DC20200219

Wednesday, February 19, 2020 1:11 PM

From last time:

2.2.2.3 2. Lax-Friedrichs fluxes: Why does it work?

- We know from the discussions FD that Lax-Friedrichs scheme is conditionally stable. That is, the added jump terms in (76) stabilize the unconditionally unstable FTCS scheme.
- Why adding the jump conditions make the scheme (conditionally) stable?
- To understand the underlying mechanism, we rearrange (77) in the following form,

$$Q_m^{n+1} = Q_m^n - \frac{k}{2h}\left[f(Q_{m+1}^n) - f(Q_{m-1}^n)\right] + \frac{1}{2}\left[Q_{m+1}^n + Q_{m-1}^n - 2Q_m^n\right] \Rightarrow$$
$$\frac{Q_m^{n+1} - Q_m^n}{k} + \frac{f(Q_{m+1}^n) - f(Q_{m-1}^n)}{2h} - \frac{h^2}{2k}\frac{Q_{m+1}^n + Q_{m-1}^n - 2Q_m^n}{h^2} = 0 \tag{80}$$

- In the present form, it is obvious that the FV (FD) equation approximates the following equation,

$$q_{,t} + \{f(q)\}_{,x} - \frac{h^2}{2k} q_{,xx} = 0 \tag{81}$$

We were discussion whether the added diffusion will cause consistency problems.

For a given h and k we are NOT solving the original PDE $q_{,t} + f(q)_{,x} = 0$

rather, we solve 81

inconsistency error.

Discretization error + Consistency error

PDE to FD stencil + PDE error

$q_{,t} + f(q)_{,x} - \frac{h^2}{2k} q_{,xx} = 0$

these are the two sources of error here

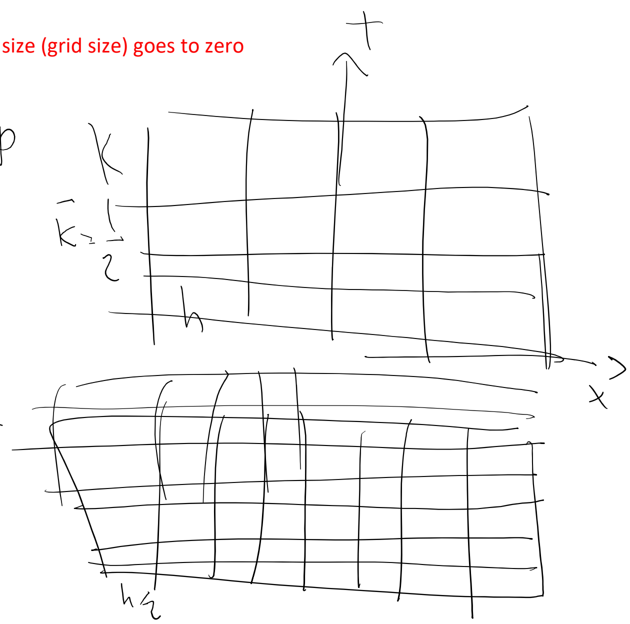
Consistency of method says we are solving the correct PDE in the limit when the element size (grid size) goes to zero

$\bar{k} = \frac{a\,k}{h}$ normalized time step

as long as we keep $\bar{k}$ fixed when we do refinement

$$\frac{h^2}{k} = \frac{h^2}{h\bar{k}/a} = h\frac{a}{\bar{k}} \longrightarrow 0$$

as $h \to 0$



This means in the limit of h -> 0 as long as we keep kBar fixed the method is consistent.

- Given that the term $\{f(q)\}_{,x}$ is first term derivative in x the original equation is hyperbolic, and
- We add the parabolic operator $-\frac{h^2}{2k}q_{,xx}$ to the original hyperbolic equation. Note that $q_{,t}-\frac{h^2}{2k}q_{,xx}$ is a parabolic equation.
- The addition of parabolic (diffusion) operator tends to damp instabilities that arise from FTCS scheme!
- This added diffusion operator results in the subtle change of $\frac{Q_m^{n+1}-Q_m^n}{k}$ in (74) (from the use of average fluxes) to $\frac{Q_m^{n+1}-\frac{1}{2}(Q_{m-1}^n+Q_{m-1}^n)}{k}$ in (79) from Lax-Friedrichs fluxes. This subtle change of the temporal difference, in fact as we saw correspond to an added diffusion operator.
- Question on consistency: We are actually solving the equation (81)

