DC20200122

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D'Alembert solution of the wave equation

Goals:

- Obtain the solution to the wave equation
- Solution of the PDE using characteristics from the PDE's canonical form

PDE $\quad u_{tt} = c^2 u_{xx} \quad -\infty < x < \infty \quad 0 < t < \infty$

ICs $\quad \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \quad -\infty < x < \infty$

The solution is,

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi$$

wave speed

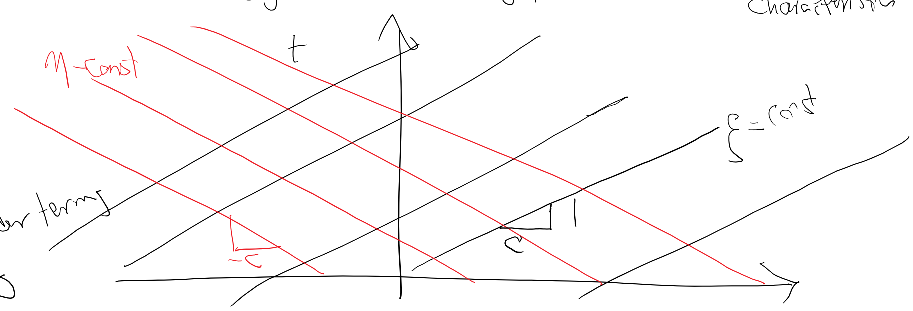
$$u_{tt} - c^2 u_{xx} = 0$$

$$A u_{tt} + B u_{tx} + C u_{xx} + \cdots = 0$$

Characteristics $\quad \frac{dy}{dx} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \pm \frac{\sqrt{4c^2}}{2} = \pm c$

$dy = \pm c\, dx \rightarrow d(y \mp cx) = 0 \quad y \mp cx = \text{const}$ for characteristics

$$\boxed{\begin{aligned} x + ct &= \eta \\ x - ct &= \xi \end{aligned}}$$



Lower order terms

$u_{\xi\eta} + \cdots = 0$

for this problem L.O.T $= 0$ $\quad (u_{tt} - c^2 u_{xx} = 0)$

$u_{\xi\eta} = 0 \rightarrow u = \phi(\eta) + \psi(\xi)$

$$\boxed{u = \phi(x+ct) + \psi(x-ct)} \quad (1)$$

how many ICs needed $\quad u_{tt} - c^2 u_{xx} = 0$

ICs (a) $u(x,0) = f(x)$

(b) $\dot{u}(x,0) = g(x)$ $\quad (2)$

(1) $\to$ $\dot{u} = c\phi'(x+ct) - c\psi'(x-ct)$

@ $t=0$ (2) $\rightarrow$

$$\boxed{\begin{aligned} \phi(x)+\psi(x) &= f(x) \quad (a) \\ c\phi'(x)-c\psi'(x) &= g(x) \quad (b) \end{aligned}} \quad (3)$$

integrate (b)

$$\boxed{\begin{aligned} c\phi(x) + c\psi(x) &= cf(x) \\ c\phi(x) - c\psi(x) &= G(x) \end{aligned}}$$

$$G(x) = \int_0^x g(x)\,dx$$

ignore the const of integration.

$$\boxed{\begin{aligned} \phi(x) &= \frac{cf(x)+G(x)}{2} \\ \psi(x) &= \frac{cf(x)-G(x)}{2} \end{aligned}}$$

