DC20200210

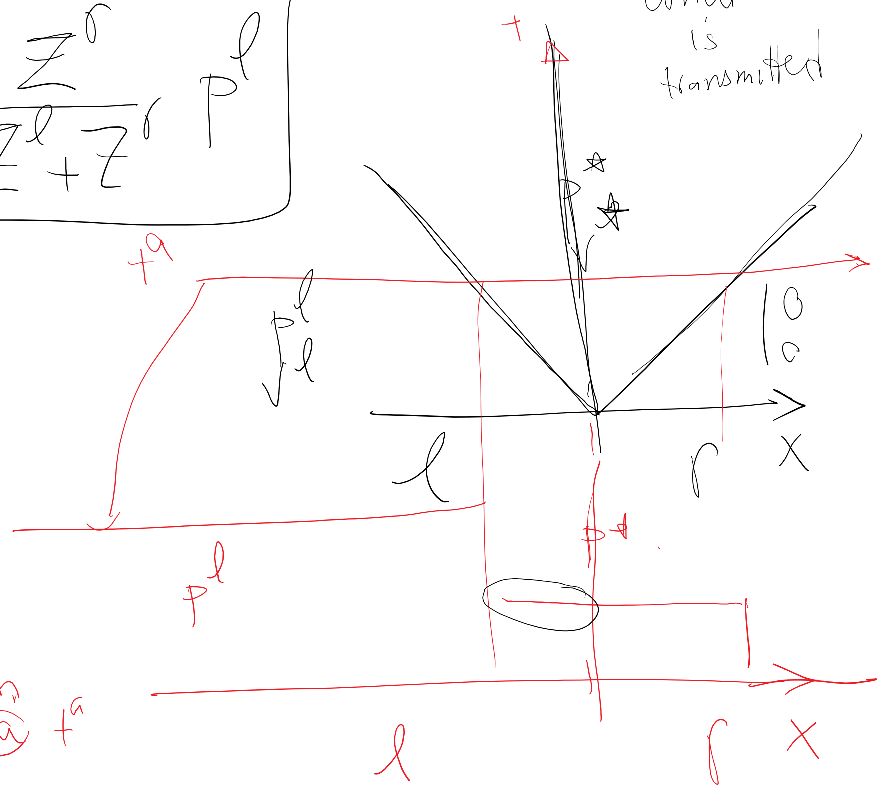
Monday, February 10, 2020 1:06 PM

p^l $v^l = \frac{p^l}{Z^l}$ interface $p^r = 0$ $v^r = 0$

$$p^* = \frac{Z^l p^r + Z^r p^l}{Z^l + Z^r} - \frac{Z^l Z^r}{Z^l + Z^r}(v^r - v^l)$$

$$= \frac{Z^r p^l}{Z^l + Z^r} - \frac{Z^l Z^r}{Z^l + Z^r}\left(0 - \frac{p^l}{Z^l}\right)$$

transmi ←
$$\boxed{p^* = \frac{2Z^r}{Z^l + Z^r} p^l}$$



$$T = \frac{p^*}{p^l}$$

$$\boxed{T = \frac{2Z^r}{Z^l + Z^r}}$$

↓ transmission coefficient

$$Z^l = Z^r \longrightarrow T = 1 \quad :)$$

Reflection coefficient:

Reflection $+ p^* = p^l$ Reflected $p = p^* - p^l$

$= p^l T - p^l = (-1+T)p^l$

Reflection coefficient $R = (-1+T)p^l \ldots = T - 1$

$= T p^r - 1 = (-1+T) p^l$

Reflection coefficient $R = \frac{(-1+T)p^l}{p^l} = T - 1$

$R = T - 1 = \frac{2Z^r}{Z^l + Z^r} - 1 \longrightarrow R_{lr} = \frac{Z^r - Z^l}{Z^r + Z^l}$

$\longrightarrow$

$l \qquad r$

likewise $R_{rl} = \frac{Z^l - Z^r}{Z^l + Z^r} = -R_{lr}$

$T_{lr} = \frac{2Z^r}{Z^l + Z^r} \qquad T_{rl} = \frac{2Z^l}{Z^l + Z^r}$

— what about oblique waves?

