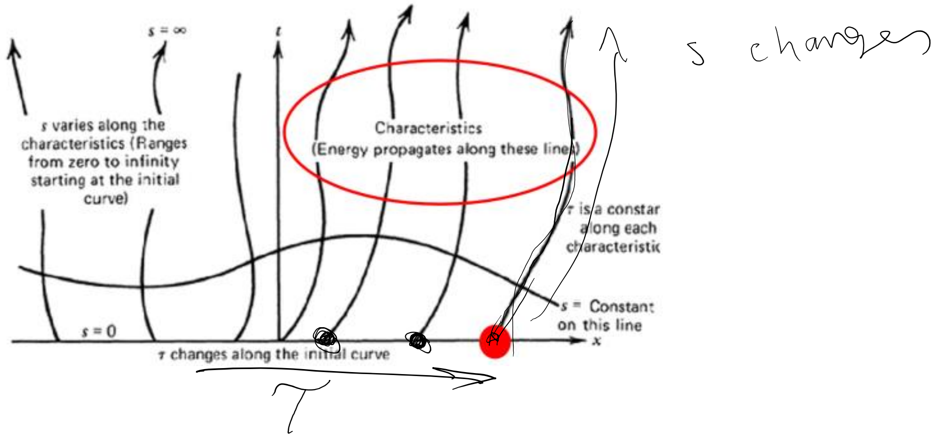
$$\text{PDE} \quad a(x,t)u_x + b(x,t)u_t + c(x,t)u = 0 \qquad -\infty < x < \infty, \quad 0 < t < \infty$$

$$\text{IC} \quad u(x,0) = \phi(x) \qquad -\infty < x < \infty$$

$a(x,t)u_x + b(x,t)u_t = \frac{du}{ds}$

$\frac{du}{ds} + cu = 0$ ODE is S we know how to solve ODEs

- The ODEs are solved along characteristic where variable s changes



$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

$$= \frac{\partial u}{\partial x}\, a(x,y) + \frac{\partial u}{\partial y}\, b(x,y)$$

original term $a(x,y)u_x + b(x,y)u_y + c(x,y)u$

$$\frac{\partial x}{\partial s} = a(x,y)$$
$$\frac{\partial y}{\partial s} = b(x,y)$$

Find s such that this eqn is satisfied

- Consider the following PDE with constant coefficients:

$$\text{PDE} \quad 1\, u_x + 1\, u_t + 2u = 0 \qquad -\infty < x < \infty \qquad 0 < t < \infty$$
$$\text{IC} \quad u(x,0) = \sin x \qquad -\infty < x < \infty$$

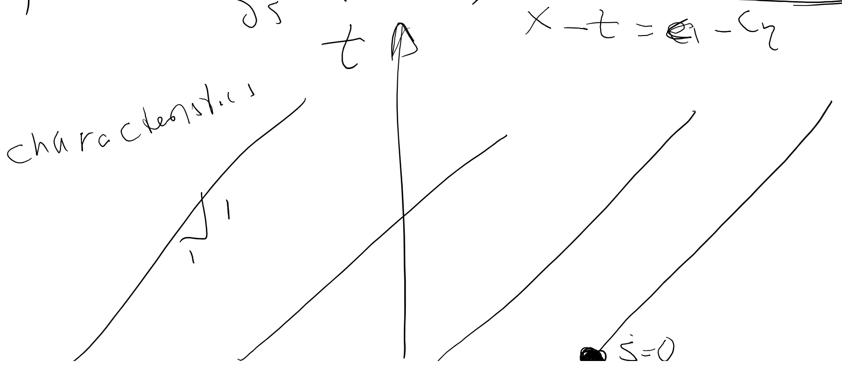
$$a(x,t)u_x + b(x,t)u_t + c(x,t)u = 0$$
$a \to 1$, $b \to 1$, $c \to 2$

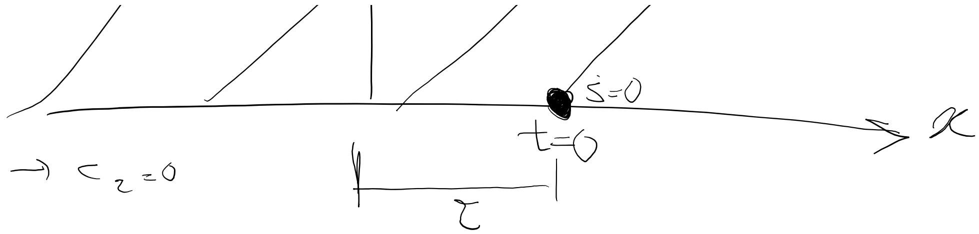
$$\frac{\partial x}{\partial s} = a(x,t) = 1, \quad \frac{\partial t}{\partial s} = b(x,t) = 1 \;\longrightarrow\; \frac{\partial x}{\partial s} = 1 \to x = s + c_1, \quad \frac{\partial t}{\partial s} = 1 \to t = s + c_2$$

$$x - t = c_1 - c_2$$

$x(s)$, $t(s)$

$$\frac{dx}{dt} = \frac{\partial x/\partial s}{\partial y/\partial s} = \frac{1}{1} = 1$$