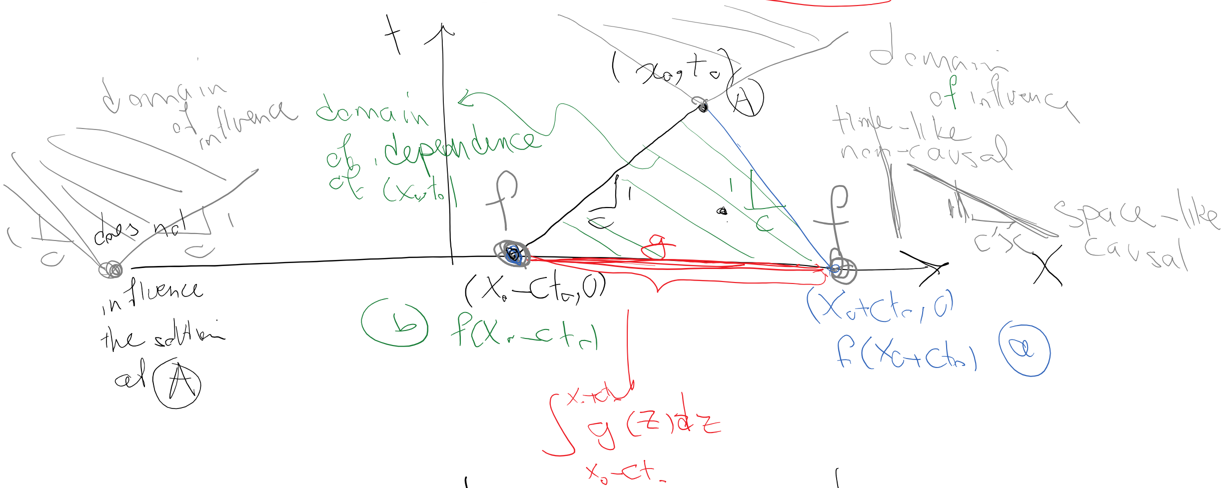
- By integrating the second equation we get

$$\left.\begin{aligned} \phi(x) + \psi(x) &= f(x) \\ -c\phi(x) + c\psi(x) &= \int_{x_0}^{x} g(\xi)\,d\xi + K \end{aligned}\right\} \Rightarrow \begin{aligned} \phi(x) &= \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(\xi)\,d\xi \\ \psi(x) &= \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(\xi)\,d\xi \end{aligned} \Rightarrow$$

$$u(\xi,\eta) = \Phi(\eta) + \psi(\xi) = \varphi(x-ct) + \psi(x+ct)$$

Right-going wave (speed c) Left-going wave (speed c)

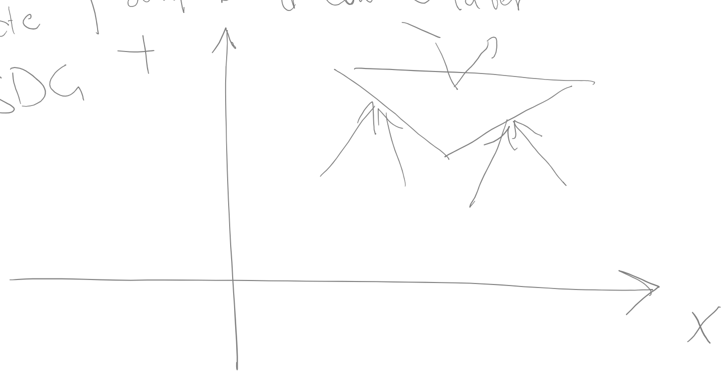
$$\boxed{u(x_0,t_0) = \frac{1}{2}\underbrace{[\overset{(b)}{f(x_0-ct_0)} + \overset{(a)}{f(x_0+ct_0)}]} + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(\xi)\,d\xi}$$

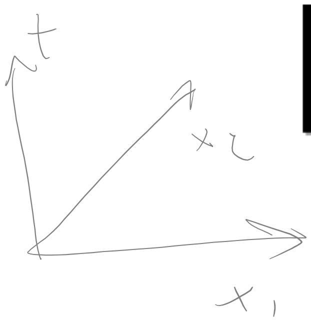


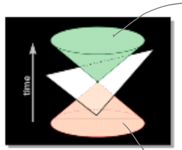
where the s'l of (x_0,t_0) is determined

side note | will be discussed later

CSDG







causality constraint

domain of influence

domain of dependence

Hyperbolicity $\checkmark$ $a(x,t,u)\,u_{,t} + b(x,t,u)\,u_{,x} = c(x,t,u)$

$\checkmark$ $Au_{xx} + Bu_{xt} + Cu_{tt} + \dots = 0$

even with more indep. arguments $x_1, x_2, x_3, \dots$

Hyperbolicity of systems of 1st order quasi-linear PDEs

- Assume we want to solve the system of semi-linear first order PDEs,

$$\text{PDE :} \qquad \mathbf{q}_{,t} + \mathbf{A}\mathbf{q}_{,x} = \mathbf{s}(\mathbf{q}, x, t) \qquad \text{(1a)}$$

$$\text{IC :} \qquad \mathbf{q}(x, 0) = \mathbf{q}_0(x) \qquad \text{(1b)}$$

where

$\mathbf{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$ vector of unknown fields

