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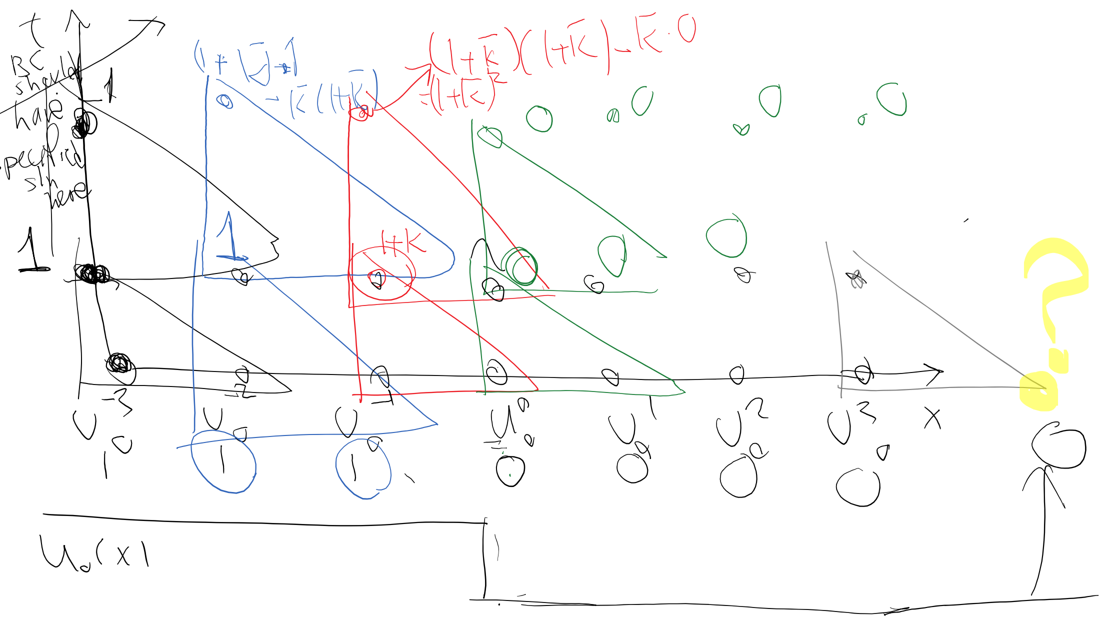
Wednesday, February 12, 2020 12:58 PM

next time step for $n=2$

FTFS

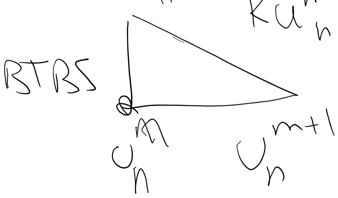


S $\quad \Delta = 1$

$n=0$

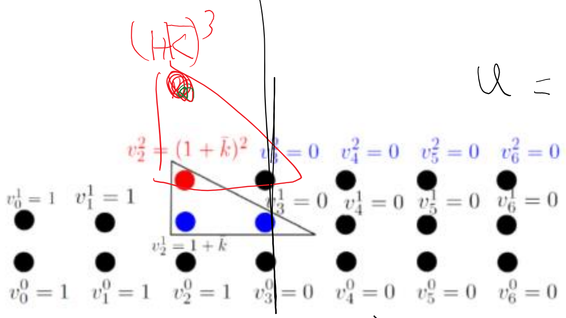
$$U^m_{n+1} = (1+\bar{k})\,U^m_n - \bar{k}\,U^{m+1}_n$$

