DC20200129

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From the last time

$$p_{,t} - \sigma_{,x} = \rho b$$

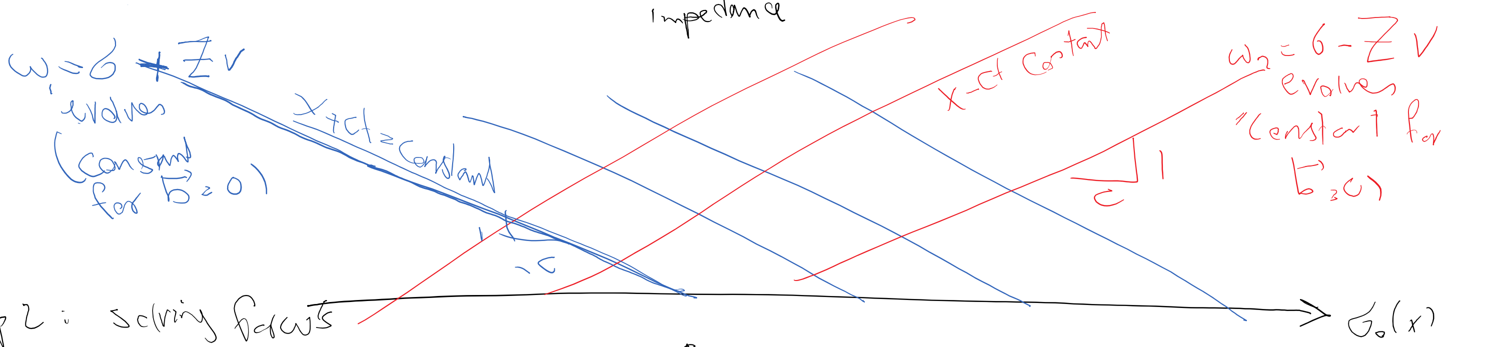
- We have two unknowns p and σ and can define

$$\mathbf{q} = \begin{bmatrix} p \\ \sigma \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 0 & -1 \\ -\frac{E}{\rho} & 0 \end{bmatrix}, \qquad \dot{q} + A q_{,x} = s$$

Step 1 $\quad q \to w$

$$L = \begin{bmatrix} c & 1 \\ -c & 1 \end{bmatrix} \begin{matrix} l_1 \\ l_2 \end{matrix} \qquad L A = \Lambda L \qquad \Lambda = \begin{bmatrix} -c & 0 \\ 0 & c \end{bmatrix} \quad c = \sqrt{\frac{E}{\rho}}$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \sigma + cp \\ \sigma - cp \end{bmatrix} = \begin{bmatrix} \sigma + c\rho v \\ \sigma - \underbrace{c\rho}_{\substack{Z \\ \text{Impedance}}} v \end{bmatrix} = \begin{bmatrix} \sigma + Z v \\ \sigma - Z v \end{bmatrix} \begin{matrix} \longrightarrow \lambda_1 = -c \\ \longrightarrow \lambda_2 = c \end{matrix}$$



Step 2: solving for w's

$$\dot{w} + \Lambda w_{,x} = S_w \longleftrightarrow \begin{cases} \dot{w}_1 - c w_{1,x} = S_{w_1} \\ \dot{w}_2 + c w_{2,x} = S_{w_2} \end{cases}$$

for $b = 0$ $\quad p_0(x)$

$S_{w_1} = S_{w_2} = 0$

here we're solving the problem for $b = 0$

$$\begin{cases} \dot{w}_1 - c w_{1,x} = 0 \\ \dot{w}_2 + c w_{2,x} = 0 \end{cases} \longleftrightarrow \boxed{\begin{aligned} w_1(x,t) &= (w_1)_0 (x + ct) \\ w_2(x,t) &= (w_2)_0 (x - ct) \end{aligned}} \quad (1)$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = L \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} c & 1 \\ -c & 1 \end{bmatrix} \begin{bmatrix} p \\ \sigma \end{bmatrix} = \begin{bmatrix} cp + \sigma \\ -cp + \sigma \end{bmatrix}$$

$$(w_1)_0 = (cp + \sigma)_0 = cp_0 + \sigma_0$$
$$(w_2)_0 = (-cp + \sigma)_0 = -cp_0 + \sigma_0 \quad (2)$$