— What about anisotropic materials (2D, 3D)

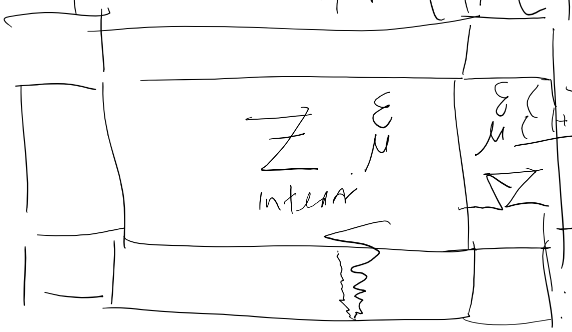
1.5.4.4 Importance of impedance Z: transmission and Reflection Coefficients

- We observed that if $Z^l = Z^r$ the Riemann solution (17) reduces to that of the same material on both sides (14).
- This implies Riemann solution only depends on Z not individual K and ρ values.
- In fact, by solving a Riemann problem with a right-going wave, we can find out how much of the wave is reflected from the interface and how much is transmitted. These values only depend on $Z^l = Z^r$.
- For example for a right-going wave with pressure value of p_0, $C_T p_0$ is transmitted and $C_R p_0$ is reflected. That is, after reflection the new pressure in the left region (region that the wave originates) is $C_R p_0$ and in the right region the pressure would be $C_T p_0$.
- The transmission and reflection coefficients for pressure field in acoustic equation are,

$$C_T = \frac{2Z^r}{Z^l + Z^r} \qquad C_R = \frac{Z^r - Z^l}{Z^l + Z^r} \tag{18}$$

- We observe when $Z^r = Z^l$ (impedance matching) $C_R = 0$ and no wave is reflected. That is, the interface behaves as if the same material is on both sides even if K, ρ are distinct.

→ idea behind perfectly matched layer (PML)



→ PML matches complex material properties but

$Z_{PML} = \sqrt{\frac{\varepsilon}{\mu}} = Z_{inside}$

no reflection

$\bar{\varepsilon} D$

$Z = \sqrt{C\rho}$

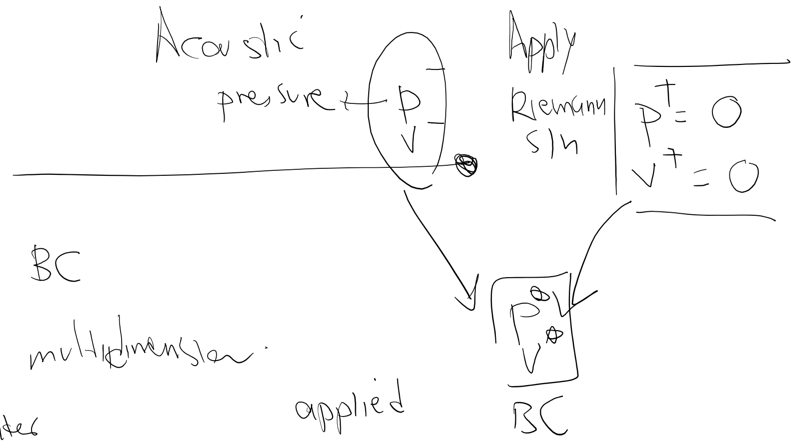
stiffness

Stiffness

no reflection

inside

1D we can form exact transmitting BC



Silver-Muller BC

not exact in multidimension.

PML is better

- Conversely, when $Z^r = Z^l$ (impedance mismatch) $C_R \neq 0$ waves get reflected from the interface.
- That explains why we call Z impedance which is the impedance to wave motion.
- For more information refer to [LeVeque, 2002] "§9.8 Variable Impedance" and "§9.10 Transmission and Reflection Coefficients".
- Figures below ([LeVeque, 2002] Fig. 9.4) show a case where $K^r \neq K^l, \rho^r \neq \rho^l$ yet since $K^r = K^l$ no wave gets reflected.