BTBS



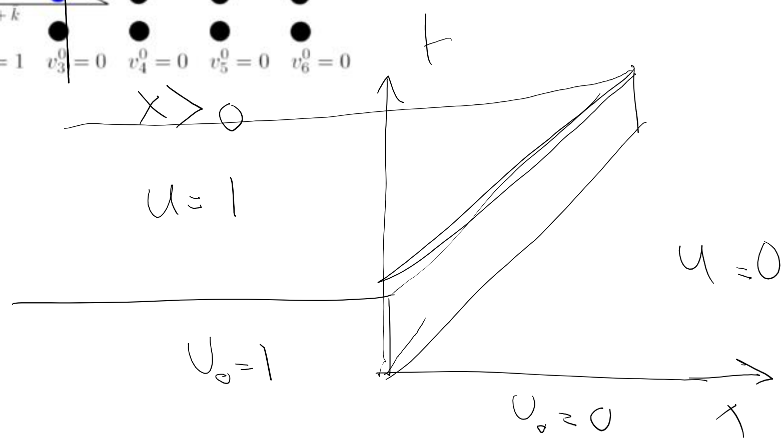
$u_t + a u_x = 0 \quad a = 1 = 0$

$u = 0$

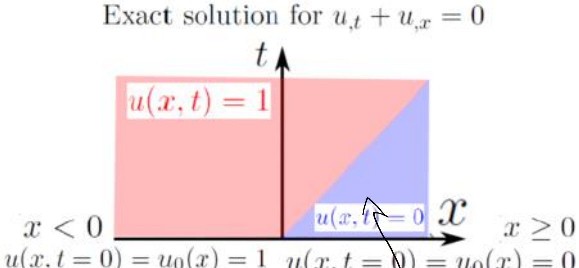


$u_t + u_{,x} = 0$

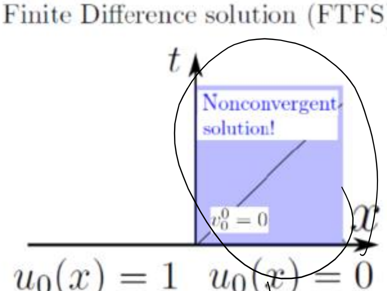
exact sln



Exact solution for $u_{,t} + u_{,x} = 0$

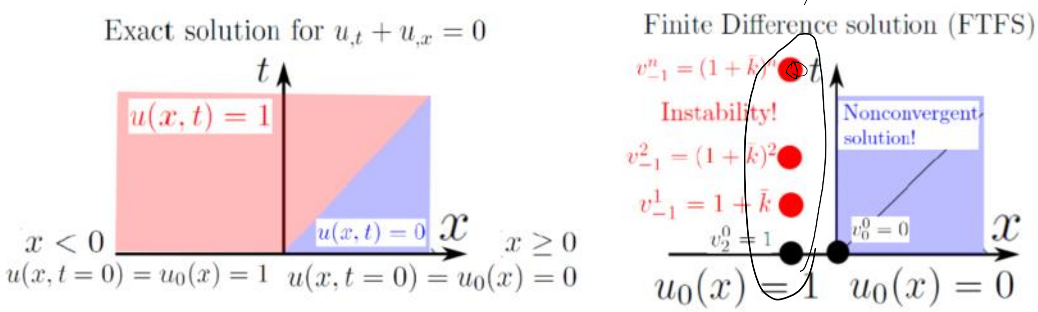


Finite Difference solution (FTFS)



don't converge to the exact sln

$(1+\bar{k})^n$



$\bar{k} = \frac{ak}{h}$, $a = 1$

For any value of k the solution blows up! This is an unconditionally unstable method!

2.1.8.3 Unconditional instability

$n = \frac{T}{k}$

- For a given time $T = nk$ we have, (given) (given)

$$v(x=-h, t=T) = v_h^k = (1+\bar{k})^n = \left(1 + \frac{ak}{h}\right)^{\frac{T}{k}} \to \infty, \text{ as } T \to \infty \tag{32}$$

  That is solution for this point approaches infinity in time. However, we know the correct solution remains bounded ($u = 1$) at this point!

- It is obvious that this problem hold for ANY time step k.
- We call this scheme UNCONDITIONALY UNSTABLE.
- On may be tempted to use very small time steps $\bar{k} \to 0$ to control the error, hoping that one may circumvent the problem. By recalling $\lim_{\bar{k}\to 0} (1+\bar{k})^{\frac{z}{\bar{k}}} = e^z$ we have,

$$v(x=-h, t=T) \approx e^{\frac{Ta}{h}}, \text{ as } k \to 0 \tag{33a}$$

That is, even in the limit of very short time steps solution blows up. In fact, the smaller the spatial grid size h, the faster the solution blows up $v(-h, T) \approx e^{\frac{Ta}{h}}$!

- This occurs for FTFS FD because the waves move to the right but FD scheme goes from the right to the left.

2.1.8.4 FD2: Conditionally stable explicit method

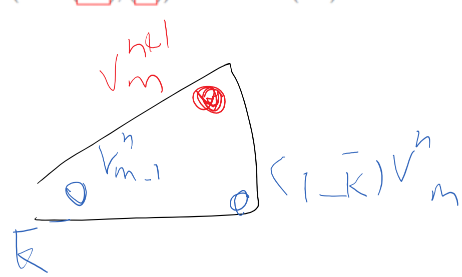
- We use the forward-time, backward-space (FTBS) from (27b) to advance the solution in time,

$$\frac{v^{n+1}_m - v^n_m}{k} + a\frac{v^n_m - v^n_{m-1}}{h} = 0 \quad \Rightarrow \tag{35a}$$

$$v^{n+1}_m = (1-\bar{k})v^n_m + \bar{k}v^n_{m-1} \quad \text{where as before (recall (31b), (28))} \tag{35b}$$

$$\bar{k} = a\lambda, \quad \lambda = \frac{k}{h}$$

Recall that $\bar{k}$ is the normalized time step.



