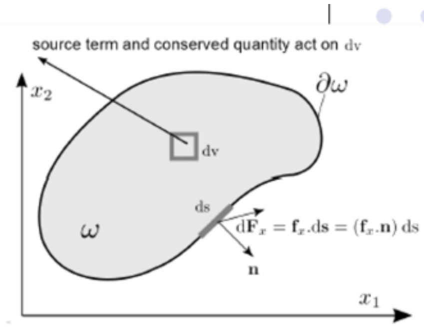
For a general conservation law let:

- $\mathbf{f}_t$: conserved quantity = temporal flux
- $\mathbf{f}_x$: total outward spatial flux
- $\mathbf{r}$: source term

then the balance law for dynamics reads:



$$\boxed{\forall\omega \subset \mathcal{D} \wedge \forall t : \int_\omega \mathbf{r}\,\mathrm{dv} - \int_{\partial\omega} \mathbf{f}_x.\mathrm{ds} = \int_\omega \mathbf{r}\,\mathrm{dv} - \int_{\partial\omega} (\mathbf{f}_x.\mathbf{n})\,\mathrm{ds} = \frac{\mathrm{d}}{\mathrm{d}t}\int_\omega \mathbf{f}_t\,\mathrm{dv}} \quad (13)$$

- Application of divergence theorem on the flux term yields,

$$\int_{\partial\omega} \mathbf{f}_x\mathbf{n}.\mathrm{d}s = \int_\omega \nabla.\mathbf{f}_x \mathrm{d}v$$

- By taking $\frac{\partial(.)}{\partial t}$ inside the integral in the balance law we have,

$$\forall\omega \subset \mathcal{D} \int_\omega \left\{\frac{\partial \mathbf{f}_t}{\partial t} + \nabla.\mathbf{f}_x - r\right\} \mathrm{d}v = 0$$

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- Since $\omega \subset \mathcal{D}$ is arbitrary, by using the localization theorem we obtain the strong form of the problem,

$$\boxed{\frac{\partial \mathbf{f}_t}{\partial t} + \nabla.\mathbf{f}_x - r = 0 \quad \text{that is in 3D} \quad \mathbf{f}_{t,t} + \mathbf{f}_{1,1} + \mathbf{f}_{2,2} + \mathbf{f}_{3,3} = r}$$

where f_1, f_2, f_3 are the components of the spatial flux $\mathbf{f}_x = [f_1, f_2, f_3]$.

- Quasilinear systems: If $\mathbf{f}_t$ or $\mathbf{f}_x$ depend on the unknown vector $\mathbf{u}$ in addition of $\mathbf{x}$ and t the system of PDE is nonlinear We can write it in the form,

$$\mathbf{f}_{t,t} + \mathbf{f}_{1,1} + \mathbf{f}_{2,2} + \mathbf{f}_{3,3} = r \quad \Leftrightarrow \quad \mathbf{A}_t\mathbf{u}_{,t} + \mathbf{A}_1\mathbf{u}_{,1} + \mathbf{A}_2\mathbf{u}_{,2} + \mathbf{A}_3\mathbf{u}_{,3} = 0 \quad \text{where}$$

$$\mathbf{A}_t(\mathbf{u}, \mathbf{x}, t) := \nabla_\mathbf{u}\mathbf{f}_t \quad \text{that is} \quad (\mathbf{A}_t)_{ij} = \frac{\partial(f_t)_i}{\partial u_j} \quad \text{similarly}$$

$$\mathbf{A}_i(\mathbf{u}, \mathbf{x}, t) := \nabla_\mathbf{u}\mathbf{f}_i \quad \text{that is} \quad (\mathbf{A}_i)_{ij} = \frac{\partial(\mathbf{f}_i)_i}{\partial u_j} \quad i \le 3$$

If $\mathbf{A}_t$ and $\mathbf{A}_i$ ($i \le 3$) do not depend on $\mathbf{u}$ the system is linear or semi-linear; otherwise it is quasi-linear.