$$q_{,t}+\{f(q)\}_{,x}-\frac{h^2}{2k}q_{,xx}=0$$

by using Lax-Friedrichs fluxes. This is clearly difference from the underlying PDE (62)

$$q_{,t}+\{f(q)\}_{,x}=0$$

Here are some questions and observations,

- The solved system (with added diffusion term) is different from the actual PDE. Is the system consistent, meaning that the solution of the new system converge to the correct solution?
- We recall that stable time stable for the hyperbolic equation scales as $k_{\max}\propto h$ (*e.g.*, for $f(q)=aq$ for advection equation we have $\bar{k}=\frac{ak}{h}\le 1$.
- Thus in the term $\frac{h^2}{2k}$ in (81) if the ratio $\frac{k}{h}$ is kept fixed for stability concerns and we let $h\to 0$ *i.e.*, by mesh refinement the numerical advection coefficient $\frac{h^2}{2k}$ approaches zero as finer grids are used.
- So, Lax-Friedrichs scheme is consistent with the underlying hyperbolic equation.
- However, the added diffusion implies that the system can by overly diffusive!
- The over damping of the Lax-Friedrichs can be seen in the comparisons shown with leapfrog scheme from FD section.

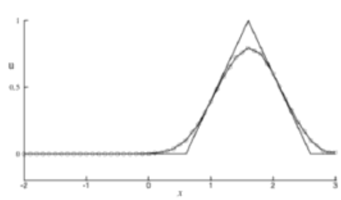


Figure 1.6. *A solution of the Lax-Friedrichs scheme,* $\lambda=0.8$.

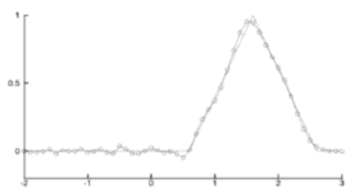


Figure 1.8. *A solution computed with leapfrog scheme,* $\lambda=0.8$.

Upstream fluxes (Riemann solutions)
This is the idea that started in FV community

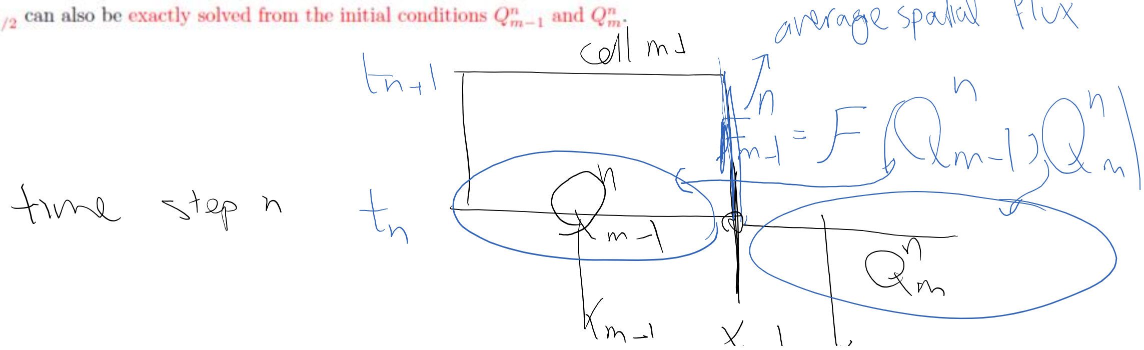
- As a reminder from (65c) for a cell on its two cell boundaries we need to define flux average values (only shown for $x_{m-1/2}$),

$$F_{m-1/2}^n=\frac{1}{k}\int_{t_n}^{t_{n+1}}f(q(x_{m-1/2},t))\,\mathrm{d}t$$

- In (66) we mentioned that for hyperbolic equations we can express this values as a function of the initial values (initial conditions) from time step n on the two sides of $x_{m-1/2}$,

