DC20200115

Wednesday, January 15, 2020 1:09 PM

Course notes:
http://rezaabedi.com/wp-content/uploads/Courses/DynamicsContinua2016/ComputerMethods_Dynamics_continua.pdf

http://rezaabedi.com/wp-content/uploads/Courses/DynamicsContinua2016/DynamicContinua.pdf

----

- Consider the problem

$$\text{PDE} \quad xu_x + u_t + tu = 0 \qquad -\infty < x < \infty \qquad 0 < t < \infty$$
$$\text{IC} \quad u(x,0) = F(x) \qquad \text{(an } \textit{arbitrary} \text{ initial wave)}$$

$$xu_x + u_t \longrightarrow \frac{du}{ds} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial s}$$
$$\frac{du}{ds} = \frac{\partial u}{\partial x}\, x + \frac{\partial u}{\partial t}\, 1 \quad \longrightarrow \quad \boxed{\frac{\partial x}{\partial s} = x \qquad \frac{\partial t}{\partial s} = 1}$$

$$a u_x + b u_t = \frac{du}{ds} = u_x x_s + u_t t_s$$

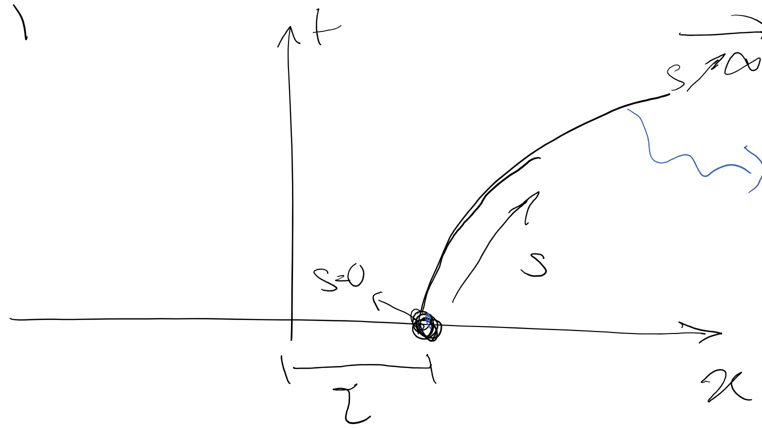
$$\boxed{a = \frac{\partial x}{\partial s} \qquad b = \frac{\partial t}{\partial s}}$$

same same

$$\frac{du}{ds} = a(x,t,u)\, u_{,x} + b(x,t,u)\, u_{,t}$$
$$\frac{du}{ds} = \frac{\partial x}{\partial s}\frac{\partial u}{\partial x} + \frac{\partial t}{\partial s}\frac{\partial u}{\partial t}$$

$$\boxed{\frac{\partial x}{\partial s} = a(x,t,s) \qquad \frac{\partial t}{\partial s} = b(x,t,s)}$$

$$\frac{dx}{ds} = x \rightarrow \frac{dx}{x} = ds \rightarrow \ln x = s + c_1 \rightarrow \boxed{x = c_2 e^{s}}$$
$$\frac{dt}{ds} = 1 \longrightarrow \boxed{t = s + c_3}$$