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$$\dot{q} + \nabla\cdot f_q = 0$$

1-array

2-array

Euler's eqn

scalar

$$q = \begin{bmatrix} \rho \\ \rho e \\ \rho v_1 \\ \rho v_2 \\ \rho v_3 \end{bmatrix}$$

vector

- If further $\mathbf{u}$ is chosen to be the temporal flux $\mathbf{f}_t$ is the primary field $\mathbf{u}$ then $\mathbf{A}_t = \mathbf{I}$ and we have,

$$\boxed{\mathbf{u}_{,t} + \mathbf{f}_{1,1} + \mathbf{f}_{2,2} + \mathbf{f}_{3,3} = r \qquad \Leftrightarrow \qquad \mathbf{u}_{,t} + \mathbf{A}_1\mathbf{u}_{,1} + \mathbf{A}_2\mathbf{u}_{,2} + \mathbf{A}_3\mathbf{u}_{,3} = 0}$$

- For a scalar system clearly $\mathbf{A}_i$ are scalar.
- In 1D $\mathbf{f}_2 = \mathbf{f}_3 = 0$ ($\mathbf{A}_2 = \mathbf{A}_3 = 0$).
- Consider a general scalar quasilinear conservation law,

$$\boxed{\begin{array}{l} u_{,t} + f_{1,1}(u) + f_{2,2}(u) + f_{3,3}(u) = r \qquad \Leftrightarrow \\ u_{,t} + g_1(u,x,t)u_{,1} + g_2(u,x,t)u_{,2} + g_3(u,x,t)u_{,3} = 0 \end{array}}$$

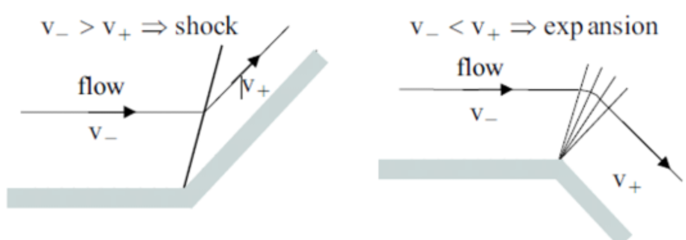
which for 1D it simply is

$$u_{,t} + f_{,x}(u) = r \qquad \Leftrightarrow \qquad u_{,t} + g(u,x,t)u_{,x} = 0$$

- If $g(u,x,t)$ depends on u the solution behavior for u can be very different from linear/semilinear first order systems and may exhibit shocks and expansion waves.

Quasi-linear first order equations and why shocks are formed:

Motivation: Shock and expansion waves:



Example: Burger's equation

spatial flux

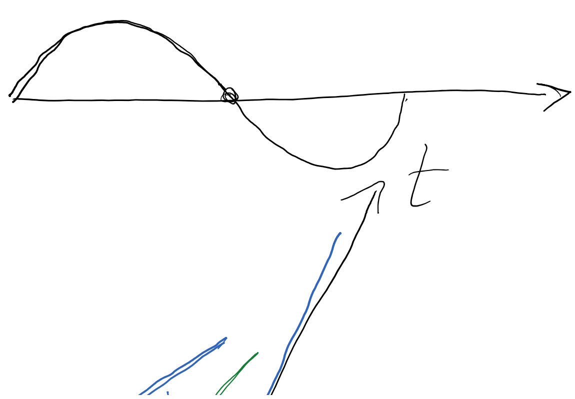
A

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0, \qquad \frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0\,.$$

$$f(u) = u^2/2 \quad \Rightarrow g(u) = f'(u) = u$$

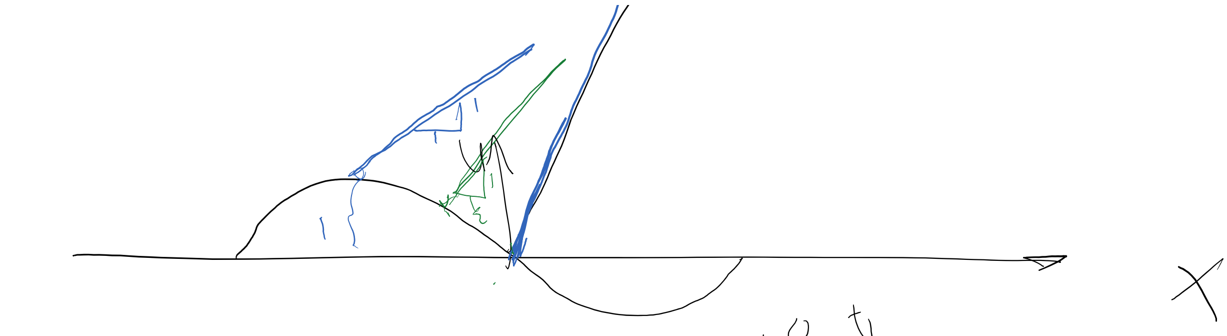
$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0$$