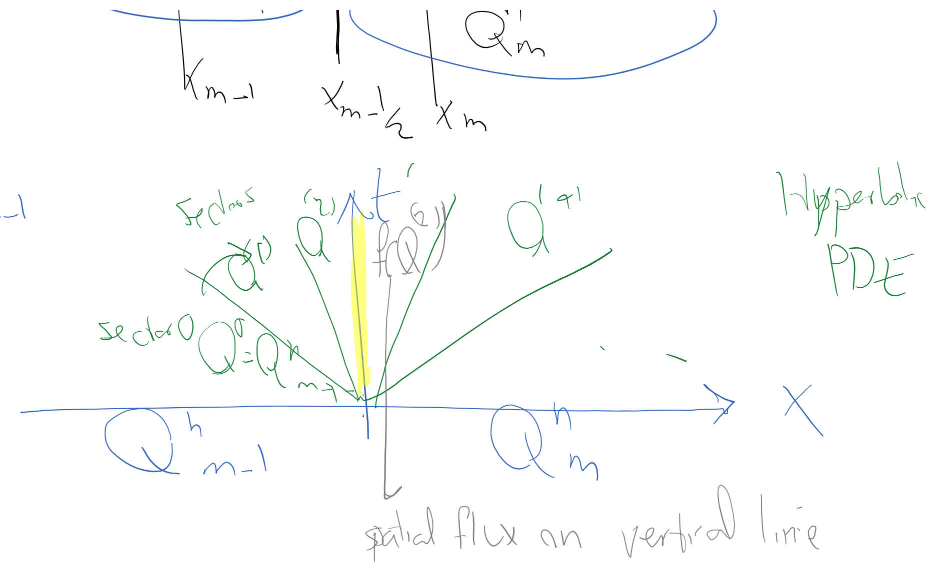
$$F_{m-1/2}^n=\mathcal{F}(Q_{m-1}^n,Q_m^n),\qquad F_{m+1/2}^n=\mathcal{F}(Q_m^n,Q_{m+1}^n)$$

- Different choices for numerical flux function F^n can be used as we observed average and Lax-Friedrichs fluxes options before.
- $F_{m-1/2}^n$ can also be exactly solved from the initial conditions Q_{m-1}^n and Q_m^n.



Riemann solution can be used to form F^n_{m-1}

How



IC

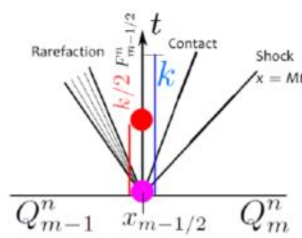
$$F^n_{m-1/2} = \frac{1}{\Delta t}\int_0^{\Delta t} f(Q^{(2)}(t))\,dt$$

2.2.2.5 Riemann problem set-up

This forms a Riemann problem set-up where,

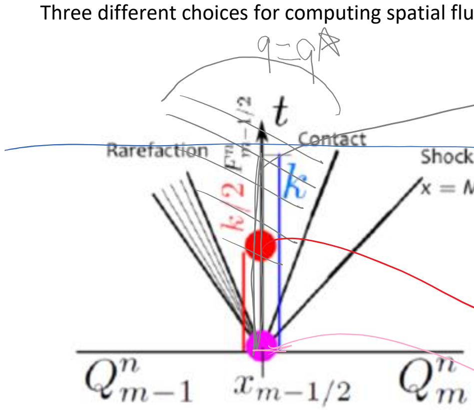
- We solve the solution for all x, t with two distinct ICs Q^n_{m-1} and Q^n_m
- Having the solution along the vertical line $x_{m-1/2}$ we can calculate $q(x_{m-1/2}, t)$ for $t_n \le t \le t_{n+1}$
- Having q along this line, we can calculate $f(q(x_{m-1/2}, t))$ for $t_n \le t \le t_{n+1}$.
- We can then calculate $F^n_{m-1/2}$ from $f(q(x_{m-1/2}, t))$,

$$F^n_{m-1/2} = \frac{1}{k}\int_{t_n}^{t_{n+1}} f(q(x_{m-1/2}, t))\,\mathrm{d}t$$



Given that we can conceptually calculate $f(q(x_{m-1/2}, t))$ for $t_n \le t \le t_{n+1}$, there are (at least) three different levels to compute/approximate $F^n_{m-1/2}$