①,② $\longrightarrow$

$$\boxed{\begin{aligned} w_1(x,t) &= cP_0(x+ct) + \sigma_0(x+ct) \\ w_2(x,t) &= -cP_0(x-ct) + \sigma_0(x-ct) \end{aligned}}$$ ③

Step 3 $\quad w \longrightarrow q \qquad w = Lq \longrightarrow q = L^{-1}w$

$$L = \begin{bmatrix} c & 1 \\ -c & 1 \end{bmatrix} \rightarrow L^{-1} = \frac{1}{2c}\begin{bmatrix} 1 & -1 \\ c & c \end{bmatrix} \longrightarrow \begin{bmatrix} p \\ \sigma \end{bmatrix} = \frac{1}{2c}\begin{bmatrix} 1 & -1 \\ c & c \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\boxed{\begin{aligned} p &= \frac{w_1 - w_2}{2c} \\ \sigma &= \frac{w_1 + w_2}{2} \end{aligned}}$$ ④

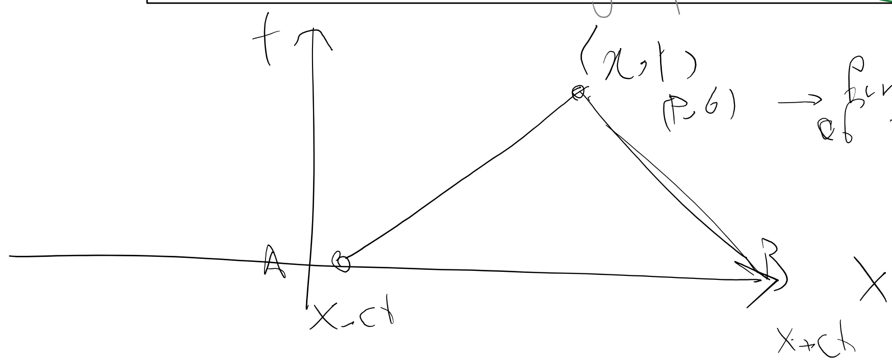
③ & ④ $\longrightarrow$

jumps

$$\boxed{\begin{aligned} (a)\quad p(x,t) &= \frac{p_0(x-ct) + p_0(x+ct)}{2} + \frac{\sigma_0(x+ct) - \sigma_0(x-ct)}{2c} \\ (b)\quad \sigma(x,t) &= \frac{c}{2}\left[p_0(x+ct) - p_0(x-ct)\right] + \frac{\sigma_0(x-ct) + \sigma_0(x+ct)}{2} \end{aligned}}$$ ⑤

averages

jumps



$(p,\sigma) \longrightarrow$ functions of p, σ @ A, B

$\begin{cases} \dot p + \sigma_{,x} = \rho b \\ \dot\sigma - \frac{E}{\rho} p_{,x} = 0 \end{cases} \longleftarrow$ if is the goal we're done. But we may need $u = \frac{1}{\rho}\int p\, dt$

$(v = \dot u)$ [5a] $\quad \varepsilon = \frac{\sigma}{E} = u_{,x} \longrightarrow u = \frac{1}{E}\int \sigma\, dx$ (b)

$\dot u \; (\quad) = v$

If we do this integral (using 5b) we get:

$$\boxed{u(x,t) = \frac{1}{2}\left(u_0(x-ct) + u_0(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(\xi)\mathrm{d}\xi}$$

Hyperbolicity condition for a system of first order PDEs:

$$\boxed{u_{xx} + u_{yy} = 0}$$

$q_1 = u_x$

$q_2 = u_y$

$$\boxed{\begin{aligned} q_{1,x} + q_{2,y} &= 0 \\ q_{2,x} - q_{1,y} &= 0 \end{aligned}}$$

$$q_{,x} + A\, q_{,y} = 0$$

plays the role of t

plays the role of x above

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Eigenvalues of A $\quad \det(A - \lambda I) = 0$