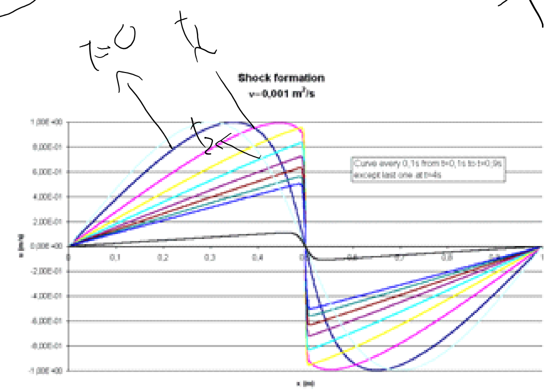
$$\underbrace{A(u) = u}_{\text{speed in advection equation}}$$



$u(0)$

IC





Example: Burger's equation (continued)

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} &= 0, \quad -\infty < x < \infty,\ t > 0 \\
u(x,0) &= \begin{cases} A, & x < 0 \\ B, & x \geq 0\,. \end{cases}
\end{aligned}
$$

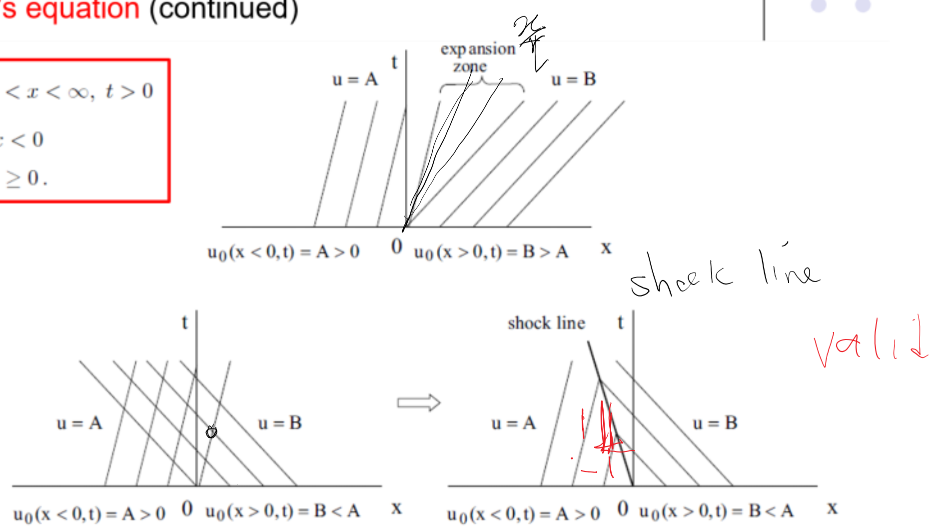


Figure IV.6 If the signal travels slower at the rear than at the front ($A < B$), the characteristic network is under-determined. Conversely, if the signal travels faster at the rear than in front ($A > B$), the characteristic network is over determined: the tentative network that displays intersecting characteristics, has to be modified to show a discontinuity line (curve).

Source: [Loret, 2008]

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Question: How to get the speed of shock?

- Quasi-linear PDE (q(u))

temporal flux

$$\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0$$

spatial flux

- We define the jump operator

$$[\![\cdot]\!] = (\cdot)_+ - (\cdot)_-$$

where + and – refer to the two sides for the jump

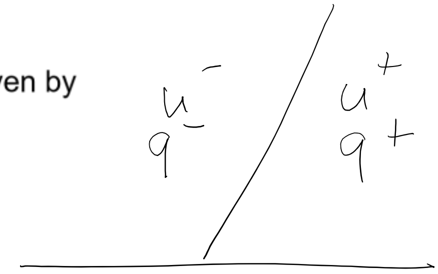
$$[\![\cdot]\!] = (\cdot)_+ - (\cdot)_-$$

where + and – refer to the two sides for the jump.

- If $X_s(t)$ is the position of the jump manifold in time, its equation is given by

$$\frac{dX_s(t)}{dt} = \frac{[\![q]\!]}{[\![u]\!]} = \frac{q_+ - q_-}{u_+ - u_-}$$

This is called the jump or Rankine-Hugoniot condition.