2.1.8.5 FD2: FTBS steps of solution

- Steps of the solution:

1. As before IC sets up values for time step 0

$v_0^0 = 1$ $v_1^0 = 1$ $v_2^0 = 1$ $v_3^0 = 0$ $v_4^0 = 0$ $v_5^0 = 0$ $v_6^0 = 0$

2. Again, similar to FTFS scheme, for time step 1, we enforce the BC on the left side,

$v_0^1 = 1$ (BC, $v(x, t = 0) = 1$)

$v_0^0 = 1$ $v_1^0 = 1$ $v_2^0 = 1$ $v_3^0 = 0$ $v_4^0 = 0$ $v_5^0 = 0$ $v_6^0 = 0$

x

$1 \cdot \bar{k} + 0(1-\bar{k}) = \bar{k}$

Inflow BC

1 $\quad \bar{k} \quad$ 0 $\quad$ 0 $\quad$ 0 $\quad$ 0

outflow boundary where we should not specify BC

$v_0^1 = 1$ (BC, $v(x, t = 0) = 1$)

$v_0^0 = 1$ $v_1^0 = 1$ $v_2^0 = 1$ $v_3^0 = 0$ $v_4^0 = 0$ $v_5^0 = 0$ $v_6^0 = 0$

$\bar{k} \cdot \bar{k} + 0(1-\bar{k}) = \bar{k}^2$

1

$v_0^2 = 1$ $v_1^2 = 1$ $v_2^2 = 1$ $v_3^2 = 2\bar{k} - \bar{k}^2$ $v_4^2 = \bar{k}^2$ $v_5^2 = 0$ $v_6^2 = 0$

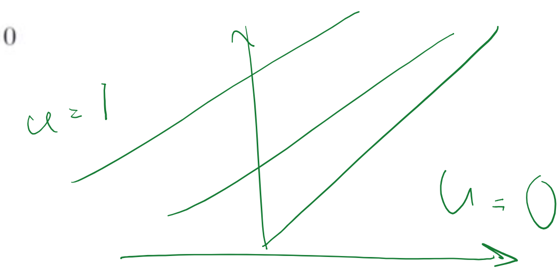
$v_0^1 = 1$ $v_1^1 = 1$ $v_2^1 = 1$ $v_3^1 = \bar{k}$ $v_4^1 = 0$ $v_5^1 = 0$ $v_6^1 = 0$

$v_0^0 = 1$ $v_1^0 = 1$ $v_2^0 = 1$ $v_3^0 = 0$ $v_4^0 = 0$ $v_5^0 = 0$ $v_6^0 = 0$

Any $\bar{k}$ That would result in exact solution in all nodes

$\bar{k} = 1$ $\quad h = k$

we capture the exact sln :)



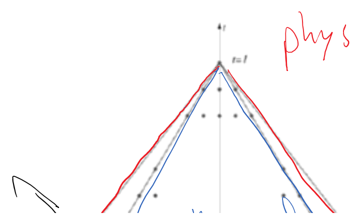
$V_{i+2}^{i} = \bar{k}^{i}$ remains bounded for $\bar{k} \le 1$

$\boxed{\text{Stability requires } \bar{k} \le 1}$ Maximum possible time step

from physical perspective

physical domain of influence

numerical method



numerical method misses the influence of these points

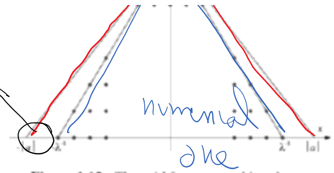
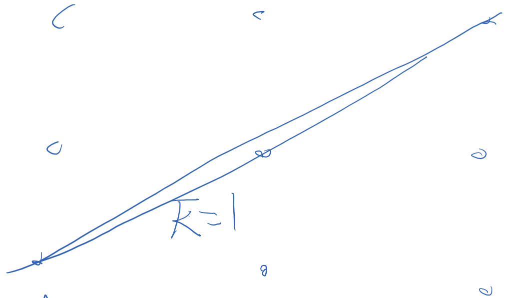
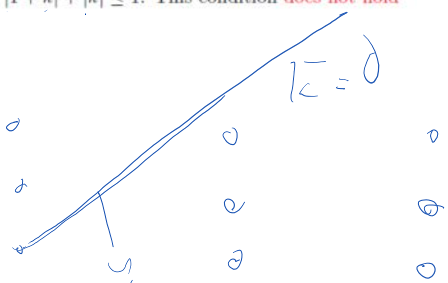


Figure 1.12. *The grid for an unstable scheme.*

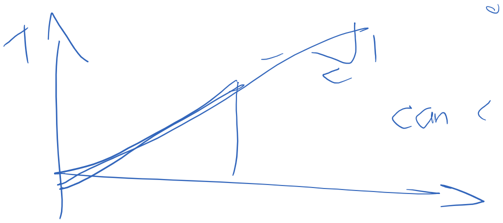
- We observe that the value v^n_{n+2} grows with the factor $\bar{k}$. That is, $v^n_{n+2} = \bar{k}^n$.
- If $|\bar{k}| > 1$ we observe that this value blows up and the method *becomes unstable.*
- We call this scheme conditionally stable: It is stable for $|\bar{k}| \le 1$.
- $\bar{k} = 1$ matches the maximum possible limit for explicit methods for hyperbolic problems. This corresponds to CFL number $= 1$ (discussed later).
- We will observe that for FD formulas of the type $v^{n+1}_m = \alpha v^n_{m-1} + \beta v^n_m$ stability is assured if $|\alpha| + |\beta| \le 1$.
- For FTBS scheme we had $v^{n+1}_m = (1-\bar{k})v^n_{m-1} + \bar{k}v^n_m \Rightarrow$ stability requires $|1-\bar{k}| + |\bar{k}| \le 1 \Rightarrow \bar{k} \le 1$.
- Similarly for FD formulas of the type $v^{n+1}_m = \alpha v^n_m + \beta v^n_{m+1}$ stability again requires $|\alpha| + |\beta| \le 1$.
- For FTFS scheme we had $v^{n+1}_m = (1+\bar{k})v^n_m + (-\bar{k})v^n_{m+1} \Rightarrow$ stability requires $|1+\bar{k}| + |\bar{k}| \le 1$. This condition does not hold for any $\bar{k}$ meaning that FTFS is unconditionally stable (for $a > 0$).