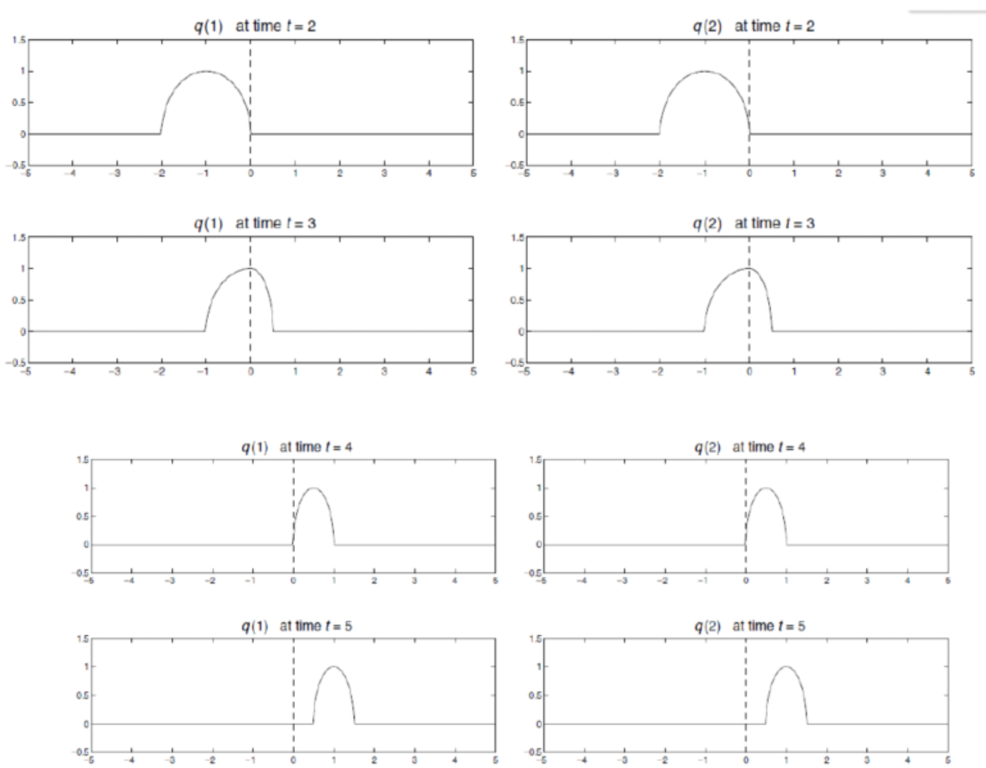


Fig. 9.4. Right-going acoustic pulse hitting a material interface (dashed line) where the sound speed changes from 1 to 0.5 but the impedance is the same. Left column: pressure. Right column: velocity. [claw/book/chap9/acoustics/interface]

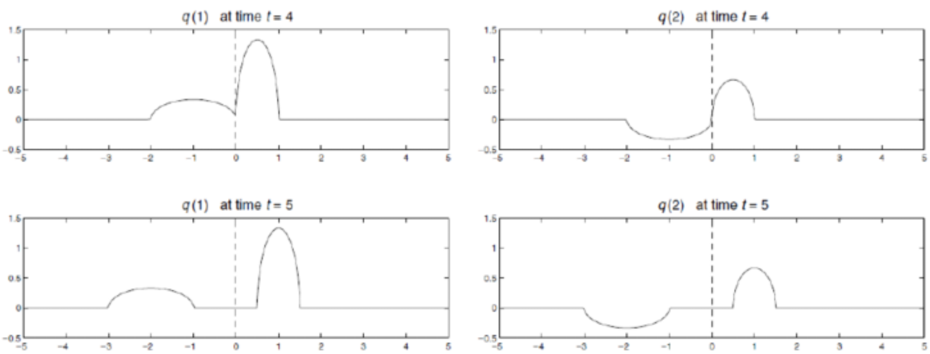


Fig. 9.5. Right-going acoustic pulse hitting a material interface (dashed line) where the sound speed changes from 1 to 0.5 and the impedance changes from 1 to 2. Part of the wave is reflected at the interface. Left column: pressure. Right column: velocity. [claw/book/chap9/acoustics/interface]

