DC20200205

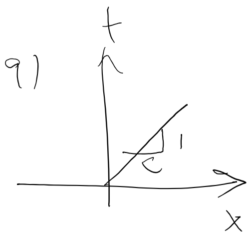
Wednesday, February 5, 2020 1:09 PM

Recall from the last time that the jump condition from the balance law can be written as follows:

$[Aq] = c[q]$ ↳ speed of jump

Semi-linear

$\dot{q} + Aq_{,x} = S(q)$



two cases

$\underline{c \neq 0}$ $\boxed{A[q] = c[q]}$

$[q]$ eigenvector of A

c " value of A

$\underline{c = 0}$ $\boxed{A^+q^+ - A^-q^- = 0}$ jump of net spatial flux (Aq) is zero

special case to be taken in general

Example of the two solution schemes, especially the 2nd one:

Acoustic equation

p = pressure $= -\sigma$

v velocity

1D Balance of linear momentum

$\rho \dot{v} - \sigma_{,x} = \rho b$

1eqn $\boxed{\dot{v} + \frac{1}{\rho} p_{,x} = b}$

$\dot{p} + K v_{,x} = 0$ combatibility

K stiffness

$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} p \\ v \end{bmatrix}$

$\dot{q}_1 + K q_{2,x} = 0$

$\dot{q}_2 + \frac{1}{\rho} q_{1,x} = 0$

assume $\underline{\rho b = 0}$

$\dot{q} + A q_{,x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\boxed{A = \begin{bmatrix} 0 & k \\ \frac{1}{\rho} & 0 \end{bmatrix}}$ spatial flux matrix

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$\underline{\text{sln method 2}}$ jump condition

$A[q] = c[q]$ Do write eigenvalue sln

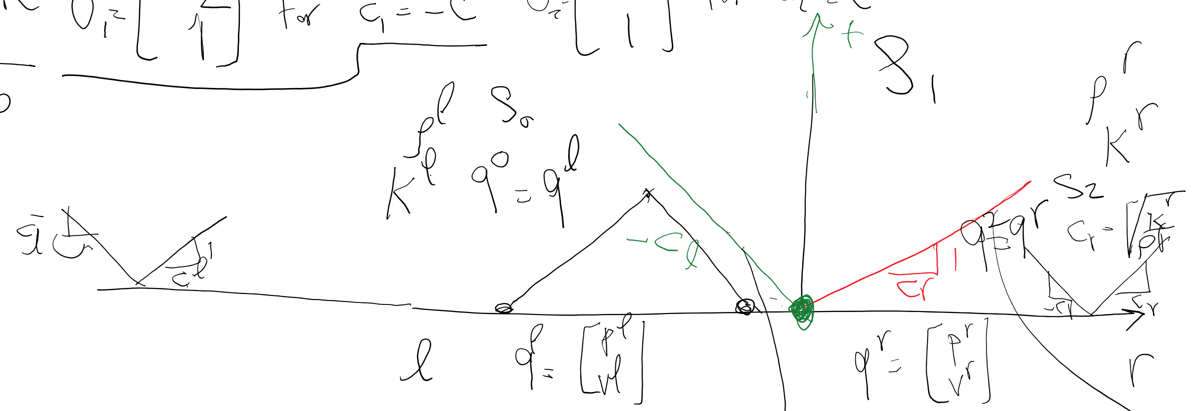
$A[q] = c[q]$ Do write eigenvalue ...

$$\underbrace{\begin{bmatrix} -c & k \\ \frac{1}{\rho} & -c \end{bmatrix}}_{\det(A - cI) = 0}\begin{bmatrix} [p] \\ [v] \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \end{bmatrix} \qquad c^2 - \frac{k}{\rho} = 0 \Rightarrow c = \pm\sqrt{\frac{K}{\rho}}$$