α β





cannot capture discontinuity

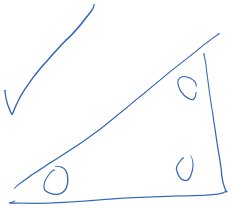


can capture discontinuity

1D

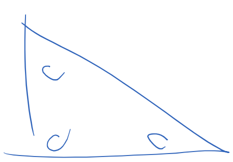
CFD = 1 is great
can capture such discontinuities.

problem



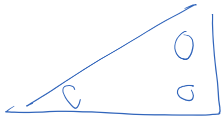
BS

$a > 0$

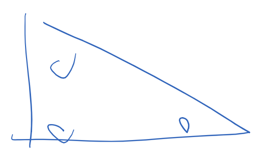


consistent with wave coming from left

this one is not



$a < 0$

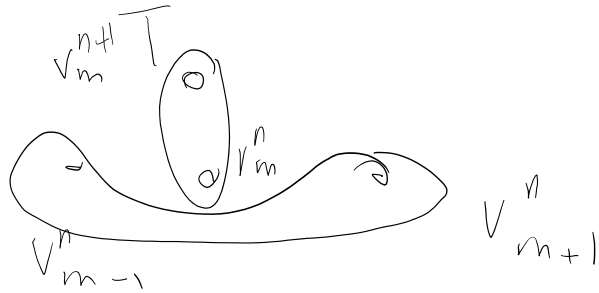


Any suggestions for a scheme working for a>0 a<0?

Central Difference

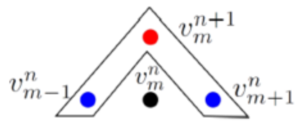
$$u_t + a u_x = 0$$

$$\frac{v_m^{n+1} - v_m^n}{k} + a\left(\frac{v_{m+1}^n - v_{m-1}^n}{2h}\right) = 0$$

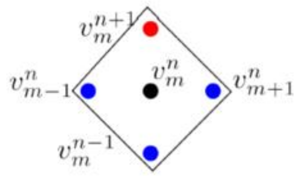


This scheme is unconditionally unstable for all a!
Von Neumann analysis (later in this course) proves this.

$$\frac{v_m^{n+1} - \frac{1}{2}\left(v_{m-1}^n + v_{m+1}^n\right)}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0 \quad \text{Lax-Friedrichs} \qquad (27d)$$



$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0 \quad \text{leapfrog} \qquad (27e)$$



more dissipative

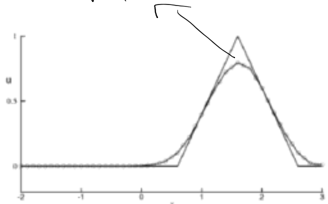


Figure 1.6. *A solution of the Lax-Friedrichs scheme,* $\lambda = 0.8$.

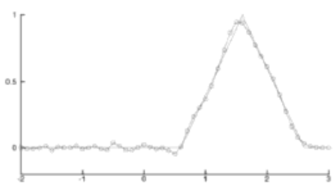


Figure 1.8. *A solution computed with leapfrog scheme,* $\lambda = 0.8$.

2.1.8.8 Development of instabilities from nonsmooth features

- If an unstable time stable is used $\bar{k} = \lambda = 1.6$ the solution will be unstable.
- If nonsmooth features exist in the solution (IC, BC, source term) instabilities often initiate from those locations (if the method is unstable):

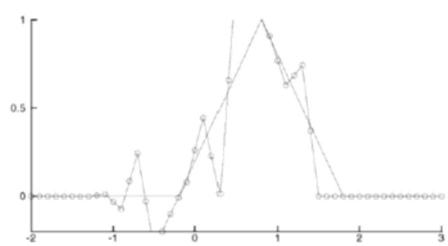


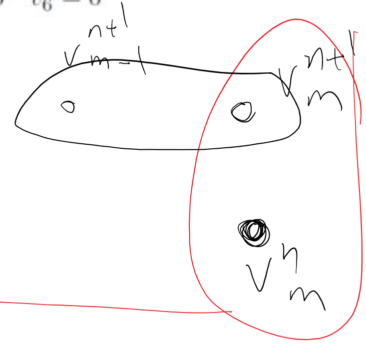
Figure 1.7. *A solution of the Lax–Friedrichs scheme,* $\lambda = 1.6$.

2.1.9 Examples for implicit methods

- We consider the same advection problem (26) $u_{,t} + a(x,t)u_{,x} = 0$, $a(x,t) = a > 0$, with IC (30) $u_0(x) = 1 - H(x)$, and BC $u(-3,t) = 1$ and the 7 point grid with $h = 1$ for the domain $x \in [-3,\ 3]$.