We solve ODE's on characteristics

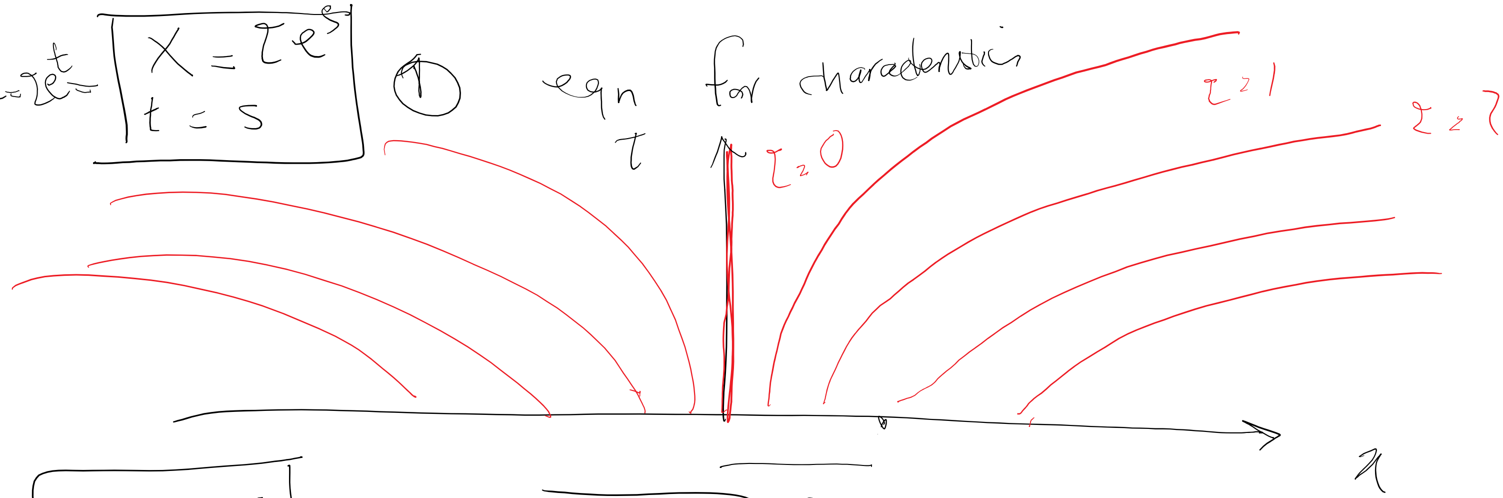
$$@\ s = 0 \quad x = \tau,\ t = 0 \longrightarrow \tau = c_2 \cdot e^0,\ 0 = 0 + c_3 \longrightarrow c_2 = \tau,\ c_3 = 0$$

$t=0$ $\qquad 0 = 0 + c_3 \longrightarrow c_3 = 0$

$x = \tau e^t$

$$\boxed{\begin{aligned} x &= \tau e^s \\ t &= s \end{aligned}}$$

(1) eqn for characteristics



$$\boxed{\begin{aligned} x &= \tau e^s \\ t &= s \end{aligned}}$$

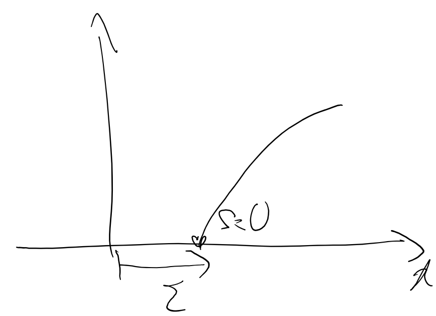
$\underbrace{xu_x + u_t}_{du/ds} + tu = 0$ PDE

$\frac{du}{ds} + tu = 0$ $\qquad t = s$ $\qquad$ need to write everything in terms of s & τ

$\frac{du}{ds} + su = 0 \longrightarrow \frac{du}{u} = -s\,ds \longrightarrow \ln u = -\frac{s^2}{2} + c_4 \longrightarrow$

(2) $\boxed{u = c_5 e^{-s^2/2}}$ $\qquad c_5 = ?$

Use IC to get c_5



$u(x, t=0) = F(x) \longrightarrow$

$u(s=0, \tau) = F(\tau)$

(2) $\Rightarrow c_5 e^{-s^2/2}\Big|_{s=0} = F(\tau) \longrightarrow c_5 = F(\tau)$