Three different choices for computing spatial flux once we have the Riemann solution:



$f(q^*)$ option 1

$F(Q^n_{m-1}, Q^n_m) = \frac{1}{\Delta t}\int_0^{\Delta t} f(q^*)\,dt$

$F(Q^n_{m-1}, Q^n_m) \approx f(q^*) (@\, t = \frac{\Delta t}{2})$

error of this Δt^2

$F(Q^n_{m-1}, Q^n_m) = f(q^*) @ (t = 0)$

error is Δt

In the absence of source terms in PDE, the solutions are constant on characteristics so the solutions will be constant in the sectors shown. -> all the above choices will coincide.

For nonzero source, we need integrate the effect source term along the characteristics for options 1 (blue) and 2 (red)

1. Exact integration of f(q)

$$F^n_{m-1/2} = \frac{1}{k}\int_{t_n}^{t_{n+1}} f(q(x_{m-1/2},t))\,\mathrm{d}t$$

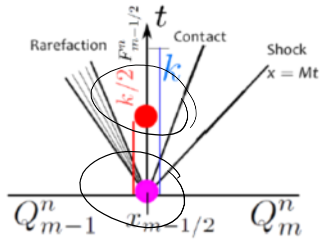
2. Using the mid-point value for f(q)

$$F^n_{m-1/2} = \frac{1}{k}\int_{t_n}^{t_{n+1}} f(q(x_{m-1/2},t))\,\mathrm{d}t \approx f(q(x_{m-1/2},\frac{t_n+t_{n+1}}{2}))$$

3. Using the start value f(q)

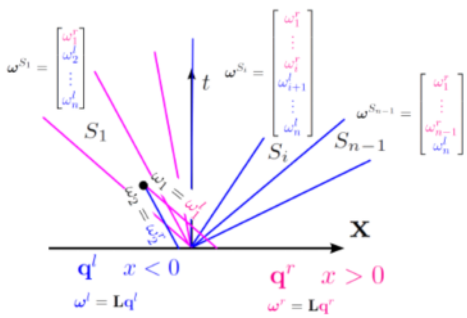
$$F^n_{m-1/2} = \frac{1}{k}\int_{t_n}^{t_{n+1}} f(q(x_{m-1/2},t))\,\mathrm{d}t \approx f(q(x_{m-1/2},t_n))$$

the advantage of the this approach is that we do not need to integrate source terms along the characteristics. For hyperbolic systems the difference between this and average option arises when source term is nonzero. This option, clearly is the least accurate one and special care must be taken in its use.



2.2.2.6 Solution of Riemann solution (Upstream solution)

- As discussed before (and shown in the previous figure) Riemann solution may involve regions with shocks and rarefaction waves for quasi-linear systems of hyperbolic PDEs.
- For semi-linear PDEs, we discussed two different approaches to derive Riemann solutions for different sectors in space time.
- The Riemann solution in each sectors are obtained by using the upstream fluxes from the characteristics coming to that sector



2.2.2.7 3. Upstream fluxes (Riemann solutions for first order hyperbolic PDEs)

- Again, we consider the hyperbolic system (62)

$$q_{,t} + \{f(q)\}_{,x} = 0 \qquad \text{PDE}$$
$$q(x,t=0) = q_0(x) \qquad \text{IC}$$