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2 General solution schemes in space (or spacetime)

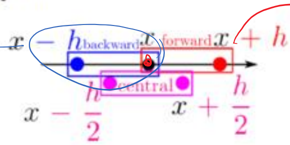
2.1.1 Finite Difference (FD) operators

- $f(x)$ scalar function of x (either space or time).
- We define the following difference operators (with spacing h)

$$\Delta_h[f](x) = f(x+h) - f(x) \quad \text{forward difference} \quad (19a)$$
$$\nabla_h[f](x) = f(x) - f(x-h) \quad \text{backward difference} \quad (19b)$$
$$\delta_h[f](x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2}) \quad \text{central difference} \quad (19c)$$

- Note: Do not mix up ∇_h with gradient operator!

backward: $\nabla_h f(x) = f(x) - f(x-h)$



$\Delta_h[f] = f(x+h) - f(x)$ forward

central: $\delta_h[f](x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$

Accuracy of FD scheme — Forward difference

$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \ldots$

$\frac{f(x+h)-f(x)}{h} - f'(x) = \frac{h}{2} f''(x) + \ldots = \frac{h}{2} f''(x^*)$

error in Forward finite difference formula for $f'(x)$

$O(h)$

Same with backward.

$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + \ldots$

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + \dots$$

$$f(x-h) = f(x) - h f'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) \dots$$

$$\underbrace{\frac{f(x+h) - f(x-h)}{2h}}_{\text{central diff}} - f'(x) = \frac{h^2}{3} f'''(x) + \dots$$

central diff approximation of $f'(x)$ $= O(h^2)$

By telescopic expansion of the stencil we can get more accurate FD approximation of f'

5 sten

$$A_2^+ \; f(x+2h) = f(x) + (\;) f'(x) + (\;) f''(x) + (\;) f'''(\;)$$

$$+ A_1^+ \; f(x+h)$$

$$+ A_0 \; f(x)$$

$$+ A_1^- \; f(x-h)$$

$$+ A_2^- \; f(x+h)$$

find A's such that $f'(x)$ is left on the RHS & cancel $f, f'', f''', f^{(4)}$

In Finite Element methods we can increase the order but the communication is always between the element and its immediate neighbors (through nodes in CFEM, edges in DG), but in FD the stencil expands by increasing the difference accuracy.

- In general n-th order forward, backward, and central differences are written as,

$$\frac{\partial^n f}{\partial x^n}(x) = \frac{\Delta_h^n[f](x)}{h^n} + \mathcal{O}(h) \quad \text{where}$$

$$\Delta_h^n[f](x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + (n-i)h) \qquad n\text{-th order forward difference}$$

$$\frac{\partial^n f}{\partial x^n}(x) = \frac{\nabla_h^n[f](x)}{h^n} + \mathcal{O}(h) \quad \text{where}$$

$$\nabla_h^n[f](x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x - ih) \qquad n\text{-th order backward difference} \qquad (24b)$$

$$\frac{\partial^n f}{\partial x^n}(x) = \frac{\delta_h^n[f](x)}{h^n} + \mathcal{O}(h^2) \quad \text{where}$$

$$\delta_h^n[f](x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + (\frac{n}{2} - i)h) \qquad n\text{-th order central difference} \qquad (24c)$$

Errors

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h)$$

$h \to 0$ discretization error $\searrow 0$

f(x+h) becomes equal to f(x) with finite precision (computer) calculation. Finite precision

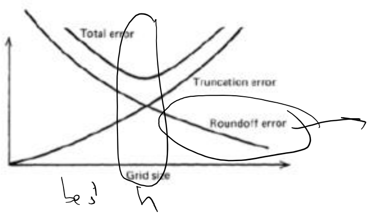
f(x+h) = 10.1235 68102 3 —

f(x) —— // —— 4

don't have access to these

2.1.2 Sources of error

- Truncation (discretization) error: This is the error of the form $\mathcal{O}(h^p)$ which is due to discretization of differential operators with order of accuracy p. The error decreases as $h \searrow$ or $p \nearrow$.
- Roundoff error: This is due to finite precision calculations in computers, where $1+\epsilon \to 1$ for ϵ being the machine epsilon. For FD $f(x+h)-f(x)$ takes the form $(1+\epsilon)-1 = 1-1 = 0$ for small enough h.