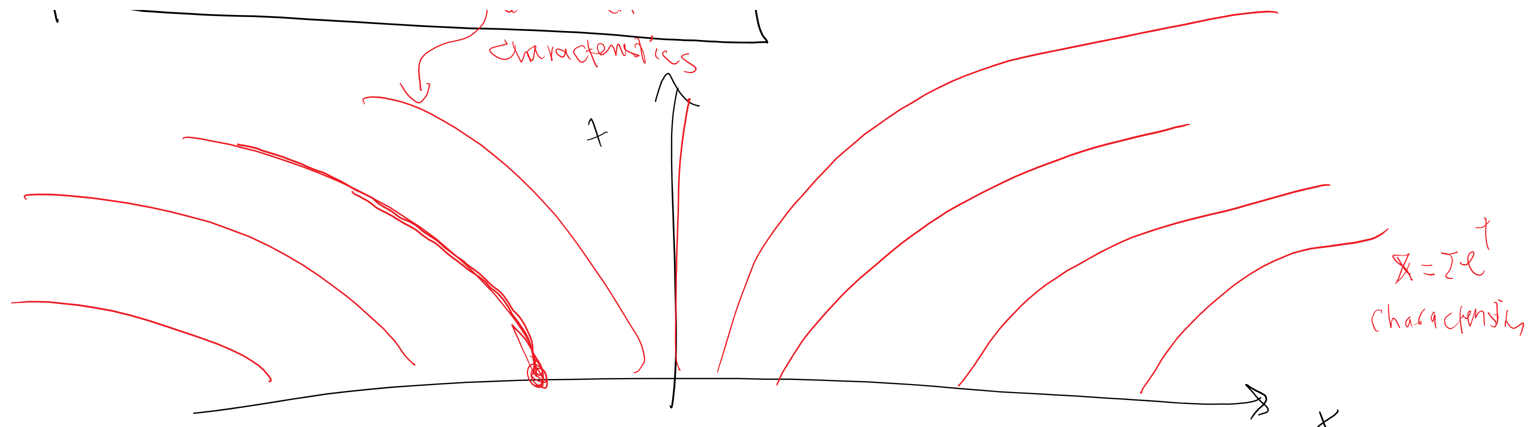
(3) $\boxed{u = F(\tau)\, e^{-s^2/2}}$

$\begin{aligned} x &= \tau e^s \\ t &= s \end{aligned} \longrightarrow \begin{aligned} \tau &= x e^{-t} \\ s &= t \end{aligned}$

$(s, \tau) \to (x, t) \qquad (x, t) \to (s, \tau)$

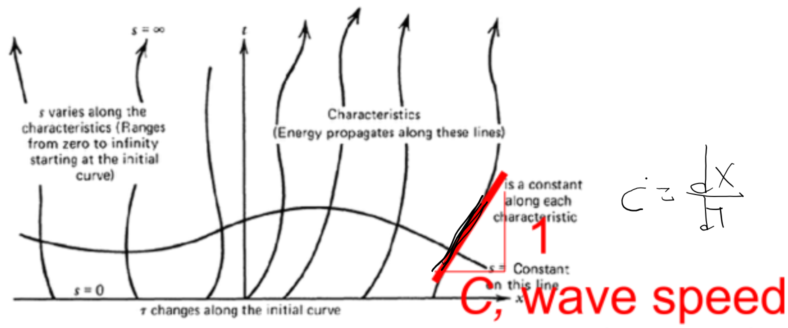
$\boxed{u(x,t) = F(xe^{-t})\, e^{-t^2/2}}$ final solution

constant on characteristics



For $x u_x + u_t = 0$ we would have had $u(x,t) = F(xe^{-t})$

- Key aspect of the first order PDEs we discussed was the solution for characteristic curves along which the solution could be obtained by the solution of an ODE.
  The spatial to temporal slope of characteristics corresponds to the wave speed.



$c = \frac{dx}{dt}$

on characteristics

$$\left.\begin{aligned} \frac{dx}{ds} &= a \\ \frac{dt}{ds} &= b \end{aligned}\right\} \rightarrow c = \frac{dx/ds}{dt/ds} = \frac{a}{b}$$

where PDE

$a u_x + b u_t - \text{Source} = 0$

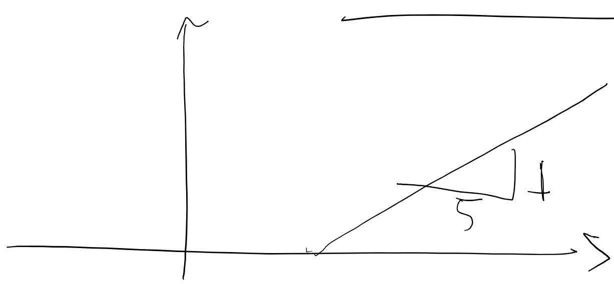
- The hyperbolicity of a PDE corresponds to having characteristic curves along which the solution propagates. For higher order PDEs we investigate if we can breakdown the PDEs into the solution of ODEs along characteristic curves. If this is possible the PDE is hyperbolic and has a finite speed of information propagation at a given point.