Source: [Loret, 2008]

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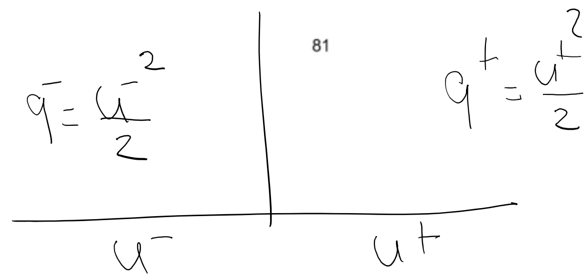
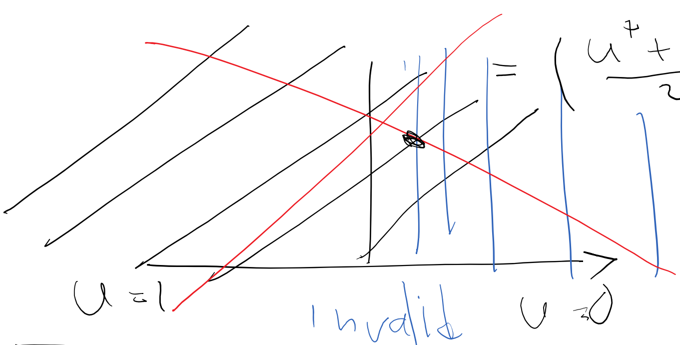
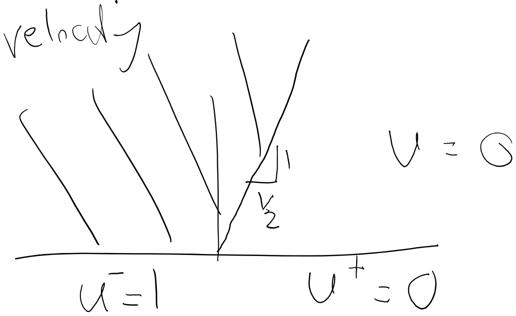
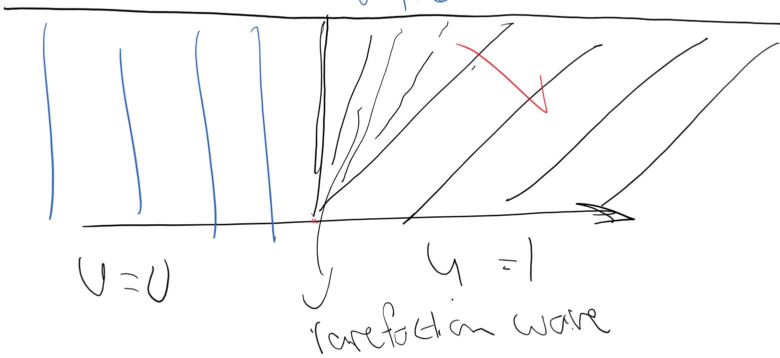


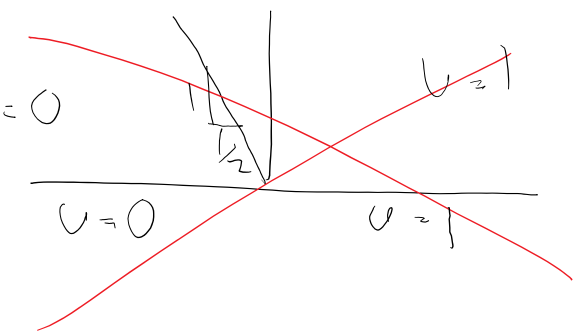
Burgers eqn

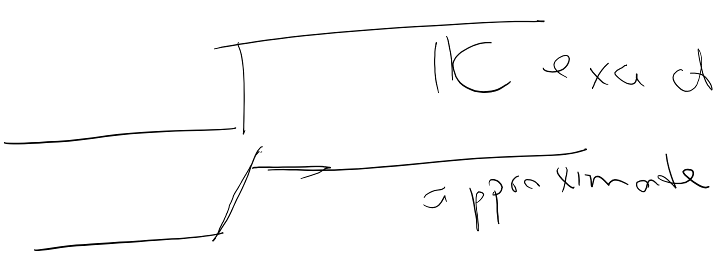
$$\text{speed of shock} = \frac{\frac{u^{+2}}{2} - \frac{u^{-2}}{2}}{u^+ - u^-} = \left(\frac{u^+ + u^-}{2}\right) \text{ average velocity}$$



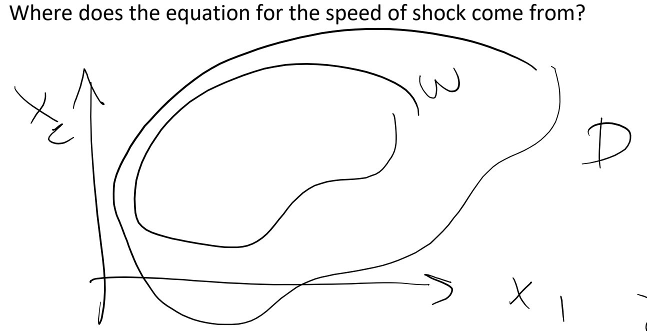








Where does the equation for the speed of shock come from?