$\rightarrow \det \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = 0 \quad \lambda^2 + 1 = 0$

$\rightarrow \boxed{\lambda = \pm i}$ Violated (2)

The characteristic values are NOT real, so the system is not hyperbolic!

need to have n lin independent ℓ^i's $\leftarrow$ $LA = \Lambda L$

$L = \begin{bmatrix} \ell^1 \\ \ell^2 \\ \vdots \\ \ell^n \end{bmatrix}$

Condition for hyperbolicity of a system of first order PDEs:

1. A is diagonalizable (means that it has n linearly independent eigenvectors for nxn A). $\longleftrightarrow$ we could not have done $w = Lq$ & $w_t + \Lambda w_{,x} = Sw$ "couldn't end up with decoupled w eqns)
2. Eigenvalues are real.

need real wave speed

- Reminder: To solver the previous system of 1st order PDEs we should have been able to obtain matrix L for diagonalizing A in terms of Λ:

$$\mathbf{LA} = \mathbf{\Lambda L} \qquad \mathbf{L} = \begin{bmatrix} \mathbf{l}^1 \\ \vdots \\ \mathbf{l}^i \\ \vdots \\ \mathbf{l}^n \end{bmatrix} = \begin{bmatrix} \mathbf{l}_1^1 & \cdots & \mathbf{l}_n^1 \\ \vdots & \ddots & \vdots \\ \mathbf{l}_1^i & \cdots & \mathbf{l}_n^i \\ \vdots & \ddots & \vdots \\ \mathbf{l}_1^n & \cdots & \mathbf{l}_n^n \end{bmatrix} \qquad \mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \cdots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

- To obtain L (diagonalizing A) we should have
  - There are n linearly independent (left) eigenvectors
  - The corresponding eigenvalues λ_i are real.
- NOTE:

$\mathbf{LA} = \mathbf{\Lambda L}$ $\quad$ $\mathbf{L}$ is left eigenvector matrix $\quad \Leftrightarrow$

$\mathbf{A} = \mathbf{L}^{-1}\mathbf{\Lambda L}$ $\quad$ $\mathbf{A}$ is diagonalizable $\quad \Leftrightarrow$

$\mathbf{AR} = \mathbf{R\Lambda}$ $\quad$ $\mathbf{R} = \mathbf{L}^{-1}$ is right eigenvector matrix

- Hyperbolicity condition requires that we can find n characteristic values for the n-tuple q where information propagates along characteristics.

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- For an $n \times n$ matrix $\mathbf{A}$ we obtain eigenvalues from the n^{th} characteristic polynomial,

$$\det(A - \lambda \mathbf{I}) = 0 \tag{14}$$

- This equation ALWAYS has n complex roots (which clearly some of all can be real). If a root λ is repeated k times we call k, the algebraic multipicity of that root.

- Given that some roots may be repeated, we list roots in ascending order,

$$\lambda_1 < \lambda_2 < \cdots < \lambda_m, \quad n^A(A) := m \leq n$$

where m is the number of distinct roots of $\mathbf{A}$ shown by $n^A(A)$. Note that some roots may be repeated multiple times.

- The algebraic multipicity of root number k λ_k is shown by $n_k^A(A)$ which in short is shown by n_k^A.

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Geometric and algebraic multiplicity

- Since (14) has n roots, even if $m < n$ (*i.e.*, some roots are repeated) we always have,

$$\sum_{k=1}^{m} n_k^A = n \qquad \text{if } m = n \quad \Rightarrow \quad n_k^A = 1 \text{ for all } k \leq n$$

- The Geometric multipicity of λ_k n_k^G is the geometric dimension (*i.e.*, number of linearly independent eigenvectors $\mathbf{u}_k^j$ of λ_k). $\mathbf{u}_k^j$ form a basis for the vector space spanned by eigenvectors of λ_k.

  Note that we can have different members in a basis of a vector space but the dimension of the vector space is independent on which basis is used.