29

wave speed is $\boxed{c(x,t) = \frac{a(x,t,u)}{b(x,t,u)}}$

$\underbrace{1}_{b} u_{,t} + \underbrace{5}_{a} u_{,x} = 0$

$c = 5$



- Consider a general 2nd order PDE

$$F(\vec{x}, u, Du, D^2u) = 0.$$

2nd order PDE

2 – independent parameters

- We restrict our attention to linear PDE with 2 independent parameters below (results can easily be generalized to semi-linear case):

$$\boxed{\underbrace{Au_{xx} + Bu_{xy} + Cu_{yy}}_{\text{highest order terms}}} + Du_x + Eu_y + Fu = G$$

lower order terms can be nonlinear

semi-linear

$A(x,y)$ $B(x,y)$ $C(x,y)$

$Az^2 + Bz + C = 0$ $\Delta = B^2 - 4AC$

where A, B, C, D, E, F, G are functions of (x, y) in general (linear PDE). The classification of PDE at a given point (x_0, y_0) is as follows:

1. Hyperbolic at a point (x_o,y_o) if $B^2(x_o,y_o) - 4A(x_o,y_o)C(x_o,y_o) > 0$.
2. Parabolic at a point (x_o,y_o) if $B^2(x_o,y_o) - 4A(x_o(x_o,y_o)C(x_o,y_o) = 0$.
3. Elliptic at a point (x_o,y_o) if $B^2(x_o,y_o) - 4A(x_o,y_o)C(x_o,y_o) < 0$.

- If 1 holds for all (x, y) the PDE is called hyperbolic for all positions (same for 2 and 3)

Examples:

- The wave equation

$$u_{tt} - u_{xx} = 0 \qquad \text{is hyperbolic.}$$

scalar wave eqn (hyperbolic)

$Au_{tt} + Bu_{tx} + Cu_{xx} + \cdots =$

$A \to 1$, $B \to 0$, $C \to -1$

$B^2 - 4AC = 4 > 0$

Laplace eqn

$\underbrace{u_{xx} + u_{yy}}_{\nabla \cdot \nabla u} + \cdots = 0$

$A = 1$ $B = 0$ $C = 1$ $B^2 - 4AC < 0$ elliptic

Fickian (Fourier) heat eqn 1D

$C\dot{T} - Ku_{,xx} = 0$

$A = -k$ $B = 0$ $C = 0$

$u_{,xt}$ $u_{,tt}$

$B^2 - 4AC = 0 \longrightarrow$ parabolic

(a) $u_t = u_{xx}$ $B^2 - 4AC = 0$ (parabolic)

(b) $u_{tt} = u_{xx}$ $B^2 - 4AC = 4$ (hyperbolic)

(c) $u_{\xi\eta} = 0$ $B^2 - 4AC = 1$ (hyperbolic)

(d) $u_{xx} + u_{yy} = 0$ $B^2 - 4AC = -4$ (elliptic)

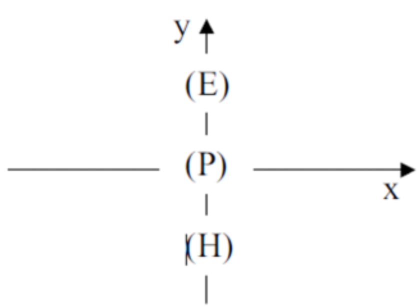
(e) $\boxed{yu_{xx} + u_{yy} = 0}$ $\underline{B^2 - 4AC = -4y}$ $\begin{cases} \text{elliptic for } y > 0 \\ \text{parabolic for } y = 0 \\ \text{hyperbolic for } y < 0 \end{cases}$

Note that PDE can change from based on position: Tricomi equation of transonic flow

$$\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} = 0\,,$$

$A = 1,\ B = 0,\ C = y \Longrightarrow B^2 - AC = -y$

PDE type based on position



Source: [Loret, 2008]

$Au_{xx} + Bu_{xy} + Cu_{yy}$

- The idea is to cast the PDE in the canonical form $(x,y) \to (\xi,\eta)$

1. $\begin{cases} \underline{u_{\xi\xi} - u_{\eta\eta}} = \Psi(\xi, \eta, u, u_\xi, u_\eta) \\ u_{\xi\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta) \end{cases}$ (two canonical forms for the hyperbolic equation)