Eigenvectors are $u_1 = \begin{bmatrix} -Z \\ 1 \end{bmatrix}$ for $\underbrace{c_1 = -c}$ $\quad u_2 = \begin{bmatrix} Z \\ 1 \end{bmatrix}$ for $c_2 = c$

$$Z = \rho c = \sqrt{K\rho}$$

$$c_\ell = \sqrt{\frac{K^\ell}{\rho^\ell}}$$



add them

$$[q]_{0 \to 1} = q^1 - q^0 \overset{\text{eigenvector for } -c_\ell}{=} \underbrace{\begin{bmatrix} -Z^\ell \\ 1 \end{bmatrix}}_{\text{any possible eigenvector for } c_1 = -c_\ell} \alpha_1$$

$$[q]_{1 \to 2} = q^2 - q^1 = \begin{bmatrix} Z^r \\ 1 \end{bmatrix} \alpha_2$$

$$q^2 - q^0 = \begin{bmatrix} -Z^\ell & Z^r \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$[q] = q^r - q^\ell = \begin{bmatrix} -Z^\ell & Z^r \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \qquad \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} -Z^\ell & Z^r \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} [q_1] \\ [q_2] \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \frac{1}{Z^\ell + Z^r}\begin{bmatrix} -1 & Z^r \\ 1 & Z^\ell \end{bmatrix}\begin{bmatrix} [p] \\ [v] \end{bmatrix}$$

$$\longrightarrow \boxed{\alpha_1 = \frac{1}{Z^\ell + Z^r}\left(-[p] + Z^r[v]\right)} \qquad [\phi] = \phi_r - \phi_\ell$$

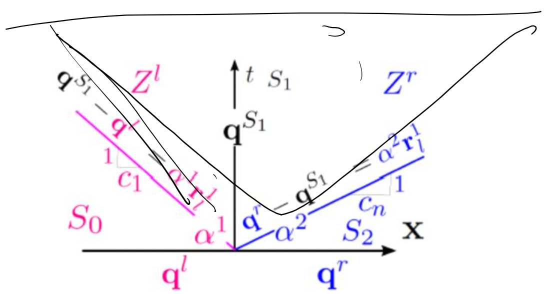
$$q^{(1)} = \begin{bmatrix} p^1 \\ v^1 \end{bmatrix} = \underbrace{\begin{bmatrix} p^l \\ v^l \end{bmatrix}}_{q^0} + \alpha_1 \begin{bmatrix} -Z^l \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} p^1 \\ v^1 \end{bmatrix} = \begin{bmatrix} p^l \\ v^l \end{bmatrix} + \frac{1}{Z^l + Z^r}\left(-[p] + Z^r[v]\right) \begin{bmatrix} -Z^l \\ 1 \end{bmatrix}$$

$$p^1 = p^l + \frac{Z^l}{Z^l+Z^r}[p] - \frac{Z^l Z^r}{Z^l+Z^r}[v] = \frac{\cancel{p^l Z^l} + p^l Z^r + Z^l p^r - \cancel{Z^l p^l}}{Z^l + Z^r} - \frac{Z^l Z^r}{Z^l+Z^r}[v]$$

$$\rightarrow \boxed{p^1 = \frac{Z^r p^l + Z^l p^r}{Z^l + Z^r} - \frac{Z^l Z^r}{Z^l+Z^r}[v]}$$

$$\mathbf{q}^m = \mathbf{q}^{S_1} = \begin{bmatrix} p^m \\ v^m \end{bmatrix} = \begin{bmatrix} \frac{Z^l p^r + Z^r p^l}{Z^l+Z^r} - \frac{Z^l Z^r}{Z^l+Z^r}(v^r - v^l) \\ -\frac{1}{Z^l+Z^r}(p^r - p^l) + \frac{Z^l v^l + Z^r v^r}{Z^l+Z^r} \end{bmatrix}$$