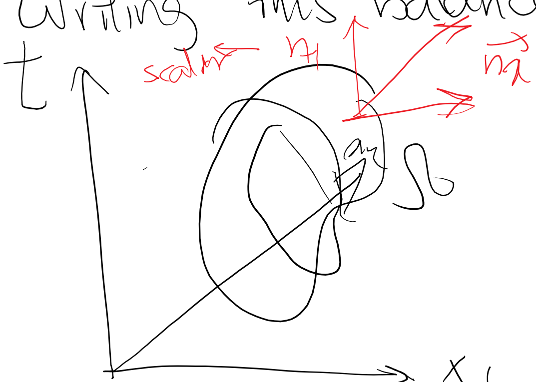
$$\frac{d}{dt}\int_\omega f_t\,dv = -\int_{\partial\omega} f_x\, n\,ds + \int_\omega r\,dv$$

$\int_\omega f_t\,dv$ → balanced quantity; $\int_{\partial\omega} f_x\, n\,ds$ → ~~outward~~ spatial flux

$$\frac{d}{dt}\int f_t\,dv + \boxed{\int_{\partial\omega} f_x\, n\,ds = \int_\omega r\,dv}$$ steady state

look alike

Writing this balance laws in space time



$$\boxed{\int_{\partial\Omega} F\cdot N\,dS = \int_\Omega r\,dV}$$

spacetime flux

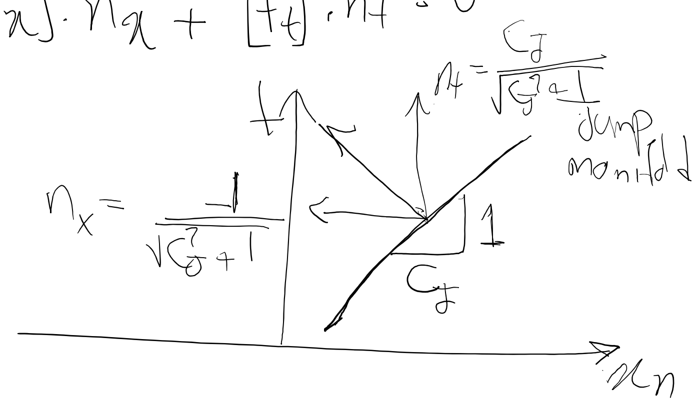
$$F = \begin{bmatrix} f_x \\ f_t \end{bmatrix} \quad N = \begin{bmatrix} \vec{n}_x \\ n_t \end{bmatrix}$$

$\omega_1 \cup \omega_2 \longrightarrow$ Jump condition

Jump condition $\boxed{[F]\cdot N = 0}$

$$\begin{bmatrix} [f_x] \\ [f_t] \end{bmatrix} \cdot \begin{bmatrix} \vec{n}_x \\ n_t \end{bmatrix} = 0 \implies [f_x]\cdot\vec{n}_x + [f_t]\cdot n_t = 0$$

c_J = speed of jump = ?



$$[f_x]\left(\frac{-1}{\sqrt{c_J^2+1}}\right) + [f_t]\cdot\frac{c_J}{\sqrt{c_J^2+1}} = 0$$

$$\boxed{c_J = \frac{[f_x]\cdot \tilde{n}_x}{[f_t]}}$$

$|\tilde{n}_x| = 1$

$\vec{N} = \begin{bmatrix} n_x \\ n_t \end{bmatrix}$

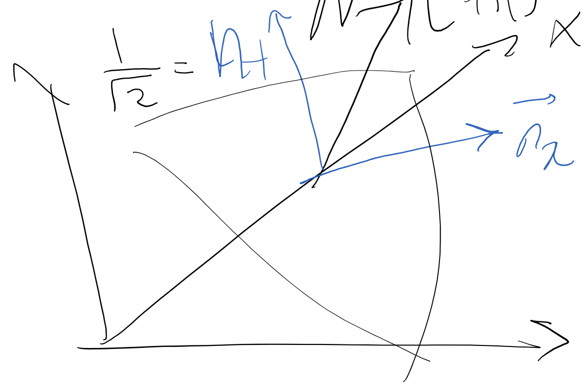
$\frac{1}{\sqrt{2}} = n_t$

$\vec{n}_x = \left(\frac{1}{2}, \frac{1}{2}\right)$

$\tilde{n}_x = \frac{n_x}{|n_x|} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$|\tilde{n}_x| = 1$