$\ddot{u} - u_{,xx} = \cdots - 0$ wave eqn

$u_{,xt} = \cdots -$

2. $u_{\eta\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta)$ (the canonical form for the parabolic equation)

$u_{,xx} = f(u, u_{,x}, u_{,t})$

$u_{,xx} = u_{,t}$

2. $u_{\eta\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta)$ (the canonical form for the parabolic equation)

$u_{,xx} = f(u, u_{,x}, u_{,t})$
$u_{,xx} = u_{,t}$

3. $u_{\xi\xi} + u_{\eta\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta)$ (the canonical form for the elliptic equation)

$u_{xx} + u_{yy} = \dots$

- We look for parameters ξ and η that cast the PDE into the hyperbolic form

$$u_{\xi\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta)$$

- By transformation $$\boxed{\begin{aligned}\xi &= \xi(x,y)\\ \eta &= \eta(x,y)\end{aligned}}$$

- By change of parameters we obtain

$u_{,x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x\\
u_y &= u_\xi \xi_y + u_\eta \eta_y\\
u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}\\
u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy}\\
u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy}
\end{aligned}$$

$u_{,xx} = \frac{\partial}{\partial x} u_{,x}$

plug into $Au_{xx} + Bu_{xy} + Cu_{yy} = \dots = \dots$

- Substituting into the original PDE $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$ we obtain

$$\bar{A}u_{\xi\xi} + \bar{B}u_{\xi\eta} + \bar{C}u_{\eta\eta} + \bar{D}u_\xi + \bar{E}u_\eta + \bar{F}u = \bar{G}$$

where

$$\begin{aligned}
0 \leftarrow \bar{A} &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2\\
\bar{B} &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y\\
0 \leftarrow \bar{C} &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2\\
\bar{D} &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y\\
\bar{E} &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y\\
\bar{F} &= F\\
\bar{G} &= G
\end{aligned}$$

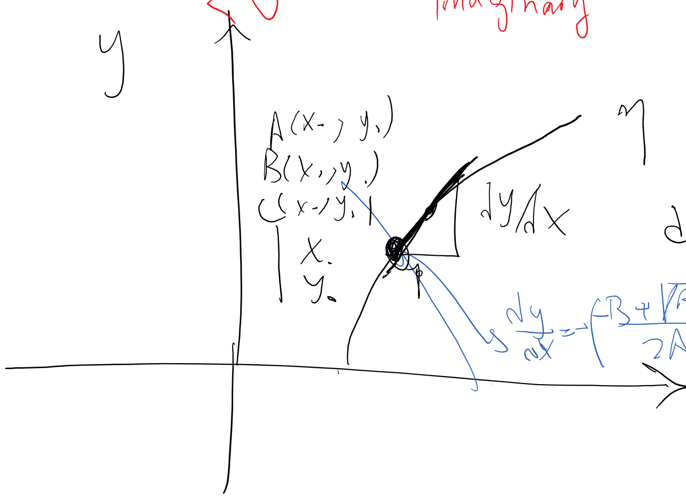
$u_{\xi\eta} = F(u, u_\xi, u_\eta)$ hyperbolic form

$\bar{A} = 0$ & $\bar{C} = 0$ $\bar{B} \neq 0$

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0 \quad \& \quad A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0$$

$$z = \frac{\xi_x}{\xi_y} = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \qquad \frac{\eta_x}{\eta_y} = \frac{-B - \sqrt{\Delta}}{2A}$$

$$\Delta = B^2 - 4AC \begin{cases} > 0 & \text{two distinct real } \frac{\xi_x}{\xi_y}, \frac{\eta_x}{\eta_y} \text{ "characteristics exist"} \\ = 0 & \text{1 characteristic} \\ < 0 & \text{imaginary} \end{cases}$$



$$d\eta = \frac{\partial \eta}{\partial x}dx + \frac{\partial \eta}{\partial y}dy = 0$$

$$\frac{dy}{dx} = -\left(\frac{-B + \sqrt{B^2 - 4AC}}{2A}\right)$$

$$\frac{dy}{dx} = -\left(\frac{\frac{\partial \eta}{\partial x}}{\frac{\partial \eta}{\partial y}}\right) = -\left(\frac{-B - \sqrt{B^2 - 4AC}}{2A}\right)$$