$$Z^l = Z^r$$

$$p^{(1)} = \{p\} - \frac{Z}{2}[v]$$

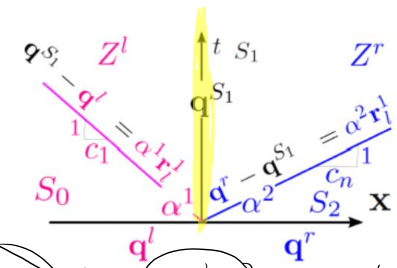
$$v^{(1)} = -\frac{1}{2Z}[p] + \{v\}$$

Elastodynamics $\sigma(-p)$ or ε

v or ρv

$q = \begin{bmatrix} \sigma \\ v \end{bmatrix}$, $\begin{bmatrix} \sigma \\ \rho v \end{bmatrix}$, $\begin{bmatrix} \varepsilon \\ v \end{bmatrix}$, $\begin{bmatrix} \varepsilon \\ \rho v \end{bmatrix}$ any of these would have been possible!

not very good at the material interface



jump of which one is zero

σ Continuous

or

ε not continuous

because of compatibility — v

linear momentum ρv

$p_{,x}$

$\rho \dot{v} - \sigma_{,x} = \rho b$

σ is spatial flux

$[\sigma] \cdot n = 0$ on $c = 0$

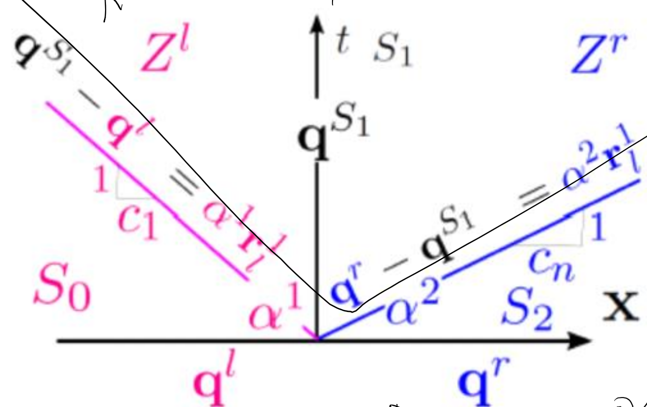
$c[f_t] + [f_x]\tilde{n}_x = 0$

$c = 0 \rightarrow [f_x] \cdot \tilde{n}_x = 0$

$\varepsilon^{ml} \neq \varepsilon^{mr}$

p^m, v^m

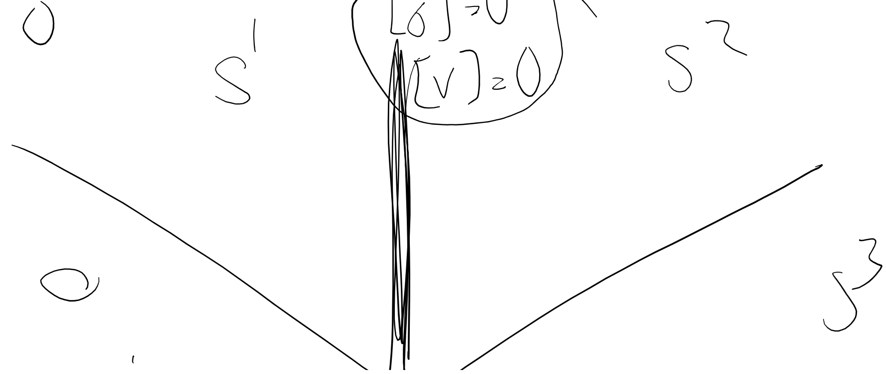
$\varepsilon^{ml} = \frac{p^m}{K^l}$ $\rho^l v^m \neq \rho^r v^m$ $\varepsilon^{mr} = \frac{p^m}{K^r}$