- We have the following observations and definitions,

$n_k^G =$ geom. multipicity λ_k	$\dim(\text{eigenspace of } \lambda_k) = \dim\{\mathbf{u} \mid \mathbf{A}\mathbf{u} = \lambda_k \mathbf{u}\}$.
$\boxed{n_k^G \leq n_k^A}$	Can be smaller if $n_k^A > 1$.
$n_k^G = 1$ if $n_k^A = 1$	Each distinct λ has ONE eigenvector direction.
$n^G(\mathbf{A}) := \sum_1^m n_k^G$	sum of the dimensions of eigenspaces.
$n^G(\mathbf{A}) \leq n^A(\mathbf{A}) = m$	
$n^G(\mathbf{A}) = n$ if $n^A(\mathbf{A}) = n$	

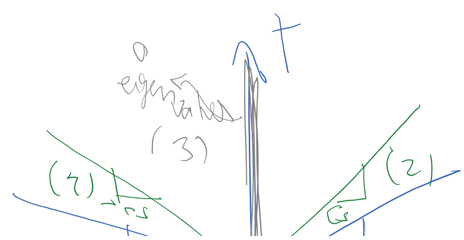
- Two important classes of diagonalizable matrices:
  - If all eigenvalues are distinct (n distinct eigenvalues -> n distinct eigenvectors): To check hyperbolicity, we also need to make sure eigenvalues are real (pertained to hyperbolicity of qDot + A q,x = S)
  - Symmetric (Hermitian if complex) are always diagonalizable even if they have repeated eigenvalues. In fact, such matrices ALWAYs have real eigenvalues with orthogonal eigenvectors. (so if A is symmetric, the systems of equations already satisfies all hyperbolicity conditions and the PDE is hyperbolic)

Strong vs. weak hyperbolic PDEs: If all eigenvalues are distinct the PDE is strongly hyperbolic, otherwise weakly.

Examples: Elastodynamics in 1D as we saw above is strongly hyperbolic.
But elastodynamic in 3D is weakly hyperbolic

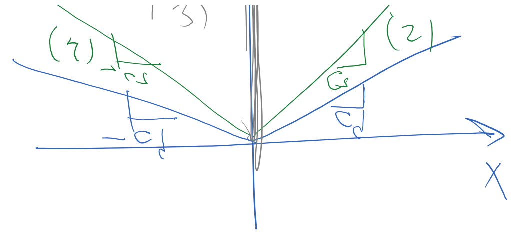
9 q's

$q = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \sigma_{11} \\ \sigma_{12} \end{bmatrix}$



9 q's

σ_{11}
σ_{22}
σ_{33}
σ_{12}
σ_{23}
σ_{31}



$\lambda = \{-c_d, -c_s, -c_s, 0, 0, 0, c_s, c_s, c_d\}$

weakly hyperbolic

- Hyperbolicity of $\mathbf{q}_{,t} + \mathbf{A}(\mathbf{q}, x, t)\mathbf{q}_{,x} = \mathbf{s}(\mathbf{q}, x, t)$ for a given point $(x^\star, t^\star)$.
  - $\mathbf{A}$ is diagonalizable iff $n^G(\mathbf{A}) = n$.
  - System is hyperbolic if $\mathbf{A}$ is diagonalizable ($n^G(\mathbf{A}) = n$) AND ALL eigenvalues are real.
  - Hyperbolicity is STRONG if $n^A(\mathbf{A}) = n$ (all characteristic values are distinct).
  - Hyperbolicity is WEAK if $n^A(\mathbf{A}) < n$ (repeated characteristic values).
  - If system is quasilinear all $\mathbf{q}$ must be considered for $\mathbf{A} = \mathbf{A}(\mathbf{q}, x^\star, t^\star)$ in definitions above.