- Which results in equations of the form

$$A[\xi_x/\xi_y]^2 + B[\xi_x/\xi_y] + C = 0$$
$$A[\eta_x/\eta_y]^2 + B[\eta_x/\eta_y] + C = 0$$

- The solutions to these equations are:

$$[\xi_x/\xi_y] = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

(characteristic equations)

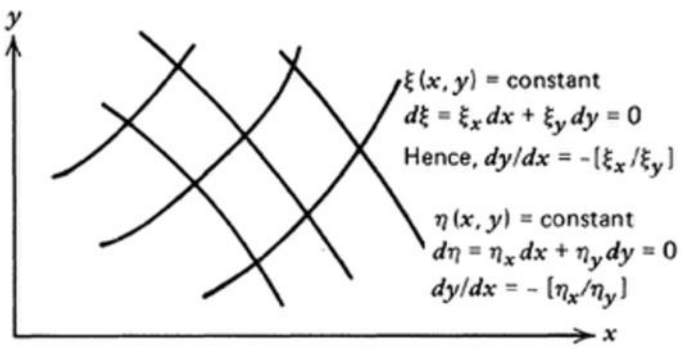
$$[\eta_x/\eta_y] = \frac{-B - \sqrt{B^2 - 4AC}}{2A}$$

- He have three cases:
  - $B^2 - 4AC > 0$: Two distinct values for ξ_x/ξ_y and η_x/η_y. We can cast the equation in hyperbolic canonical form.
  - $B^2 - 4AC = 0$: ONLY one distinct values for ξ_x/ξ_y and η_x/η_y. We cannot cast the equation in hyperbolic form, but can cast in parabolic form.
  - $B^2 - 4AC < 0$: NO REAL roots ξ_x/ξ_y and η_x/η_y. We cannot cast the equation in hyperbolic canonical form, but can cast in elliptic form.

37

- ξ and η are called the characteristic parameters (similar to the first ordr PDE)

- By solving the previous page 2nd order equation we can find how the contour lines (constant values) for ξ and η look like in (x, y) space.



38

- A constant coefficient hyperbolic example:

$$u_{xx} - 4u_{yy} + u_x = 0$$

Comments:

Ux term does not contribute to the form of characteristics and PDE type (lower order term)

$u_x = V$
$2u_y = W$

$$\begin{cases} V_{,x} - 2W_{,y} + V = 0 \\ W_{,x} - 2V_{,y} = 0 \end{cases}$$

$(U_{,xy} - U_{,xy} \neq 0)$

$2v_y = W$ $\quad \begin{cases} \dots \\ w_{,x} - 2v_{,y} = 0 \end{cases}$ $\quad (U_{,xy} - U_{,xy} \neq 0)$

$\rightarrow$ characteristics for this

$u_{xx} - 4u_{yy} \dots$

$A = 1 \quad B = 0 \quad C = -4$

$$\left(-\frac{dy}{dx}\right) = \left(\frac{\eta_x}{\eta_y}\right) \Big/ \frac{\xi_x}{\xi_y} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \pm 2$$

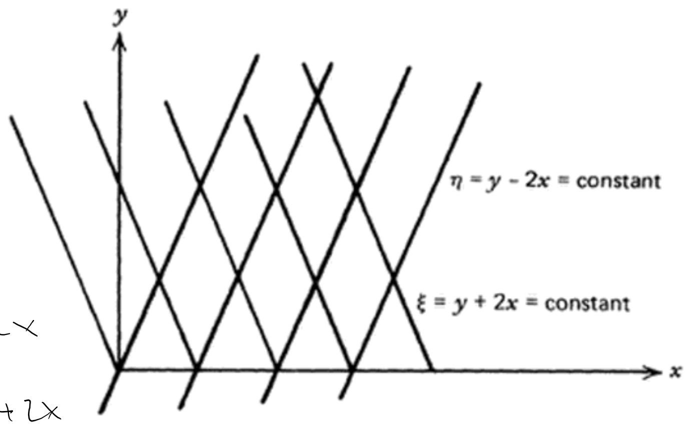
on characteristic

$$\frac{dy}{dx} = \pm 2$$

$\rightarrow y = \pm 2x + c$

$y = 2x + c$ or $\underset{\eta}{c} = y - 2x$

$y = -2x - c \rightarrow \underset{\xi}{(-c)} = y + 2x$



- Consider the PDE

$$y^2 u_{xx} - x^2 u_{yy} = 0 \; x > 0 \; y > 0$$

which is a hyperbolic equation in the first quadrant.