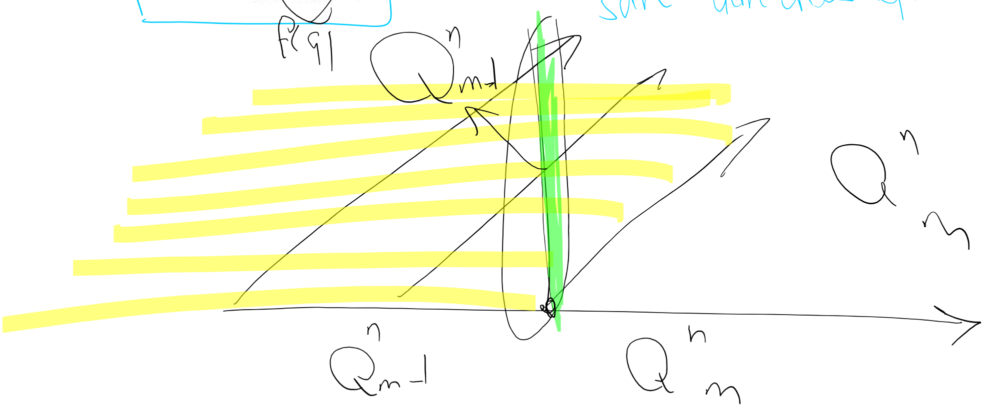
- The Riemann solution for a general nonlinear flux $f(q)$ may not be trivial.
- For nonlinear flux, we either directly solve the nonlinear Riemann solution, or use approximate Riemann solutions similar to (76).
- We consider that we are solving the linear advection equation where $f(q) = aq$ (71),

$$\boxed{q_{,t} + aq_{,x} = 0}$$

f(q)

solve advection eqn with FV

$a > 0$



Q_{m-1} $\quad$ Q_m

$$F(Q^n_{m-1}, Q^n_m) = f(Q^n_{m-1}) = aQ^n_{m-1}$$

Similarly $F(Q^n_m, Q^n_{m+1}) = aQ^n_m$

$$Q^{n+1}_m = Q^n_m - \frac{k}{h}\left[F(Q^n_m, Q^n_{m+1}) - F(Q^n_{m-1}, Q^n_m)\right] \quad (\star)$$

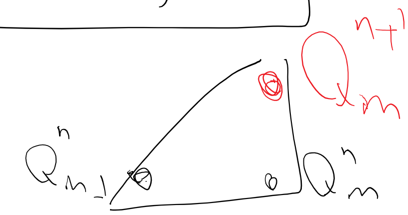
$$= Q^n_m - \frac{k}{h}\left(aQ^n_m - aQ^n_{m-1}\right)$$

$$\rightarrow \boxed{\frac{Q^{n+1}_m - Q^n_m}{k} + a\left(\frac{Q^n_m - Q^n_{m-1}}{h}\right) = 0}$$

FT $\dot{Q}$ $\qquad$ BS $Q_{,x}$

$$\dot{Q} + aQ_{,x} = 0$$

BSFT $\quad a > 0$



We saw that this was conditionally stable for a > 0

What if a < 0

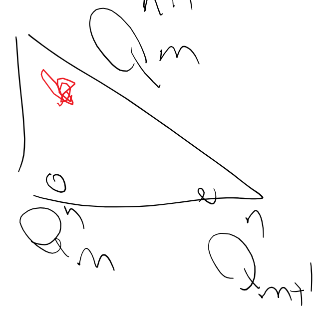
$$F(Q^n_{m-1}, Q^n_m) = aQ^n_m$$

Similarly $F(Q^n_m, Q^n_{m+1}) = aQ^n_{m+1}$

plug in $(\star) \longrightarrow Q^{n+1}_m = Q^n_m - \frac{k}{h}\left(aQ^n_{m+1} - aQ^n_m\right)$

$$\frac{Q^{n+1}_m - Q^n_m}{k} + a\left(\frac{Q^n_{m+1} - Q^n_m}{h}\right) = 0$$

$\dot{Q}$ FT $\qquad Q_{,x}$ FS



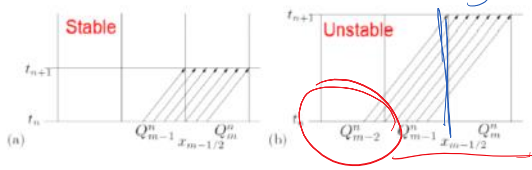
This is conditionally stable for $a < 0$

So, by choosing the upstream value of fluxes in Riemann solution, FV always takes the correct solution for spatial flux at the interface. This allows us for example to solve an advection equation where the sign of a(x) changes (No change with FS, BS, CS - FT FD schemes!)