2.9 Hyperbolicity of Linear Systems (Also for semi-linear case)

Definition 2.1. *A linear system of the form*

$$q_t + Aq_x = 0$$

is called hyperbolic *if the $m \times m$ matrix A is diagonalizable with real eigenvalues.*

We denote the eigenvalues by

$$\lambda^1 \le \lambda^2 \le \cdots \le \lambda^m$$

The matrix is diagonalizable if there is a *complete* set of eigenvectors, i.e., if there are nonzero vectors $r^1, r^2, \ldots, r^m \in \mathbb{R}^m$ such that

$$Ar^p = \lambda^p r^p \quad \text{for } p = 1, 2, \ldots, m, \tag{2.70}$$

and these vectors are linearly independent. In this case the matrix

$$R = [r^1|r^2|\cdots|r^m],$$

Source: [LeVeque, 2002, 2.9]

$$R^{-1}AR = \Lambda \quad \text{and} \quad A = R\Lambda R^{-1},$$

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balanced quantity

spatial flux

$q_{,t} + f(q)_{,x} = 0$

$q_{,t} + \frac{\partial f(q)}{\partial q} q_{,x} = 0$

$A(x, t, q)$

Quasi-linear system $q_t(x,t) + f(q(x,t))_x = 0.$

$q_t + A(q, x, t)q_x = 0$ where $A(\mathbf{q}, x, t) = \nabla_{\mathbf{q}} f(\mathbf{q}, x, t)$

is said to be *hyperbolic* at a point (q, x, t) if the matrix $A(q, x, t)$ satisfies the hyperbolicity condition (diagonalizable with real eigenvalues) at this point.

The nonlinear conservation law ( ) is *hyperbolic* if the Jacobian matrix $f'(q)$ appearing in the quasilinear form ( ) satisfies the hyperbolicity condition for each physically relevant value of q.

Quasi-linear system $q_t(x,t) + f(q(x,t))_x = 0.$

$q_t + A(q,x,t)q_x = 0$ where $A(\mathbf{q},x,t) = \nabla_{\mathbf{q}} f(\mathbf{q},x,t)$

$q_{,t} + f(q)_{,x} = 0$

$q_{,t} + \frac{\partial f(q)}{\partial q} q_{,x} = 0$

$A(x,t,q)$

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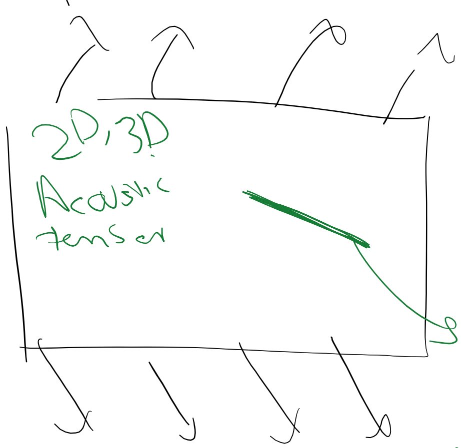
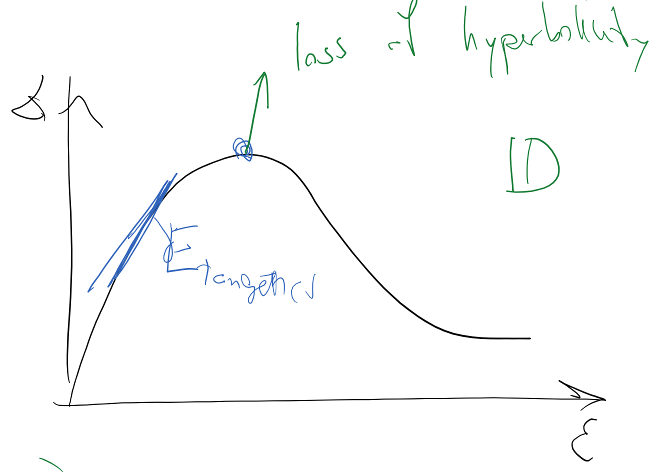
Source: [LeVeque, 2002, 2.11]

Examples of pointwise hyperbolicity





crack or localization occurs in the region of loss of hyperbolicity

Systems of 1st order PDEs
More than 2 independent parameters (2D, 3D)

- Consider the system,

$$\mathbf{q}_{,t} + \mathbf{A}^1\mathbf{q}_{x_1} + \mathbf{A}^2\mathbf{q}_{x_2} + \mathbf{A}^3\mathbf{q}_{x_3} = s(\mathbf{q},\mathbf{x},t) \quad (a)$$

where $\mathbf{x} = (x_1, x_2, x_3)$.