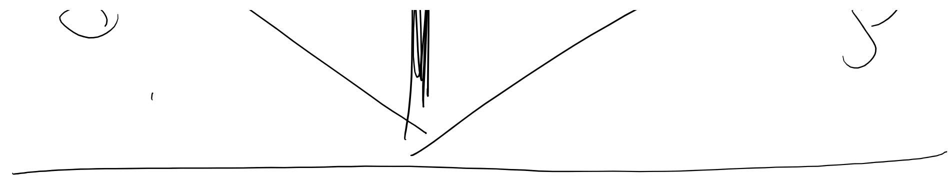


If we had chosen $q = \begin{bmatrix} p/K \\ v \end{bmatrix}, \begin{bmatrix} p/K \\ \rho v \end{bmatrix}, \begin{bmatrix} p \\ \rho v \end{bmatrix}$

we had to ensure interface matching conditions where satisfied @ $c = 0$

S^1 $[\sigma] = 0$ $[v] = 0$ S^2





Message: Often it's better to use quantities that do not jump in material interface as components of q (as we chose p and v in this case).
If not, we need to make sure we satisfy the correct jump conditions at the material interface (c = 0).

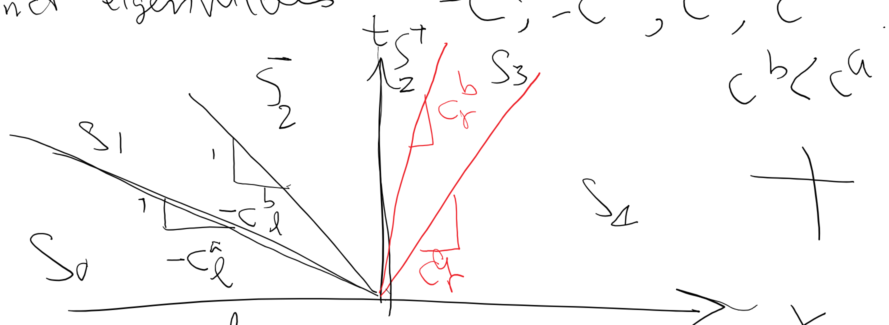
What material property appears in the solution of region 1 (K, rho, and Z)? So the solution only depends on Z not individual values of K and rho.
Basically, wave transmission between two media only depends on impedance.

How does this generalize to more number of parameters?

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + A_{4\times 4}\, q_{,x} = 0$$

A has 4 distinct eigenvalues $-c^a, -c^b, c^b, c^a$

$c^b < c^a$



U_i's & c_i's are left eigenvectors & values of A:

$$\begin{cases} q^{(1)} - q^{(0)} = U_0^l\, \alpha_0 \\ q^{(2)-} - q^{(1)} = U_1^l\, \alpha_1 \\ q^{(2)-} = q^{(2)+} \\ q^{(3)} - q^{(2)+} = U_2^r\, \alpha_2 \\ q^{(4)} - q^{(3)} = U_3^r\, \alpha_3 \end{cases}$$

Same matching condition between $q^{(2)-}$ & $q^{(2)+}$
if q is chosen wisely $q^{(2)-} = q^{(2)+}$

$$q^{(4)} - q^{(3)} = U_3^r \alpha_3 \qquad \bar{U}$$

$$q^r - q^l = q^{(4)} - q^{(0)} = \overbrace{\left[\underbrace{U_0^l \,\middle|\, U_1^l}_{\text{eigen vectors for eigenvalues from } l} \,\middle|\, \underbrace{U_2^r \,\middle|\, U_3^r}_{\text{eigen vectors for eigenvalues from } R} \right]}^{\bar{U}} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