$t = s + c_2$ at $s=0$, $t=0 \longrightarrow c_2 = 0$

$$\boxed{t = s}$$

$x = s + c_1$ at $s=0$

$\tau = 0 + c_1 \longrightarrow c_1 = \tau$

$$\boxed{\begin{aligned} x &= s + \tau \\ t &= s \end{aligned}}$$

s: parameter on characteristic line

τ: starting point of characteristic on x axis

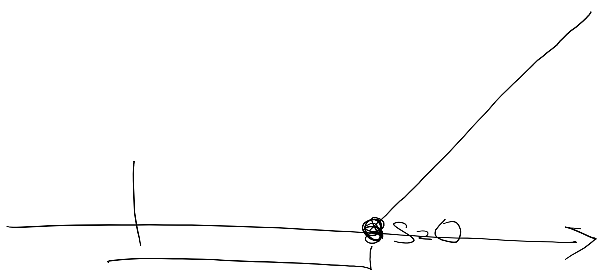
$\underbrace{u_x + u_y}$ $+ 2u = 0$

$\frac{du}{ds} + 2u = 0 \longrightarrow \frac{du}{u} = -2ds \longrightarrow \ln u = -2s + A$

ODE :)

$$\boxed{u = C_3 e^{-2s}}$$

Use IC to get c_3:



$u(x, t=0) = \sin x = \sin\tau$

$u(s, \tau) = C_3(\tau) e^{-2s} = C_3(\tau) e^{-2\cdot 0} = C_3(\tau)$ $\quad x = \tau$

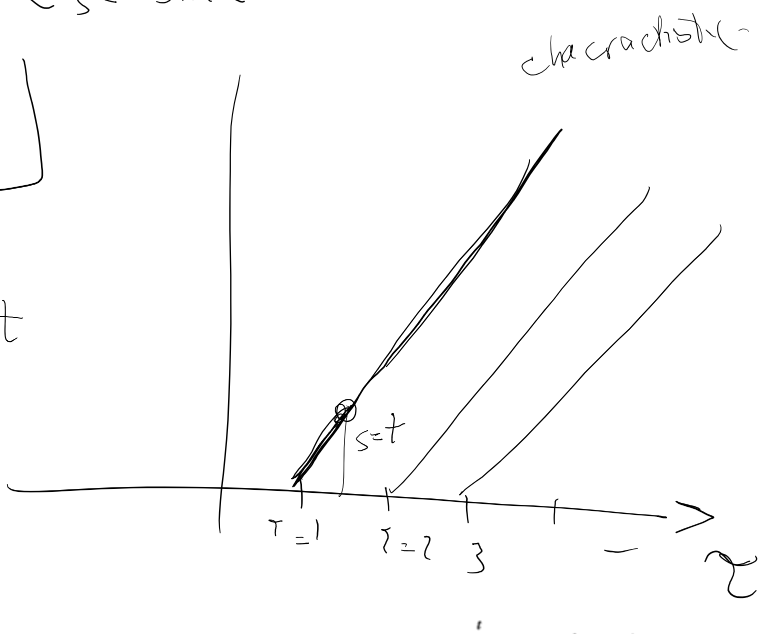
@ $s=0$ $\quad x = \tau$ $\quad C_3 = \sin\tau$

$$\boxed{u(s, \tau) = \sin\tau e^{-2s}}$$

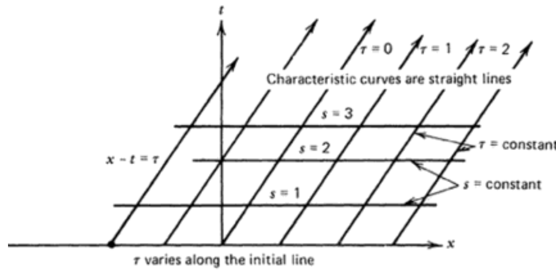
$u(x,t) = ?$

$$\begin{aligned} x &= s + \tau \\ t &= s \end{aligned} \longrightarrow \begin{aligned} \tau &= x - t \\ s &= t \end{aligned}$$

$$\boxed{u(x,t) = \sin(x-t) e^{-2t}}$$



$u(x,t) = \sin(x-t)e$



$\frac{du}{ds}$ = source

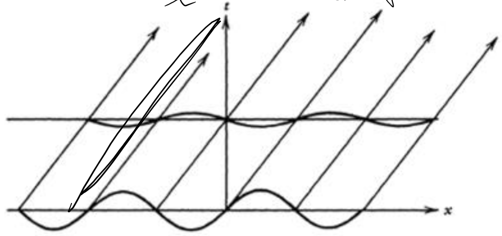
- So the solution to

$$\text{PDE} \quad u_x + u_t + 2u = 0 \quad -\infty < x < \infty \quad 0 < t < \infty$$
$$\text{IC} \quad u(x,0) = \sin x \quad -\infty < x < \infty$$

Is

$$u(x,t) = \sin(x-t)e^{-2t}$$

$x - t$ = constant



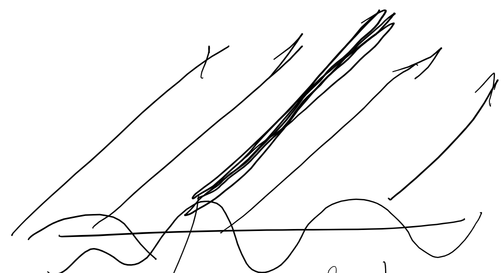
Which is a sine wave moving with speed 1 and damping in time

$\frac{du}{ds} = 0$

$u_x + u_t = 0$

$u(x,0) = \sin x$

$u(x,t) = \sin(x-t)$



on characteristic sln remains constant