$v_0^0 = 1 \quad v_1^0 = 1 \quad v_2^0 = 1 \quad v_3^0 = 0 \quad v_4^0 = 0 \quad v_5^0 = 0 \quad v_6^0 = 0$

- The stencil for backward-time backward space (BTBS) scheme is (29a),

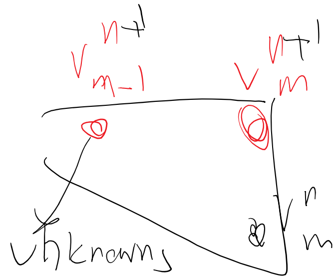


$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_m^{n+1} - v_{m-1}^{n+1}}{h} = 0 \quad \Rightarrow$$

$$v_t \Rightarrow \frac{v_m^{n+1} - v_m^n}{k} + a \underbrace{\frac{v_m^{n+1} - v_{m-1}^{n+1}}{h}}_{u_x} = 0$$

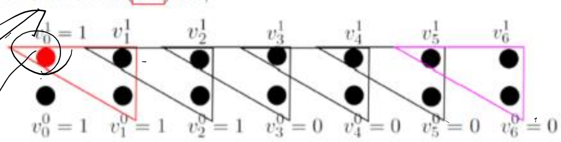
$$v_m^{n+1} - v_m^n + \bar{k}\left(v_m^{n+1} - v_{m-1}^{n+1}\right) = 0$$

$$\boxed{(1+\bar{k})v_m^{n+1} - \bar{k}v_{m-1}^{n+1} = v_m^n}$$



- Stages of solutions:
  1. IC is set as before for FTFS & FTBS.
  2. Boundary condition on the left boundary is set as $v_0^1 = 1$.
  3. The equations for points 1 to 6 based on (36a) are,

Inflow Boundary Condition $v = 1$



$=1$ from BC

$$\left.\begin{aligned}
(1+\bar{k})v_1^1 - \bar{k}v_0^1 &= v_1^0 \\
(1+\bar{k})v_2^1 - \bar{k}v_1^1 &= v_2^0 \\
(1+\bar{k})v_3^1 - \bar{k}v_2^1 &= v_3^0 \\
(1+\bar{k})v_4^1 - \bar{k}v_3^1 &= v_4^0 \\
(1+\bar{k})v_5^1 - \bar{k}v_4^1 &= v_5^0 \\
(1+\bar{k})v_6^1 - \bar{k}v_5^1 &= v_6^0
\end{aligned}\right\} \Rightarrow \mathbf{A}\mathbf{v}^1 := \mathbf{v}^0 + \mathbf{b}^1, \quad \text{where}$$

6 unknowns   6 eqns

$$\mathbf{v}^n := \begin{bmatrix} v_1^n \\ v_2^n \\ v_3^n \\ v_4^n \\ v_5^n \\ v_6^n \end{bmatrix}, \quad \mathbf{b}^n = \bar{k}\begin{bmatrix} u(0,t_n) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} (1+\bar{k}) & 0 & 0 & 0 & 0 & 0 \\ \bar{k} & (1+\bar{k}) & 0 & 0 & 0 & 0 \\ 0 & \bar{k} & (1+\bar{k}) & 0 & 0 & 0 \\ 0 & 0 & \bar{k} & (1+\bar{k}) & 0 & 0 \\ 0 & 0 & 0 & \bar{k} & (1+\bar{k}) & 0 \\ 0 & 0 & 0 & 0 & \bar{k} & (1+\bar{k}) \end{bmatrix}$$

$$Av^1 = v^0 + b^1 \longrightarrow v^1 = A^{-1}v^0 + A^{-1}b^1$$

same $\bar{k}$ $\quad Av^2 = v^1 + b^2 \longrightarrow v^2 = A^{-1}v^1 + A^{-1}b^2 = A^{-1}(A^{-1}v^0 + A^{-1}b^1) + A^{-1}b_2$

$$= A^{-2}v^0 + A^{-2}b^1 + A^{-1}b_2$$

$$\mathbf{v}^n = \mathbf{A}^{-n}\mathbf{v}^0 + \mathbf{A}^{-1}\mathbf{b}^n + \cdots + \mathbf{A}^{-n}\mathbf{b}^1 \quad \Rightarrow \tag{39a}$$

$$\mathbf{v}^n = \mathbf{A}^{-n}\mathbf{v}^0 + \left\{\mathbf{A}^{-1} + \cdots \mathbf{A}^{-n}\right\}\mathbf{b}^1 = \mathbf{A}^{-n}\mathbf{v}^0 + \mathbf{A}^{-1}\left(\mathbf{I} - \mathbf{A}^{-n}\right)\left(\mathbf{I} - \mathbf{A}^{-1}\right)^{-1} b^1 \text{ for constant BC } u(-3,t) = 1 \text{ at } x = -3\mathbf{b}^1 \tag{39b}$$

$$v^n = A^{-n} v^0 \qquad \text{for zero BC}$$

v^n: sln at step n

v^0: IC

- When does $\mathbf{v}^n$ blow-up, *i.e.*, tend to infinity?
- Assume for the moment that $\mathbf{A}^{-1} = \mathbf{D}$ is diagonal (most general case by using Jordan decomposition is discussed in §5.3).