✓ here it

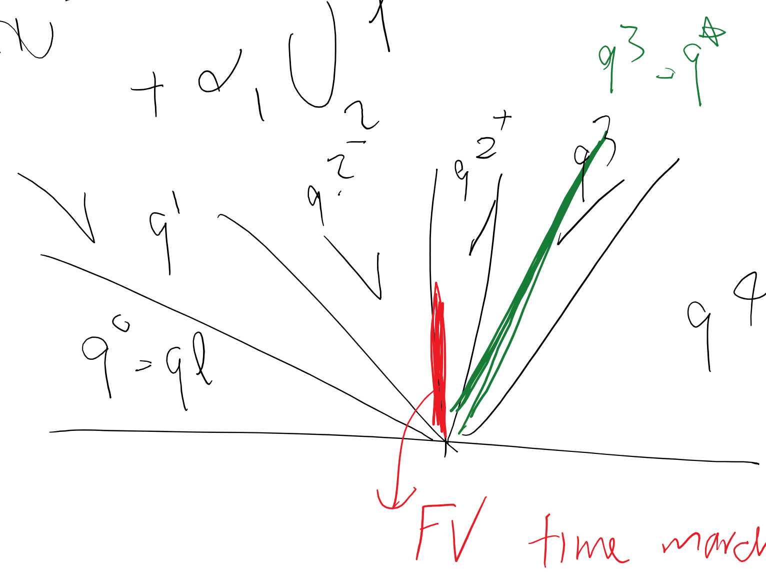
$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \bar{U}^{-1} \begin{bmatrix} q_1^r - q_1^l \\ q_2^r - q_2^l \\ \vdots \\ q_4^r - q_4^l \end{bmatrix}$$

α's known

$$q^{(1)} = \underbrace{q^l}_{q^{(0)}} + \alpha_0 U_1^l \quad ✓$$

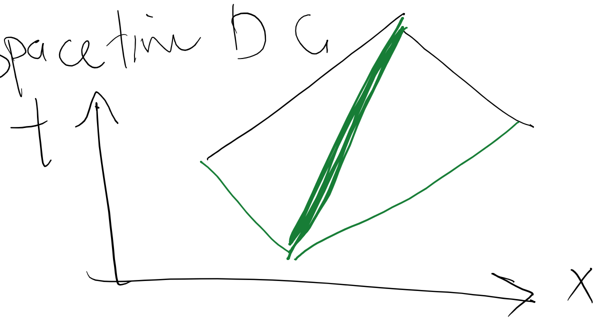
$$q^{(2)-} = q^{(1)-} + \alpha_1 U_2^l$$

$$\vdots$$



time marching schemes
the * value is the solution on this line

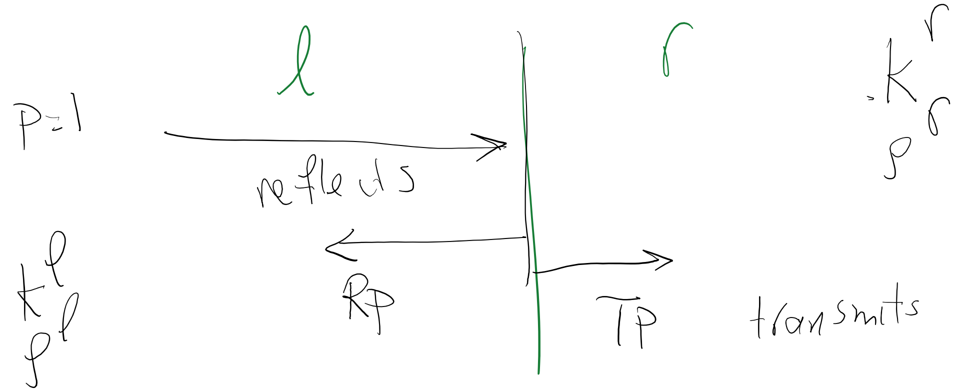
Spacetime DG



Transmission and Reflection coefficients:

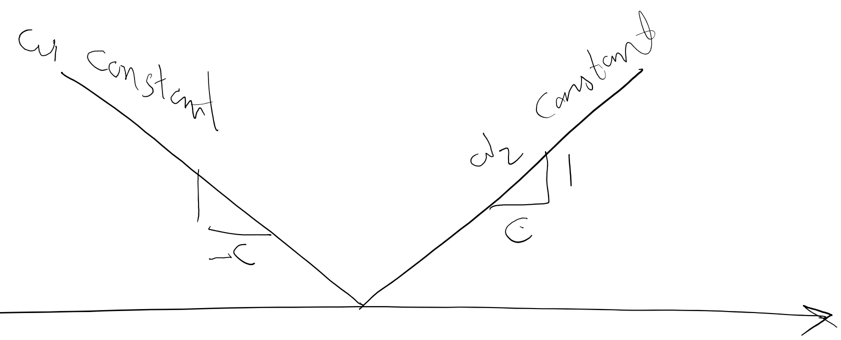
$R = ?$

$T = ?$



$$A = \begin{bmatrix} 0 & K \\ \frac{1}{\rho} & 0 \end{bmatrix}$$

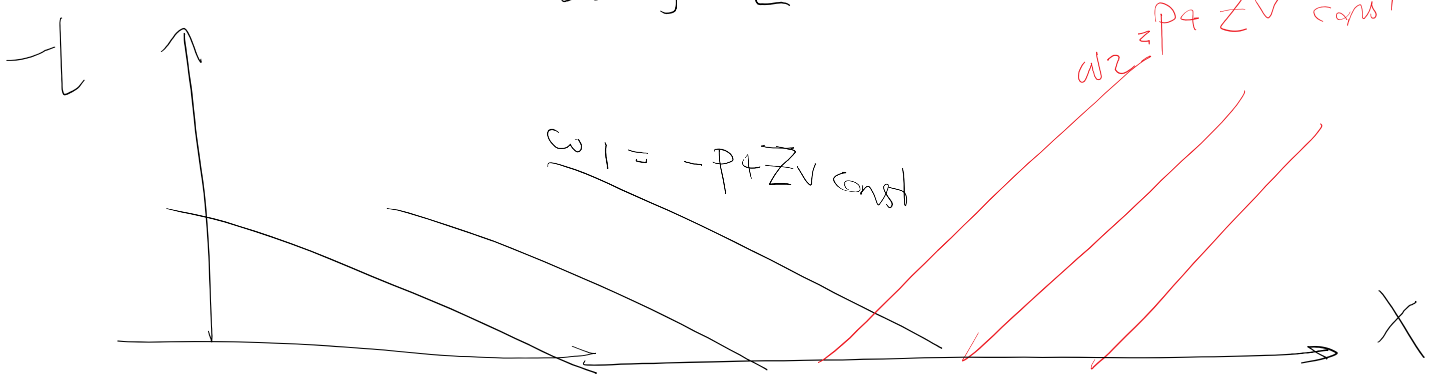
left eigenvectors are



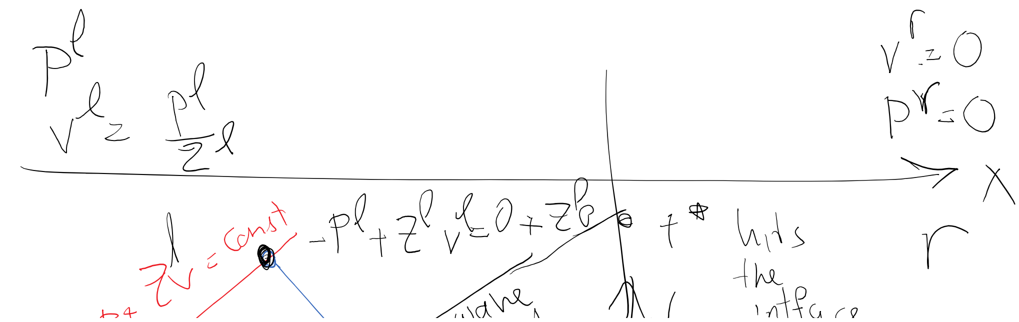
$$L = \begin{bmatrix} -1 & Z \\ 1 & Z \end{bmatrix} \begin{array}{l} l_1 \\ l_2 \end{array}$$

for eigenvalues $-c$, c

$$\omega = Lq = \begin{bmatrix} -1 & Z \\ 1 & Z \end{bmatrix} \begin{bmatrix} p \\ v \end{bmatrix} = \begin{bmatrix} -p + Zv \\ p + Zv \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