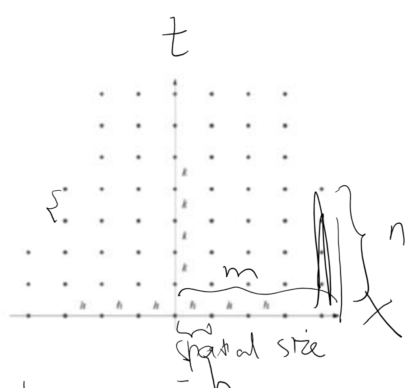


Finite Difference grids for spacetime

2.1.3 Finite Difference grids

- FD grids can be nonuniform and be constructed for 2D, 3D problems. We only discuss the uniform grids, although extension to nonuniform grids is relatively trivial.
- For 1D × time problems the following notation is often used:
  - Spatial size: h (or Δx)
  - Temporal size: k (or Δt)
  - Temporal to spatial ratio $\lambda := \frac{k}{h}$.
- In 2D and 3D h_x, h_y, h_z (or $\Delta x, \Delta y, \Delta z$) are used.
- The FD difference notation is,

$$u_n^m := u(mh, nk) \tag{25}$$



λ = temporal to spatial ratio $= \frac{k}{h}$

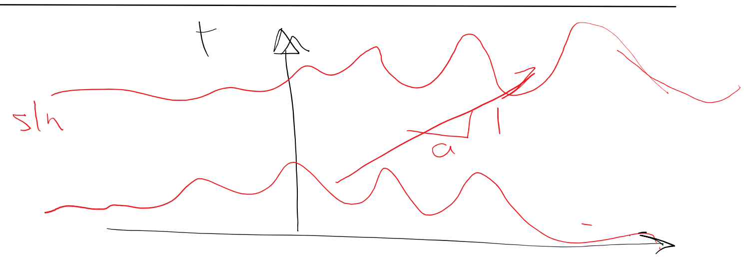
$u^{m}_{n} = u(mh, nk)$ (m: space, n: time)

Finite Difference schemes for solving advection equation:

$a > 0$

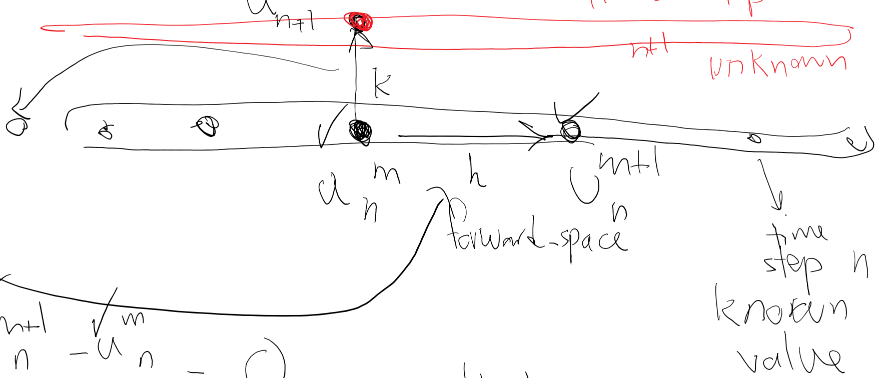
$U_t + a U_x = 0$

$u(x, t=0) = U_0(x)$



$u(x, t=0) = u_0(x)$

Forward space – forward time



$$u_t \approx \frac{u^m_{n+1} - u^m_n}{k}$$

$$u_x \approx \frac{u^{m+1}_n - u^m_n}{h}$$

$$\underbrace{\frac{u^m_{n+1} - u^m_n}{k}}_{u_t} + a\,\frac{u^{m+1}_n - u^m_n}{h} = 0$$