$$\mathbf{v}^n = \mathbf{D}^n\mathbf{v}^0 \quad \text{where} \quad \mathbf{D} := \begin{bmatrix} d_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_6 \end{bmatrix} \quad \text{where } d_1, \cdots, d_6 \text{ are diagonal values of } \mathbf{D} \tag{41}$$

2.1.9.3 Spectral radius of a matrix

- Based on these equations we get,

$$\mathbf{v}^n = \mathbf{D}^n\mathbf{v}^0 = \begin{bmatrix} d_1^n & 0 & 0 & 0 & 0 & 0 \\ 0 & d_2^n & 0 & 0 & 0 & 0 \\ 0 & 0 & d_3^n & 0 & 0 & 0 \\ 0 & 0 & 0 & d_4^n & 0 & 0 \\ 0 & 0 & 0 & 0 & d_5^n & 0 \\ 0 & 0 & 0 & 0 & 0 & d_6^n \end{bmatrix}\mathbf{v}^0 \tag{42}$$

- Let $n \to \infty$ (*i.e.*, $t_n = nk \to \infty$). When the solution goes to infinity at t_n (*i.e.*, components of $\mathbf{v}^n$ go to infinity)?
- Answer: $\boxed{\text{If } |d_i| > 1 \text{ for ANY } i \text{ the solution blows up!}}$

In general spectral radius of A^{-1} $(\rho(A^{-1}) < 1)$

$= 1$? we'll discuss it later

why

$$v^{n+1} = B v^n \quad \text{zero BC} \longrightarrow v^n = B^n v^0$$

$B = A^{-1}$

$B w_i = \lambda_i w_i$

eigenvalues (λ_i, w_i)

$$B\left[w_1 | \dots | w_n\right] = \left[w_1 | \dots | w_n\right]\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$\underbrace{\boxed{BW = W\Lambda}}_{\text{if } B \text{ is diagonalizable}}$$

$$BW = W\Lambda \longrightarrow \boxed{B = W\Lambda W^{-1}}$$

$$B^2 = (W\Lambda \underbrace{W^{-1})(W}_{I}\Lambda W^{-1}) = W\Lambda^2 W^{-1}$$

$$\boxed{B^m = W\Lambda^m W^{-1}}$$

λ^m

$$\Lambda^m = \begin{bmatrix} \lambda_1^m & & \\ & \ddots & \\ & & \lambda_n^m \end{bmatrix}$$

blows up if for any i $|\lambda_i| > 1$

If all $|\lambda_i| \leq 1$ the method is stable!

$$\boxed{\text{if } B \text{ is diagonalizable for stability } \rho(B) = \max_{i=1,\dots,n} |\lambda_i(B)| \leq 1 \qquad v^{n+1} = B v^n}$$

$\mathbf{v}^n = \mathbf{A}^{-n}\mathbf{v}^0$ for zero BC on the left (40)

$B = A^{-1}$ eigenvalues of B are $< 1 \implies$ diagonalizable

eigenvalues of A ≥ 1
A diagonalizable

$$\mathbf{A} = \begin{bmatrix} (1+\bar{k}) & 0 & 0 & 0 & 0 & 0 \\ \bar{k} & (1+\bar{k}) & 0 & 0 & 0 & 0 \\ 0 & \bar{k} & (1+\bar{k}) & 0 & 0 & 0 \\ 0 & 0 & \bar{k} & (1+\bar{k}) & 0 & 0 \\ 0 & 0 & 0 & \bar{k} & (1+\bar{k}) & 0 \\ 0 & 0 & 0 & 0 & \bar{k} & (1+\bar{k}) \end{bmatrix} \quad (37b)$$

eigenvalues of A
Lower triangular matrix
diagonals are eigenvalues

$\{(1+\bar{k}), 1+\bar{k}, \dots (1+\bar{k})\}$

$(1+\bar{k}) \geq 1$ $\qquad \bar{k} \geq 0$ Always true

$$\boxed{\text{whether diagonalizable or not } \quad v^{n+1} = B v^n \quad \boxed{\rho(B) < 1} \quad \text{is always stable}}$$

$\bar{k} > 0 \qquad \rho(B) = \rho(A^{-1}) < 1$

This is unconditionally stable.