$n_x^2 + n_t^2 = 1$



Does it match

$$\frac{dX_s(t)}{dt} = \frac{[\![q]\!]}{[\![u]\!]} = \frac{q_+ - q_-}{u_+ - u_-}$$

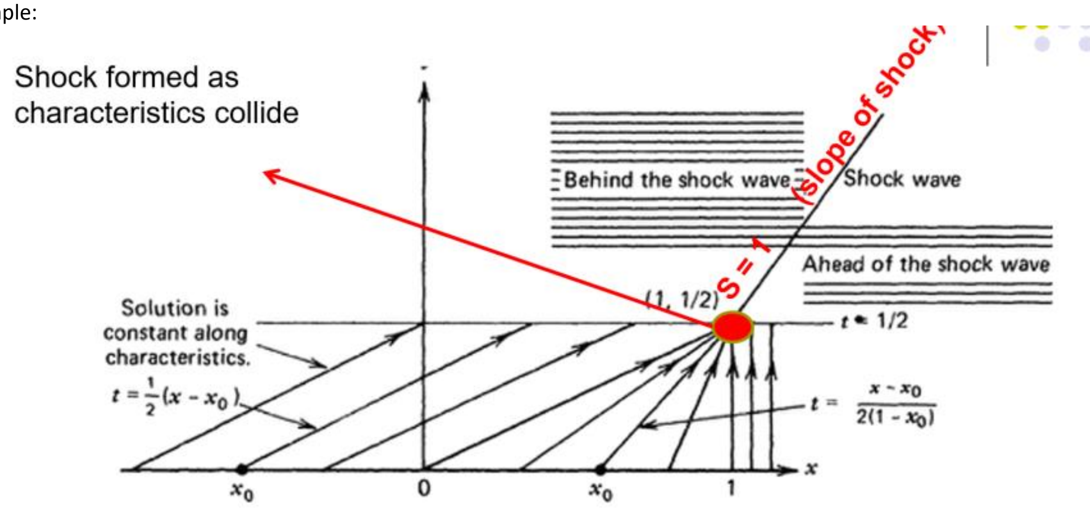
c_J $\quad [f_x]\cdot\tilde{n}_x \quad [f_t]$

$\dot{u} + q_x = 0$

$f_t \quad f_x$

- Jumps cannot propagate with arbitrary speed and we cannot have arbitrary jumps in fx and ft. Their jumps are related and dete rmines the speed of jump
- For semi-linear sys. Of convs law the speeds of discontinuities are the wave speeds.

Example:



$$S = \frac{f(u_R) - f(u_L)}{u_R - u_L} = \frac{0-1}{0-1} = 1$$

Flux = u^2

Source: [Farlow, 2012, lesson 28]

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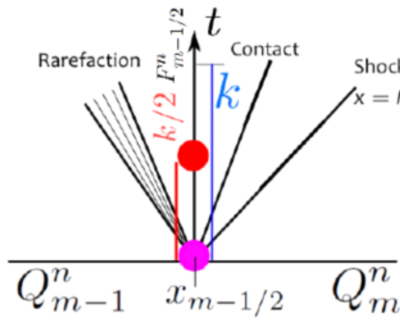
1.5.1 Introduction and Motivation

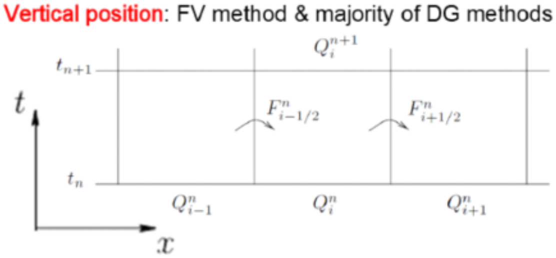
- In many instances we need to find the flux on an extruded facet in space time. For example, in Finite Volume (FV) method:
  - PDE from the balance law is $q_{,t}(x,t) + f_{,x}(q(x,t)) = 0$.
  - We need to calculate spatial flux $f(q(x,t))$ average
  - which for the boundary between cells $m-1$ and m is represented / approximated by numerical flux $F^n_{m-1/2}$.
- We solve a local problem with initial conditions Q^n_{m-1}, Q^n_m to find the value for solution at position in the figure. This is called a Riemann solution. Sometimes, we opt to choose substitutes or approximations for Riemann solution as it solution is to expensive or not available.