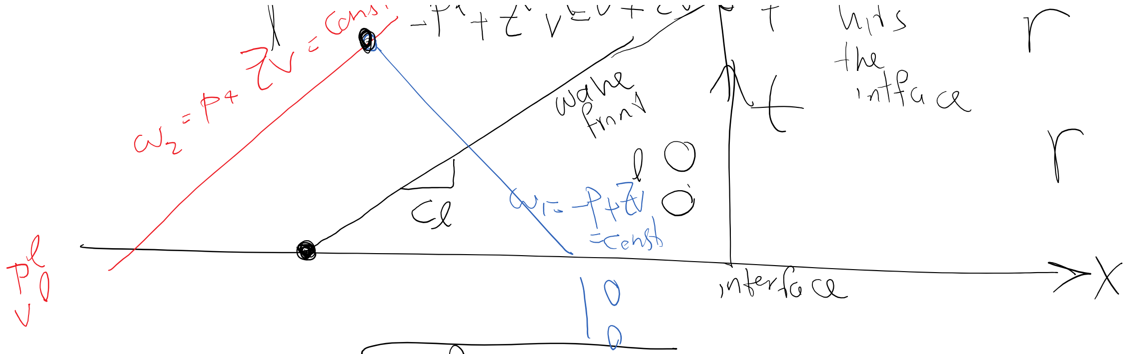


Right-going wave



wave

l r



r

$$-p^l + Z^l v^l = 0 \Longrightarrow \boxed{v^l = \frac{p^l}{Z^l}}$$ right going wave

Interface

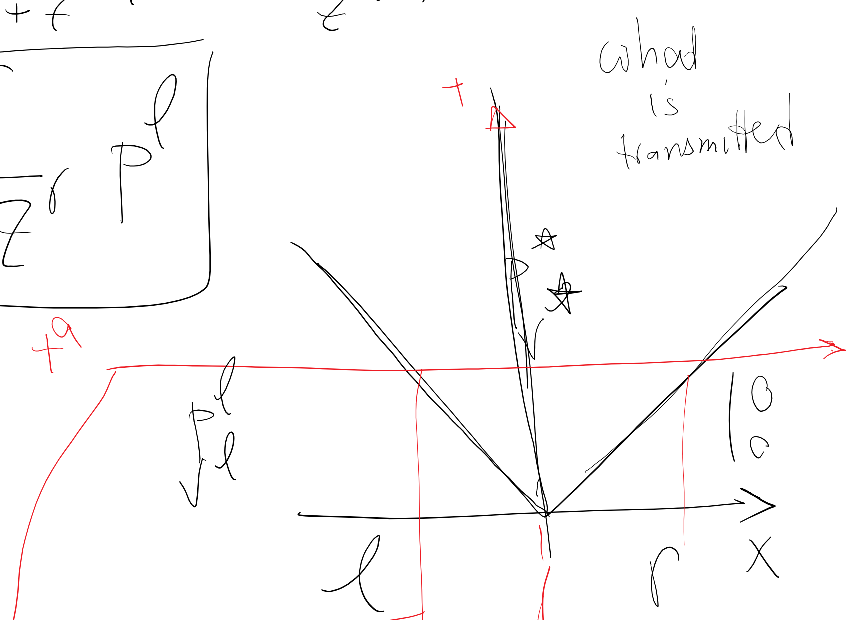
p^l $\qquad$ $p^r = 0$

$v^l = \frac{p^l}{Z^l}$ $\qquad$ $v^r = 0$

$$p^* = \frac{Z^l p^r + Z^r p^l}{Z^l + Z^r} - \frac{Z^l Z^r}{Z^l + Z^r}\left(v^r - v^l\right)$$

$$= \frac{Z^r p^l}{Z^l + Z^r} - \frac{Z^l Z^r}{Z^l + Z^r}\left(0 - \frac{p^l}{Z^l}\right)$$

transmi $\leftarrow$ $$\boxed{p^* = \frac{2 Z^r}{Z^l + Z^r} p^l}$$



$$T = \frac{p^*}{p^l}$$

$$T \quad 2 Z^r$$

$$\boxed{T = \frac{2Z^r}{Z^l + Z^r}}$$

transmission coefficient

$$Z^l = Z^r \longrightarrow T = 1$$