Explicit methods and CFL condition

2.2.4 Stability limit of explicit finite volume methods (hyperbolic PDEs)

$\mathcal{F}(Q^n_{m-1}, Q^n_m)$



if $k > ah$ physically Q^n_{m-2} also effects $\mathcal{F}$ which we are ignoring

- For FD scheme we showed that for a first order hyperbolic PDE the stable time step must have been $\bar{k} \leq 1$, *i.e.*, the CFL condition was satisfied. FV update equations looked similar to FD onces, *e.g.*, FTBS, FTCS, FTFS, Lax-Friedrichs.
- This ensures that the numerical speed of information propagation is greater than or equal to the physical wave speed.
- Figure above shows that if CFL condition is violated on vertical boundary of cell $x_{m-1/2}$ information no longer is only influenced by the two side averages Q^n_{m-1} and Q^n_m but also influenced by Q^n_{m-2}.
- This manifests itself in instability as the influence of Q^n_{m-2} is not included in the numerical flux $F^n_{m-1/2} = \mathcal{F}(Q^n_{m-1}, Q^n_m)$.

FV for parabolic PDEs

2.2.5 FV example for 2ndorder PDEs: Parabolic equation

- Consider the parabolic equation (50) (written for q),

$$q_{,t} - Dq_{,xx} = r \tag{89}$$

- This can be written as,

$$q_{,t} - (Dq_{,x})_{,x} = r \tag{90}$$

$f(q_{,x}) = -Dq_{,x}$

spatial flux

for diffusion spatial flux is a function of $q_{,x}$ instead of q

spatial flux $f(q, q_{,x})$

advection hyperbolic part

Diffusion parabolic part

advection diffusion eqn

$$q_{,t} + \{f(q_{,x})\}_{,x} = r, \qquad \text{where} \qquad f(q_{,x}) = -Dq_{,x} \tag{91}$$

- This is the same as the PDE form of balance law (63), *i.e.*, (62) $q_{,t} + \{f(q)\}_{,x} = 0$,
- with the difference that the flux function f depends on $q_{,x}$ rather than q.
- In general for transient problems (hyperbolic, parabolic) we can express PDE and the flux function f in the form,

$$q_{,t} + \{f(q, q_{,x})\}_{,x} = r \tag{92}$$

FV example for 2ndorder PDEs: Parabolic equation

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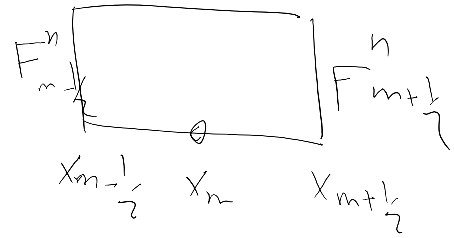
- The formulation of FV is similar to the hyperbolic case. That is, we obtain,

$$Q^{n+1}_m = Q^n_m - \frac{k}{h}(F^n_{m+1/2} - F^n_{m-1/2}) \text{ where} \tag{93a}$$

$$F^n_{m\pm1/2} \approx \frac{1}{\tau}\int_{t_n}^{t_{n+1}} f(q(x_{m\pm1/2}, t), q_{,x}(x_{m\pm1/2}, t))\, dt \quad \text{some approximation of the average flux along } x_{m\pm1/2} \tag{93b}$$

$$Q_m^{n+1} = Q_m^n - \frac{k}{h}\left(F^n_{m+1/2} - F^n_{m-1/2}\right), \text{ where} \tag{93a}$$

$$F^n_{m\pm1/2} \approx \frac{1}{k}\int_{t_n}^{t_{n+1}} f\left(q(x_{m\pm1/2},t), q_{,x}(x_{m\pm1/2},t)\right) dt \quad \text{some approximation of the average flux along } x_{m\pm1/2} \tag{93b}$$

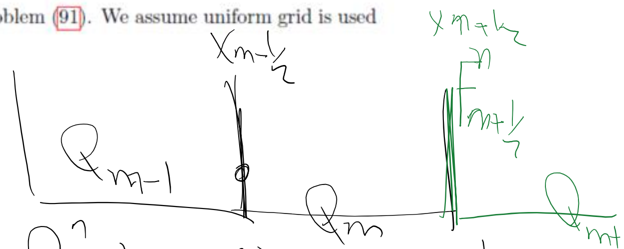


- Similar to (66), we can approximate average flux F, with numerical flux function $\mathcal{F}$, which depends only on inflow (previous step t_n) values:

$$F^n_{m-1/2} = \mathcal{F}(Q^n_{m-1}, Q^n_m), \qquad F^n_{m+1/2} = \mathcal{F}(Q^n_m, Q^n_{m+1}) \tag{94}$$