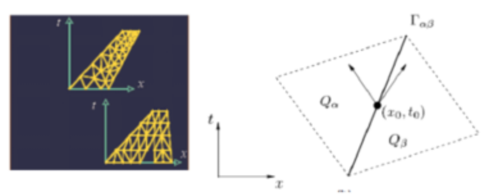
- The Riemann solution set-up in the figure is for a the Euler's equation, where for constant states at the left and right we obtain different regions in spacetime with distinct solutions.

- The red dot is the position where we position we may seek the solution for a FV scheme, Discontinuous Galerkin (DG), *etc.* (to represent the average of flux on the cell boundary).
- For quasilinear systems we may have complex solutions with rarefaction waves and shocks.
- This explains why we may use approximate Riemann solutions.
- Riemann solution may be required for nonvertical directions as well.

Vertical position: FV method & majority of DG methods



Nonvertical position: Unstructured spacetime grids



1.5.2 Approach 1: Using characteristic values (linear PDEs)

- Characteristic values $\omega = \mathbf{L}\mathbf{q}$ are constant (or solved as ODEs) along characteristic directions
- Primary field $\mathbf{q}$ has n components

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$$

- Transfer to characteristic variables and directions For the system on $\mathbf{q}$ we transfer to characteristics by,

$$\text{For } \dot{\mathbf{q}} + \mathbf{A}\mathbf{q}_{,x} = 0 \quad \mathbf{L}\mathbf{A} = \mathbf{\Lambda}\mathbf{L}, \quad \text{for } \Lambda = \mathrm{diag}(c_1, \cdots, c_n)$$

ω

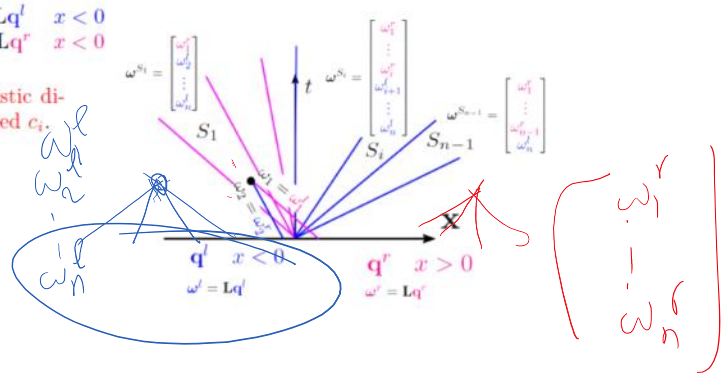
- Eigenvalues (wave speeds) are $c_1 \le c_2 \le \cdots \le c_n$

- Initial conditions are

$$\mathbf{q}(x,0)=\mathbf{q}_0=\begin{cases}\mathbf{q}^l & x<0\\ \mathbf{q}^r & x>0\end{cases} \quad \Rightarrow \omega(x,0)=\omega_0=\begin{cases}\omega^l=\mathbf{L}\mathbf{q}^l & x<0\\ \omega^r=\mathbf{L}\mathbf{q}^r & x<0\end{cases}$$

- Characteristic values $\omega=\mathbf{L}\mathbf{q}$ are constant along characteristic directions; *i.e.*, ω_i is constant along the wave moving with speed c_i.
- Thus, for sample segments S^1, S^i, S^{n-1} we have,

$$\omega^{S_1}=\begin{bmatrix}\omega_1^r\\ \omega_2^l\\ \vdots\\ \omega_n^l\end{bmatrix} \quad \omega^{S_i}=\begin{bmatrix}\omega_1^r\\ \vdots\\ \omega_i^r\\ \omega_{i+1}^l\\ \vdots\\ \omega_n^l\end{bmatrix} \quad \omega^{S_{n-1}}=\begin{bmatrix}\omega_1^r\\ \vdots\\ \omega_{n-1}^r\\ \omega_n^l\end{bmatrix}$$



- Transfer back to primary variables and fluxes is by using $\mathbf{L}$: $\quad \mathbf{q}^{S_i}=\mathbf{L}^{-1}\omega^{S_i}$