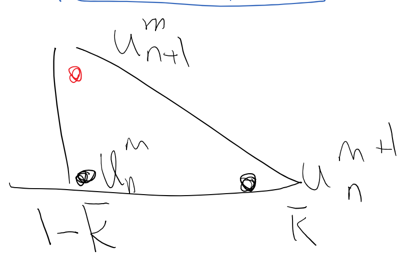
multiply both sides by k

$$u^m_{n+1} = u^m_n - \left(\frac{ka}{h}\right)\left(u^{m+1}_n - u^m_n\right)$$

$\bar{k} = \lambda a$ normalized time step

$$\bar{k} = \lambda a, \quad \lambda = \frac{k}{h}$$

$$\boxed{u^m_{n+1} = (1+\bar{k})\,u^m_n - \bar{k}\,u^{m+1}_n}$$

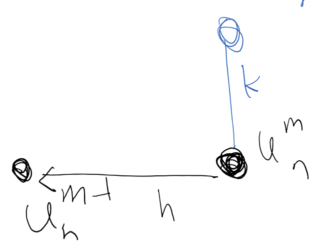


Backward in space, forward in time

$$u_t \approx \Delta_h u \overset{?}{=} \frac{u^m_{n+1} - u^m_n}{k}$$
forward

$u^m_{n+1} = ?$

$$u_x \approx \nabla_h u = \frac{u^m_n - u^{m-1}_n}{h}$$
backward



$$u_t + a u_x = 0 \rightarrow \frac{u^m_{n+1} - u^m_n}{k} + a\,\frac{u^m_n - u^{m-1}_n}{h} = 0$$

multiply by k

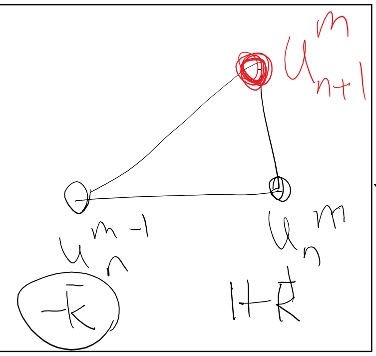
$$u^m_{n+1} = u^m_n - \left(\frac{ak}{h}\right)\left(u^m_n - u^{m-1}_n\right)$$

$\bar{k}$

m u^m_{n+1}

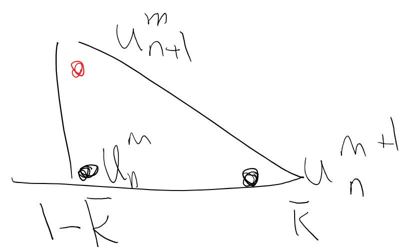
$$u_{n+1}^{m} = \bar{k}\, u_n^{m-1} + (1-\bar{k})\, u_n^{m}$$

FTBS



$$u_{n+1}^{m} = (1+\bar{k})\, u_n^{m} - \bar{k}\, u_n^{m+1}$$

FTFS



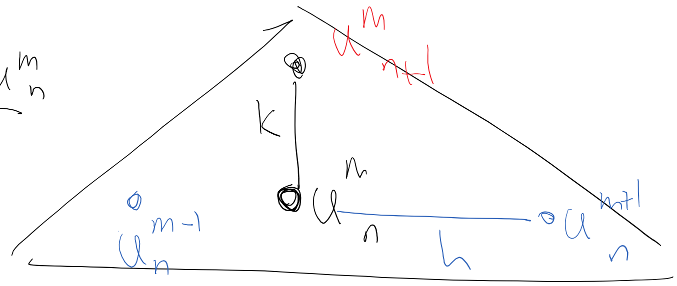
(3) Forward time central space

$$u_t = \frac{u_{n+1}^{m} - u_n^{m}}{k}$$

$$u_x \approx \delta_h u = \frac{u_n^{m+1} - u_n^{m-1}}{2h}$$

$$u_t + a u_x = 0$$

$$\frac{u_{n+1}^{m} - u_n^{m}}{k} + a\,\frac{u_n^{m+1} - u_n^{m-1}}{2h} = 0$$



(4) Lax - Fredrich's method

$$u_t = \left[u_{n+1}^{m} - \frac{1}{2}\left(u_n^{m-1} + u_n^{m+1}\right)\right] / k$$