- We find the characteristics by the equation

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = -\frac{x}{y}$$

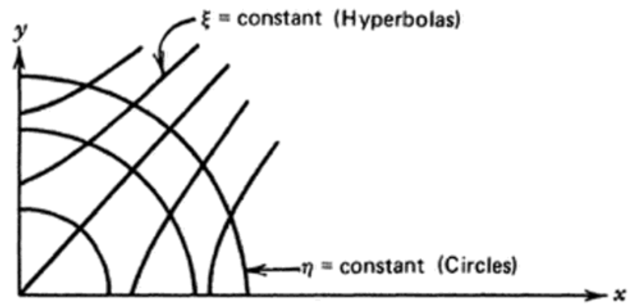
$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = \frac{x}{y}$$

$y\,dy + x\,dx = 0 \quad \frac{y^2}{2} + \frac{x^2}{2} = C$

- By solving these equations and moving x, y to the RHS we obtain

$y^2 - x^2 = \text{constant}$ $\quad\Rightarrow\quad$ $\xi = y^2 - x^2$

$y^2 + x^2 = \text{constant}$ $\quad\quad$ $\eta = y^2 + x^2$



- To obtain the form of the equation in the canonical form

$$\overline{A}u_{\xi\xi} + \overline{B}u_{\xi\eta} + \overline{C}u_{\eta\eta} + \overline{D}u_{\xi} + \overline{E}u_{\eta} + \overline{F}u = \overline{G}$$

we compute

$$\begin{aligned}
\overline{A} &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
\overline{B} &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
\overline{C} &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
\overline{D} &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
\overline{E} &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
\overline{F} &= F \\
\overline{G} &= G
\end{aligned}$$

where $\overline{A}$ and $\overline{C}$ are zero (why?)

- To obtain

$$u_{\xi\eta} = \frac{-(x^2 + y^2)u_{\xi} + (y^2 - x^2)u_{\xi}}{8x^2y^2}$$

- And by solving (x, y) in terms of ξ and η we obtain:

$$u_{\xi\eta} = \frac{\eta u_{\xi} - \xi u_{\eta}}{2(\xi^2 - \eta^2)}$$

- For a second order hyperbolic PDE in the form

observed that hyperbolic PDEs can be written in this form

$$u_{\xi\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta)$$ Hyperbolic 1st canonical form

- by the change of parameter

$$\boxed{\begin{aligned} \alpha &= \alpha(\xi,\eta) = \xi + \eta \\ \beta &= \beta(\xi,\eta) = \xi - \eta \end{aligned}}$$

- We cast it into the 2nd canonical form:

$$u_{\alpha\alpha} - u_{\beta\beta} = \psi(\alpha, \beta, u, u_\alpha, u_\beta)$$ Hyperbolic 2nd canonical form

$u_{tt} - c^2 u_{xx} = 0$

Note: In fact for elliptic PDEs by the same form of transformation we can cast the PDE into elliptic canonical form:

$$u_{\alpha\alpha} + u_{\beta\beta} = \psi(\alpha, \beta, u, u_\alpha, u_\beta)$$ Elliptic canonical form

For the derivation of this form and the parabolic canonical form refer to lesson 41 of [Farlow, 2012]:

$$u_{\alpha\alpha} = \psi(\alpha, \beta, u, u_\alpha, u_\beta)$$ Parabolic canonical form

44

Generalize the previous result (2nd order PDE) but instead of only 2 independent parameters, we can have more than 2