1.5.3 Approach 2: Using Jump shapes determined by right eigenvectors

- Again, consider the system,

$$\dot{\mathbf{q}}+\mathbf{A}\mathbf{q}_x=0$$

$$\underbrace{q}_{f_t}+\underbrace{f(q)_{,x}}_{f_x}=0 \qquad f_t+\nabla\cdot f_x=S$$

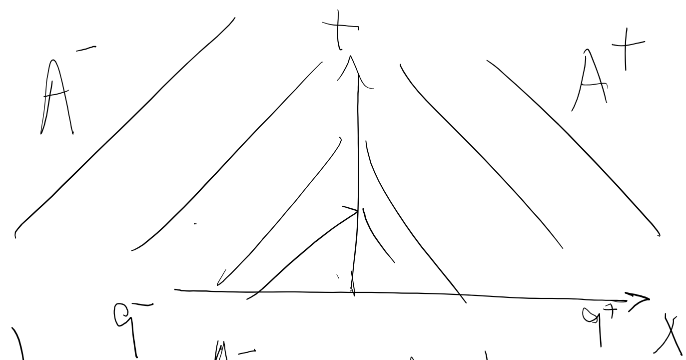
1D first & semi-linear

$$c_J=\frac{[f_x]}{[f_t]}=\frac{[Aq]}{[q]}$$

Jump seed $c_J=\dfrac{[Aq]}{[q]}$

case 1 if $c_J\neq 0$ then $[A]=0$

(we deal with A^- $c_J<0$
A^+ $c_J>0$)



call it A:

$$c_J=\frac{[Aq]}{[q]}=\frac{A[q]}{[q]}$$

$$\underbrace{A[q]=c_J[q]}$$

vector

matrix

RIGHT eigenvector of A

right eigen value of A

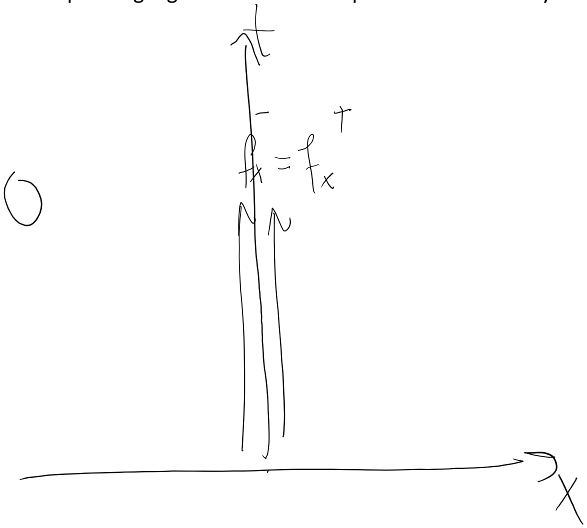
DC Page 8

For c != 0, right eigenvectors are the shapes of the discontinuity across a discontinuity line and corresponding eigenvalues are the speed of discontinuity.

For c = 0:

$$[f_x] \cdot n_x = c[f_t]$$

$$[Aq] = c[f_t] \qquad c = 0$$

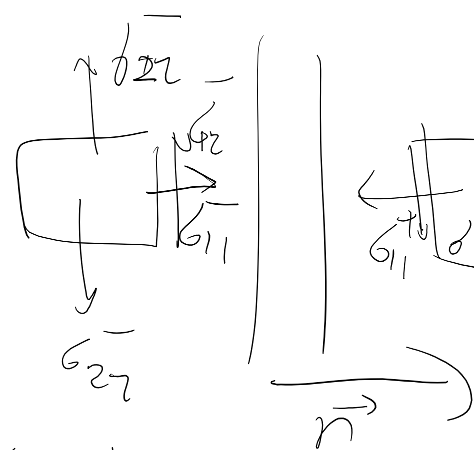


$$\boxed{[Aq] = 0}$$

$$A^- q^- = A^+ q^+$$

$$f_{x_n}^- = f_{x_n}^+$$

$$\begin{array}{c} \\ \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \end{array} \begin{array}{c} n \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array} =$$



net flux $\begin{bmatrix} \sigma_{11} \\ \sigma_{21} \end{bmatrix}$ continuous $[\sigma_{22}] \neq 0$

$$\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix} \qquad \sigma = C_{2\times2\times2\times2}\,\varepsilon \qquad [\varepsilon] \neq 0$$

$$[v] = 0 \qquad [p] \neq 0 \qquad p = \rho V$$