In this case, since A is lower triangular, the solution for next v is trivial

$$\mathbf{v}^n := \begin{bmatrix} v_1^n \\ v_2^n \\ v_3^n \\ v_4^n \\ v_5^n \\ v_6^n \end{bmatrix}, \quad \mathbf{b}^n = \bar{k}\begin{bmatrix} u(0,t_n) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} (1+\bar{k}) & 0 & 0 & 0 & 0 & 0 \\ \bar{k} & (1+\bar{k}) & 0 & 0 & 0 & 0 \\ 0 & \bar{k} & (1+\bar{k}) & 0 & 0 & 0 \\ 0 & 0 & \bar{k} & (1+\bar{k}) & 0 & 0 \\ 0 & 0 & 0 & \bar{k} & (1+\bar{k}) & 0 \\ 0 & 0 & 0 & 0 & \bar{k} & (1+\bar{k}) \end{bmatrix} \tag{37b}$$

$$v_0^1 = 1 \qquad \text{BC}$$

$$(1+\bar{k})v_1^1 - \bar{k}v_0^1 = v_1^0 \quad \Rightarrow \quad v_1^1 = \frac{\bar{k}}{1+\bar{k}}v_0^1 + \frac{1}{1+\bar{k}}v_1^0 = \frac{\bar{k}}{1+\bar{k}}1 + \frac{1}{1+\bar{k}}1 = 1$$

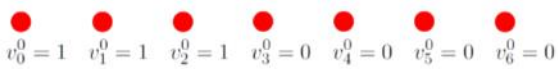
$$(1+\bar{k})v_2^1 - \bar{k}v_1^1 = v_2^0 \quad \Rightarrow \quad v_2^1 = \frac{\bar{k}}{1+\bar{k}}v_1^1 + \frac{1}{1+\bar{k}}v_2^0 = \frac{\bar{k}}{1+\bar{k}}1 + \frac{1}{1+\bar{k}}1 = 1$$

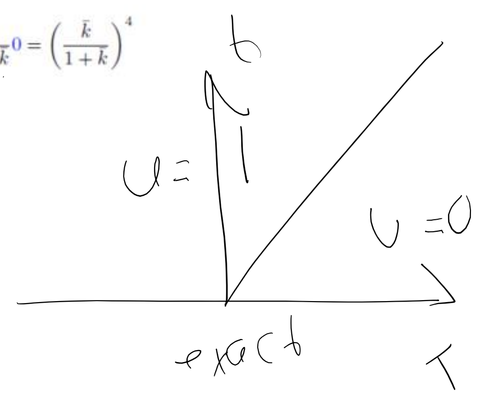
$$(1+\bar{k})v_3^1 - \bar{k}v_2^1 = v_3^0 \quad \Rightarrow \quad v_3^1 = \frac{\bar{k}}{1+\bar{k}}v_2^1 + \frac{1}{1+\bar{k}}v_3^0 = \frac{\bar{k}}{1+\bar{k}}1 + \frac{1}{1+\bar{k}}0 = \frac{\bar{k}}{1+\bar{k}}$$

$$(1+\bar{k})v_4^1 - \bar{k}v_3^1 = v_4^0 \quad \Rightarrow \quad v_4^1 = \frac{\bar{k}}{1+\bar{k}}v_3^1 + \frac{1}{1+\bar{k}}v_4^0 = \frac{\bar{k}}{1+\bar{k}}\frac{\bar{k}}{1+\bar{k}} + \frac{1}{1+\bar{k}}0 = \left(\frac{\bar{k}}{1+\bar{k}}\right)^2$$

$$(1+\bar{k})v_5^1 - \bar{k}v_4^1 = v_5^0 \quad \Rightarrow \quad v_5^1 = \frac{\bar{k}}{1+\bar{k}}v_4^1 + \frac{1}{1+\bar{k}}v_5^0 = \frac{\bar{k}}{1+\bar{k}}\left(\frac{\bar{k}}{1+\bar{k}}\right)^2 + \frac{1}{1+\bar{k}}0 = \left(\frac{\bar{k}}{1+\bar{k}}\right)^3$$

$$(1+\bar{k})v_6^1 - \bar{k}v_5^1 = v_6^0 \quad \Rightarrow \quad v_6^1 = \frac{\bar{k}}{1+\bar{k}}v_5^1 + \frac{1}{1+\bar{k}}v_6^0 = \frac{\bar{k}}{1+\bar{k}}\left(\frac{\bar{k}}{1+\bar{k}}\right)^3 + \frac{1}{1+\bar{k}}0 = \left(\frac{\bar{k}}{1+\bar{k}}\right)^4$$





In 2D and 3D FD of PDEs in general, we don't get this nice 1 node solve at a time

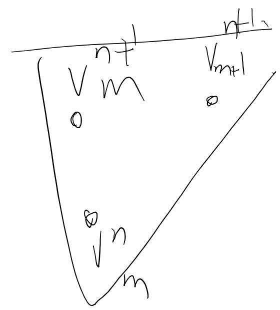
Global coupling of implicit methods make them more difficult to parallelize.