- For a second order linear hyperbolic PDE with n independent variables:

$$\boxed{\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}} + \sum_{i,j=1}^{n} b_i \frac{\partial u}{\partial x_i} + c\,u + g = 0.$$

$$5u_{,x_1x_1} - 3u_{,x_1x_3} + 6u_{,x_2x_3} - 2u_{,x_2x_2} + 7u_{,x_3x_3}$$

$$= \underbrace{5}_{a_{11}}u_{,x_1x_1} - \underbrace{\tfrac{3}{2}}_{a_{13}=a_{31}}u_{,x_1x_3} - \tfrac{3}{2}u_{,x_3x_1} + \underbrace{3}_{a_{23}}u_{,x_2x_3} + \underbrace{3}_{=a_{32}}u_{,x_3x_2} + \ldots$$

$$a = \begin{bmatrix} 5 & 0 & -3/2 \\ 0 & -2 & 3 \\ -3/2 & 3 & 7 \end{bmatrix}$$

other examples

$u_{,tt} - 4u_{,xx} - u_{,yy} = 0$

$$a = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ λ's $= \{-4, -1, 1\}$

(x, y, t)

all except 1 positive & the remaining is − (or opposite) hyperbolic

$u_{,xx} + u_{,yy} + u_{,zz} = 0$

The classification is as follows:

- (H) for ($Z = 0$ and $P = 1$) or ($Z = 0$ and $P = n - 1$)
- (P) for $Z > 0$ ($\Leftrightarrow \det a = 0$)

& the remarks is → (or opposite)

The classification is as follows:

- (H) for ($Z = 0$ and $P = 1$) or ($Z = 0$ and $P = n - 1$)
- (P) for $Z > 0$ ($\Leftrightarrow \det \mathbf{a} = 0$)
- (E) for ($Z = 0$ and $P = n$) or ($Z = 0$ and $P = 0$)
- (ultraH) for ($Z = 0$ and $1 < P < n - 1$)

where

- Z: nb. of zero eigenvalues of a
- P: nb. of strictly positive eigenvalues of a

The alternatives in the (H) and (P) definitions are due to the fact that multiplication by -1 of the equation leaves it unchanged.

Source: [Loret, 2008]

Elliptic: $u_{,xx} + u_{,yy} + u_{,zz} = 0$

all λ's are + or −

$$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \lambda's = (1,1,1)$$

$u_{,t} - u_{,xx} - u_{,yy} = 0$ Diffusion

parabolic

at least 1 λ is zero.

$$a = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 0 \end{bmatrix} \quad (x, y, t)$$

$\lambda's = (0, -1, -1)$

Apply this definition to

$$Au_{xx} + Bu_{xy} + Cu_{yy} \; a \; - \; - \; = 0$$

$$a = \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix}$$

λ's are obtained from $\det(a - \lambda \underline{I}) = 0$

$$\det \begin{bmatrix} A - \lambda & B/2 \\ B/2 & C - \lambda \end{bmatrix} = 0$$

$$\lambda^2 - (A + C)\lambda + \left(AC - \frac{B^2}{4}\right) = 0$$

$\lambda_1, \lambda_2 > 0$ $\qquad AC - \frac{B^2}{4} > 0 \qquad B^2 - 4AC < 0$ elliptic

$\lambda_1, \lambda_2 < 0$ (1+, 1−) $\qquad AC - \frac{B^2}{4} < 0 \qquad B^2 - 4AC > 0$ hyperbolic

$\lambda_1 \lambda_2 = 0 \qquad AC - \frac{B^2}{4} = 0 \qquad$ " $= 0$ parabolic

Classification of second order PDEs: More than 2 independent variables

Canonical form after coordinate transformation (refer to Loret chapter 3)

- Elliptic:

$$\sum_{i=1}^{n} u_{x_i x_i} + \ldots = 0.$$

$$\mathbf{a} = \mathrm{diag}(1, 1, \cdots, 1) \quad \Rightarrow$$
$$\lambda_i = 1 \quad (i \leq n)$$

- Hyperbolic:

$$u_{x_1 x_1} - \sum_{i=2}^{n} u_{x_i x_i} + \ldots = 0.$$

$$\mathbf{a} = \mathrm{diag}(-1, 1, \cdots, 1) \quad \Rightarrow$$
$$\lambda_1 = -1, \lambda_i = 1 \quad (2 \leq i \leq n)$$

- Parabolic:

$$\sum_{i=2}^{n} u_{x_i x_i} + \ldots = 0.$$

$$\mathbf{a} = \mathrm{diag}(0, 1, \cdots, 1) \quad \Rightarrow$$
$$\lambda_1 = 0, \lambda_i = 1 \quad (2 \leq i \leq n)$$

Source: [Loret, 2008]