average (rep for u_n^{m})

$$u_x = \frac{u_n^{m+1} - u_n^{m-1}}{2h}$$

$$u_t + a u_x = 0 \;\; \ldots$$

(5) Leap from : Another tweak to FTCS

$$u_t = \frac{u_{n+1}^{m} - u_{n-1}^{m}}{2k}$$

(... Time ... Central space)

$$U_t = \frac{\quad\quad}{2K}$$

(Central Time – Central space)
= Leap frog

$$U_x = \frac{u_n^{m+1} - u_n^{m-1}}{2h}$$

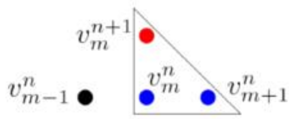
plug in $u_t + a u_x = 0$

u_n^{m-1} u_n^{m} u_n^{m+1}

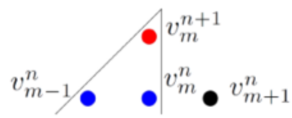
u_{n-1}^{m}

- We can discretize (26) with the following **explicit** schemes (v is used for discretized solution),

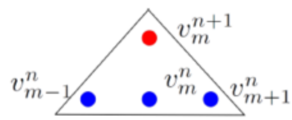
$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^n - v_m^n}{h} = 0 \quad \text{forward-time, forward-space} \tag{27a}$$



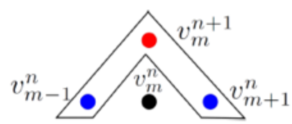
$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_m^n - v_{m-1}^n}{h} = 0 \quad \text{forward-time, backward-space} \tag{27b}$$

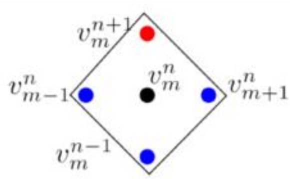


$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0 \quad \text{forward-time, central-space} \tag{27c}$$



$$\frac{v_m^{n+1} - \frac{1}{2}\left(v_{m-1}^n + v_{m+1}^n\right)}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0 \quad \text{Lax-Friedrichs} \tag{27d}$$





$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0 \quad \text{leapfrog} \tag{27e}$$

- **Multistep schemes**: If for the value of v_m^{n+1} more than time step n are required we call the scheme **multi-step**. For example **leapfrog** scheme is multi-step.
- We cannot directly apply multi-step schemes to compute first few time step values (*e.g.*, v_m^1 for leapfrog scheme.
- Solution for the first few steps:
  - First few steps are solved with with a single step approach, *e.g.*, (27d).
  - It is assumed first few step values are given (which in turn should be initialized with another method).

Sample problem for FTFS

2.1.8 Examples for explicit methods

1. Consider the advection equation (26) $u_{,t} + a(x,t)u_{,x} = 0$.
2. A Right-going wave with constant speed a is chosen. That is, $a(x,t) = a > 0$.
3. We use a step function initial condition; *cf.* (26b),

$$u_0(x) = \begin{cases} 1 & x < 0 \\ 0 & x \geq 0 \end{cases} \tag{30}$$

4. Domain of computation & grid size:
   (a) Computation domain: $x \in [-3,\ 3]$.
   (b) FD grid size $h = 1$

   So, there will be 7 grid points from point 0, 1, $\cdots$, 7, corresponding to $x = -3, -2, \cdots, 3$, respectively.

5. Boundary Condition(s) (BC):
   (a) First order PDE in space $\Rightarrow$ only one spatial boundary condition.
   (b) Right going wave, BC is specified on the upstream of waves @ $x = -3$. BC is $u(x = -3, t) = 1$.
   (c) No BCs for $x = 3$ for this 1st PDE. (Discussed further below).