$\mathbf{A}$ $n \times n$ flux matrix

$\mathbf{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$ source term (can be nonlinear in $\mathbf{q}$

n number of fields

n-coupled equations. By using eigen-decomposition of A we can decouple them:)

$q_t + A q_{,x}$

turn it to Diagonal matrix

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} + \begin{bmatrix} d_{11} & & \\ & \ddots & \\ & & d_{nn} \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}_{,x} = \begin{bmatrix} \dots \end{bmatrix}$$

decoupled

$$\begin{cases} w_1 + d_{11} w_{1,x} = \dots \\ w_2 + d_{22} w_{2,x} = \dots \\ \vdots \\ \vdots \end{cases}$$

Imagine we can find L such that this holds for diagonal D:

matrix

$L A = D$ diagonal

$LA = DL$

$q_{,t} + A q_{,x} = S$ premultiply by L

$$(Lq)_{,t} + \underbrace{LA}_{DL} q_{,x} = LS$$

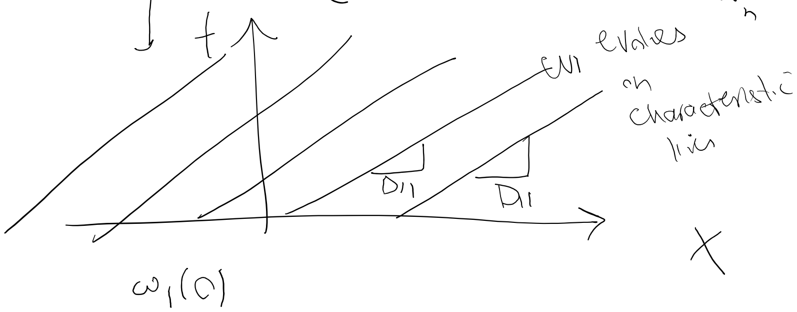
$$\underbrace{(Lq)}_{\omega}{}_{,t} + D \underbrace{(Lq)}_{\omega}{}_{,x} = LS$$

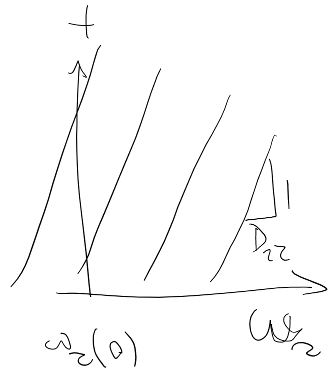
characteristics

$$\boxed{\begin{gathered} \omega_{,t} + D\omega_{,x} = S_{\omega} \\ S_{\omega} = LS \\ \omega = Lq \end{gathered}}$$

Now we have

$$\begin{cases} \omega_1 + D_{11}\, \omega_{1,x} = S_{\omega_1} \\ \omega_2 + D_{22}\, \omega_{2,x} = S_{\omega_2} \\ \vdots \\ \omega_n + D_{nn}\, \omega_{n,x} = S_{\omega_n} \end{cases}$$





Solve this

$$\omega_i(x, 0) = Lq(x, 0) = Lq_0(0)$$

Final step?

(3) ω ✓ $q = ?$ $\omega = Lq \rightarrow$

$$q(x,t) = L^{-1}\omega(x,t)$$

☒

vector

$LA = DL$

$$LA = DL$$

vector #

direction

$$\begin{bmatrix} l_1^1 & l_2^1 & l_3^1 & \cdots & l_n^1 \\ \hline & & \vec{l}^2 & & \\ \hline & & \vdots & & \\ & & \vec{l}^n & & \end{bmatrix} A = D \begin{bmatrix} l^1 \\ \hline l^2 \\ \hline \vdots \\ l^n \end{bmatrix} = \begin{bmatrix} D_{11}\, l^1 \\ D_{22}\, l^2 \\ \vdots \\ D_{nn}\, l^n \end{bmatrix}$$

row vector

left eigenvector

left eigenvalue

$$\boxed{\vec{l}^i A = D_{ii} \vec{l}^i}$$

typical eigenvalue problem

$$A r^i = D_{ii} r^i$$

right eigenvalue

column vector

right eigenvector

Left eigen problem

$$l A = \lambda l \longrightarrow A^T(l^T) = \lambda l^T$$

l^T is Right eigenvector of A^T

Since eigenvalues of A & A^T are the same ($\det(A-\lambda I) = \det(A^T - \lambda I^T)$)