idea turn (a) to

where $\mathbf{x} = (x_1, x_2, x_3)$.

turn (a) to (b)

- In general we cannot solve this system by diagonalizing the system and solving ODEs as a system with 2 independent parameters

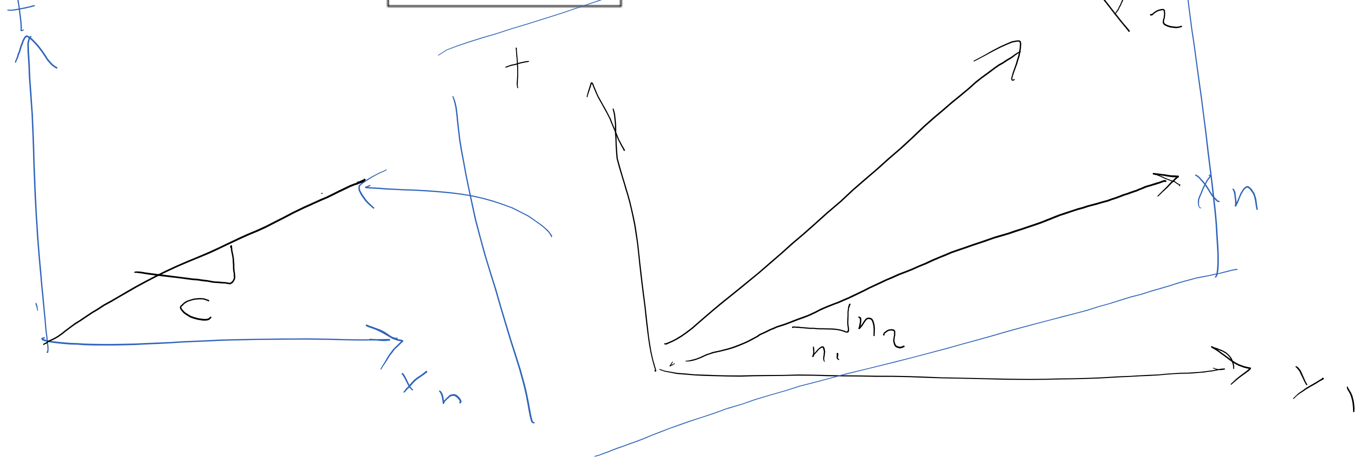
$$\mathbf{q}_{,t} + \mathbf{A}^1 \mathbf{q}_{x_1} = s(\mathbf{q}, x_1, t)$$

however, even in 2D & 3D if the IC, BC are 1D and the form of matrices accommodates the direction of solution implied by IC and BC we can basically solve a 1D problem.

Idea look at (a) for wave propagation along the direction n and see if the PDE in x_n (x along n) and t is hyperbolic

- The hyperbolicity is investigated by seeking planar waves in direction $\mathbf{n} = (n_1, n_2, n_3)$:

$$\boxed{\mathbf{q} = \mathbf{U} f(\mathbf{n}.\mathbf{x} - ct)} \qquad (13)$$



where

$\mathbf{U} = [U_1\ U_2\ \cdots U_n]^{\mathrm{T}}$	wave shape (mode
$\mathbf{n} = (n_1, n_2, n_3)$	wave direction
c	wave speed
f	a scalar function ($\mathbf{U}$) turns f into the vector form $\mathbf{q}$

- By plugging (13) in (12) we obtain,

$$\boxed{(\mathbf{A}^n - cI)\,\mathbf{U} = 0, \quad \text{where} \quad \mathbf{A}^n := n_1\mathbf{A}^1 + n_2\mathbf{A}^2 + n_3\mathbf{A}^3}$$

That is we are solving an eigenvalue problem for $\mathbf{A}^n$ exactly similar to 1D case.

- Hyperbolicity condition:

System (12) admits propagating planar waves for **arbitrary** directions $\mathbf{n}$ $\Leftrightarrow$ $\mathbf{A}^n$ is diagonalizable for **arbitrary** directions $\mathbf{n}$

- Clearly, the same procedure works for 2D and higher dimensions as well.
- For more discussion refer to [LeVeque, 2002, section 18] (particularly 18.5).