2.1.9.6 Backward-time forward-space: A conditional stable implicit method

- For Backward-time forward-space (BTFS) scheme we have, *cf.* (29b)

$$\underbrace{\frac{v_m^{n+1} - v_m^n}{k}}_{u} + a\underbrace{\frac{v_{m+1}^{n+1} - v_m^{n+1}}{h}}_{u_{,x}} = 0 \quad \Rightarrow$$

BTFS

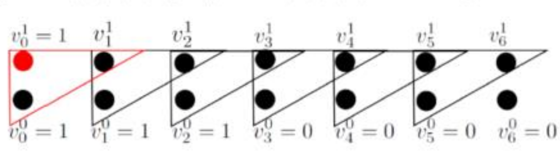


$$v_m^{n+1} - v_m^n + \bar{k}\, v_{m+1}^{n+1} - \bar{k}\, v_m^{n+1} = 0$$

$$\boxed{(1-\bar{k})v_m^{n+1} + \bar{k}\, v_{m+1}^{n+1} = v_m^n}$$

unknowns → known

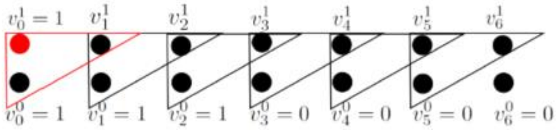
- By writing the equations for points 0, 1, 2, 3, 4, 5 (compare to 1, 2, 3, 4, 5, 6 for BTBS),



$$
\mathbf{A} = \begin{bmatrix}
\bar{k} & 0 & 0 & 0 & 0 & 0 \\
(1-\bar{k}) & \bar{k} & 0 & 0 & 0 & 0 \\
0 & (1-\bar{k}) & \bar{k} & 0 & 0 & 0 \\
0 & 0 & (1-\bar{k}) & \bar{k} & 0 & 0 \\
0 & 0 & 0 & (1-\bar{k}) & \bar{k} & 0 \\
0 & 0 & 0 & 0 & (1-\bar{k}) & \bar{k}
\end{bmatrix}
$$

eigenvalues are $\bar{k}$

eigenvalues of $B = A^{-1} = \frac{1}{\bar{k}} \rightarrow \boxed{\rho(A^{-1}) = \frac{1}{\bar{k}}}$



$\rho(A^{-1}) = \frac{1}{\bar{k}} < 1 \rightarrow \boxed{\bar{k} > 1}$ for stability

for this implicit scheme time step should be large enough

$(\bar{k} > 1)$

Backward-time forward-space: **A conditional stable implicit method** 93

- Thus, all eigenvalues of $\mathbf{A}$ are $\bar{k}$ meaning that all eigenvalues of $\mathbf{A}^{-1}$ are $1/\bar{k}$ and

$$\rho(\mathbf{A}^{-1}) = \frac{1}{\bar{k}} \quad \Rightarrow$$

$$\text{BTFS scheme is stable if } \rho(\mathbf{A}^{-1}) = \frac{1}{\bar{k}} \leq \quad \Leftrightarrow \quad \bar{k} \geq 1 \tag{49}$$

- That is,

The IMPLICIT method of BTFS is CONDITIONALLY STABLE and large enough steps ($\bar{k} \geq 1$) must be taken for stability.

- This is a good example of an implicit method that IS NOT unconditionally stable. That is, it does not have the main advantage of most implicit methods (unconditional stability) yet is more expensive than explicit ones (in 2D and 3D) for this problem.
- The cause of this problem is again the wave (right-going with $a > 0$) not being consistent with FD grid. Although we cannot always make such arguments and stability of a method should be directly evaluated.
- Likewise BTBS method will only be conditionally stable for left-going wave.

Higher order PDEs: parabolic PDEs (Diffusion equation) 95

2.1.10 Higher order PDEs: 2$^{\text{nd}}$order parabolic & hyperbolic PDEs

- Consider the solution of the parabolic PDE,

$$u_{,t} - Du_{,xx} = r$$

diffusion coefficient

$[D]$ physical viscosity

Unit $= \frac{L^2}{T}$

FTime

$v_{,t} = \frac{v_m^{n+1} - v_m^n}{k}$

v_m^{n+1}

k

v_m^n

v_{m-1}^n v_{m+1}^n

CS

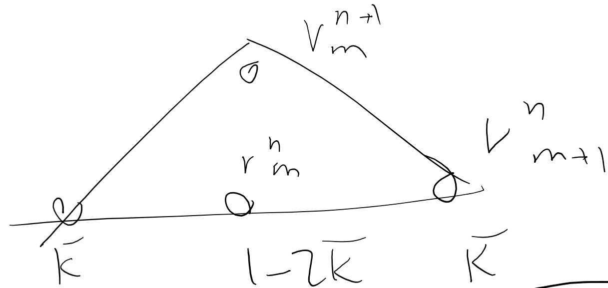
$u_{,xx} = \frac{v_{m+1}^n + v_{m-1}^n - 2v_m^n}{h^2}$

$v^{n+1} \quad v^n \quad (k)[v^n \quad + v^n_{m-1} - 2v^n_m]$

$$v_m^{n+1} - v_m^n + \underbrace{\left(\frac{kD}{h^2}\right)}_{\bar{k}}\left[v_{m+1}^n + v_{m-1}^n - 2v_m^n\right] = 0$$

$$\boxed{v_m^{n+1} = (1-2\bar{k})\, v_m^n + \bar{k}\, v_{m-1}^n + \bar{k}\, v_{m+1}^n}$$

$\bar{k} \le \underbrace{\bar{k}^*}_{\text{number}}$



$$\frac{kD}{h^2} \le \bar{k}^* \longrightarrow \boxed{k \le \frac{\bar{k}^* h^2}{D}}$$

stable time step $\propto h^2$

not h for parabolic PDEs