Left & Right eigenvalues are the same.

Basically we need to solve Left eigenvalue problem for A:

Procedure for solving (1),

$$\begin{aligned} &\text{PDE}: && \mathbf{q}_{,t} + \mathbf{A}\mathbf{q}_{,x} = \mathbf{s}(\mathbf{q}, x, t) \\ &\text{IC}: && \mathbf{q}(x, 0) = \mathbf{q}_0(x) \end{aligned}$$

- Step 1: Solve for n left eigenvectors $\mathbf{l}^i$ eigenvalues λ_i of $\mathbf{A}$

$$\mathbf{l}^i\mathbf{A} = \lambda^i \mathbf{l}^i \quad \text{(no summation on } i\text{)}$$

And form left-eigenvalue matrix $\mathbf{L} = [\mathbf{l}^1 \ \cdots \ \mathbf{l}^n]^{\mathrm{T}}$, $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$.

- Step 2: Define characteristic values, and corresponding source terms, and IC

Characteristic variables	$\omega = \mathbf{L}\mathbf{q} \quad \Rightarrow$
Source term vector	$\mathbf{s}^{\omega}(\omega, x, t) = \mathbf{L}\mathbf{s}(\mathbf{L}^{-1}\omega, x, t)$
IC	$\omega(x, 0) = \omega_0(x) = \mathbf{L}\mathbf{q}_0(x)$

Solve n-decoupled first order PDEs in ω_i

$$\left.\begin{array}{ll}\text{PDE}: & \omega_{i,t} + \lambda_i \omega_{i,x} = \mathbf{s}_i^{\omega}(\omega, x, t) \\ \text{IC}: & \omega_i(x, 0) = \omega_{i0}(x)\end{array}\right\} \quad i \leq n \quad \text{(no summation on } i) \qquad (4)$$

Note: If the source term $\mathbf{s}(\mathbf{q}, x, t)$ explicitly depends on $\mathbf{q}$ equations (4) for ω_i are coupled through the source term $\mathbf{s}^{\omega}$ but we still have n characteristics and the solution is simpler in this space.

- Step 3: Once the solution $\omega(x, t)$ is obtained we solve for $\mathbf{q}(x, t)$ from,

$$\mathbf{q}(x, t) = \mathbf{A}^{-1}\omega(x, t)$$

Example 1:
Elastodynamic problem in 1D:

Example: 1D elastodynamics problem

Example: Consider the solution to 1D solid mechanics problem corresponding to the conservation law $\sigma_{,x} + \rho b = p_{,t}$ where $\sigma = E\epsilon$ is stress, ρ is density, b is body force, $p = \rho v$ is linear momentum, $\epsilon = u_{,x}$ is strain, $v = u_{,t}$ is velocity, and u is displacement.

To solve this problem as a system of first order PDEs we follow these steps:

- Write equation of motion in conservation law form,

$$p_{,t} - \sigma_{,x} = \rho b \qquad (5)$$

$p = \rho v = \rho \dot{u}$

method 1 $\quad \dot{(\rho \dot{u})} - \sigma_{,x} = \rho b$

$\sigma = E\varepsilon = E u_{,x} \qquad$ E const.

$\rho \ddot{u} - E u_{,xx} = \rho b$

$\ddot{u} - \underbrace{\frac{E}{\rho}}_{c^2} u_{,xx} = \cancel{b}\ 0 \qquad$ Also assume $b=0$

$c = \sqrt{\frac{E}{\rho}}$

Use D'Alembert s'n

$$u(x,t) = \frac{1}{2}\left(u_0(x-ct) + u_0(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(z)\, dz$$

<u>Method 2</u> solve it as a system of 1st order PDEs

$$\dot{p} - \sigma_{,x} = \rho b \qquad q = \begin{bmatrix} p \\ \sigma \end{bmatrix}$$

need this eqn ← $\dot{\sigma} + ?(\sigma_{,x}) + ?(p_{,x}) = \ldots$

is needed $\dot{q} + A q_{,x} = 0$

$$\sigma = E\varepsilon = E u_{,x} \rightarrow \dot{\sigma} = E \dot{u}_{,x} = E v_{,x} = \frac{E}{\rho} p_{,x}$$

$$\boxed{\begin{aligned} \dot{p} - \sigma_{,x} &= \rho b \\ \dot{\sigma} - \frac{E}{\rho} p_{,x} &= 0 \end{aligned}} \qquad \dot{q} + A q_{,x} = S \qquad A = \begin{bmatrix} 0 & -1 \\ -\frac{E}{\rho} & 0 \end{bmatrix} \quad S = \begin{bmatrix} \rho b \\ 0 \end{bmatrix}$$

Step 1: Solve the eigen problem for A

$$A = \begin{bmatrix} 0 & -1 \\ -\frac{E}{\rho} & 0 \end{bmatrix} \qquad \frac{E}{\rho} = c^2 \qquad A = \begin{bmatrix} 0 & -1 \\ -c^2 & 0 \end{bmatrix}$$

$$[l_1 \;\; l_2] \begin{bmatrix} 0 & -1 \\ -c^2 & 0 \end{bmatrix} = \lambda [l_1 \;\; l_2] \qquad [-c^2 l_2 \;\; -l_1] = [\lambda l_1 \;\; \lambda l_2]$$

$$\begin{aligned} -c^2 l_2 &= \lambda l_1 \\ -l_1 &= \lambda l_2 \end{aligned} \qquad \underbrace{\begin{bmatrix} \lambda & c^2 \\ 1 & \lambda \end{bmatrix}}_{\det = 0} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \lambda^2 - c^2 = 0 \rightarrow \lambda = \pm c$$

$$\boxed{\lambda = -c} \qquad \begin{aligned} -c^2 l_2 &= -c l_1 \\ -l_1 &= -c l_2 \end{aligned} \qquad l_2 = 1 \quad l_1 = -c$$

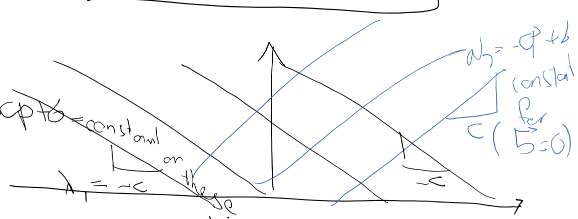
Similarly $\lambda = c$ $\qquad l_2 = 1 \quad l_1 = c$

- **Find L and Λ as,**

$$\mathbf{L} = \begin{array}{c} l^1 \\ l^2 \end{array}\begin{bmatrix} c & 1 \\ -c & 1 \end{bmatrix}, \quad \mathbf{\Lambda} = \mathrm{diag}(-c, c) = \begin{bmatrix} -c & 0 \\ 0 & c \end{bmatrix} \quad (\lambda_1, \lambda_2)$$

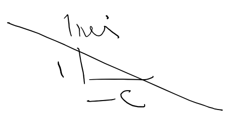
$$LA = \Lambda L \qquad w = Lq \qquad \boxed{\dot{w} + \Lambda w_{,x} = S_w}$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} c & 1 \\ -c & 1 \end{bmatrix} \underbrace{\begin{bmatrix} p \\ \sigma \end{bmatrix}}_{q} = \begin{bmatrix} cp + \sigma \\ -cp + \sigma \end{bmatrix}$$



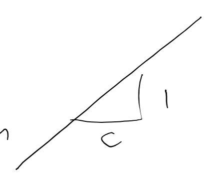
$$w_1 = (\underbrace{c\rho}) v + \sigma = Z v + \sigma$$

impedance Z $\qquad$ const on this



$$w_2 = \sigma - Z v$$

on these lines



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$(I_b = 0)$