- We will demonstrate how we can apply FV method for the solution of diffusion problem (91). We assume uniform grid is used $h_m = h$ and D is constant.

$$f(q) = -D q_{,x}$$



Decent choice for use central difference

$$F^n_{m-1} = \mathcal{F}(Q^n_{m-1}, Q^n_m) \approx -D q_{,x} \Big|_{@\, x_{m-1/2}}$$

$$F^n_{m-1/2} = -D\left(\frac{Q_m - Q_{m-1}}{h}\right) \qquad F^n_{m+1/2} = -D\left(\frac{Q_{m+1} - Q_m}{h}\right)$$

$$F^n_{m+1/2} - F^n_{m-1/2} = \frac{-D}{h}\left(Q_{m+1} - Q_m - Q_m + Q_{m-1}\right)$$

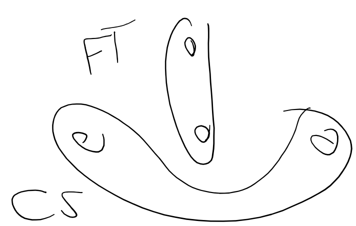
$$Q^{n+1}_m = Q^n_m - \frac{k}{h}\left(F^n_{m+1/2} - F^n_{m-1/2}\right) = Q^n_m - \frac{k}{h}\left(\frac{-D}{h}\left(Q^n_{m+1} + Q^n_{m-1} - 2Q^n_m\right)\right)$$

$$\rightarrow \boxed{Q^{n+1}_m = Q^n_m + \frac{Dk}{h^2}\left(Q^n_{m+1} + Q^n_{m-1} - 2Q^n_m\right)}$$ Update eqn

we can also write this as

$$\boxed{\underbrace{\frac{Q^{n+1}_m - Q^n_m}{k}}_{\text{FT } \dot{Q}} - D\underbrace{\left(\frac{Q^n_{m+1} + Q^n_{m-1} - 2Q^n_m}{h^2}\right)}_{\text{CS } Q_{,xx}} = 0}$$

$$\dot{Q} - DQ_{,xx} = 0$$

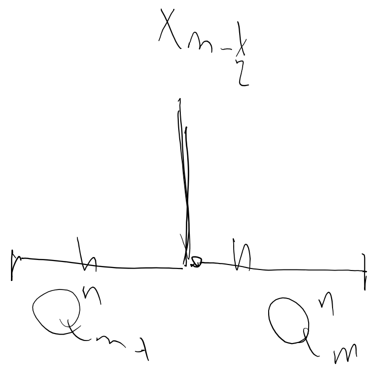


Any better approach for forming numerical fluxes for this problem?
Hyperbolic PDEs Riemann solutions provide a very good way to compute these spatial fluxes and we noticed that for advection e quation because of that we always got a stable difference formula.

Q: Is there a way to form Riemann solutions for parabolic PDEs

$k \propto h^2$

→ we can ignore this part (not affecting stability)



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An explicit discontinuous Galerkin scheme with local time-stepping for general unsteady diffusion equations

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conditions read as

$$\frac{\partial^2}{\partial \xi_1^2} w^{\pm} - \frac{s}{\mu^{\pm}} w^{\pm} = -\frac{u^{\pm} + \xi_1 u^{\pm}_{\xi_1}}{\mu^{\pm}}. \tag{A.2}$$

Solutions of the ordinary differential Eq. (A.2) are

$$w^{\pm}(\xi_1, s) = c_1^{\pm} \exp\left(-\sqrt{\frac{s}{\mu^{\pm}}}\xi_1\right) + c_2^{\pm} \exp\left(\sqrt{\frac{s}{\mu^{\pm}}}\xi_1\right) + \frac{u^{\pm}}{s} + \frac{u^{\pm}_{\xi_1}}{s}\xi_1, \tag{A.3}$$

where $c_1^{\pm}$ and $c_2^{\pm}$ are arbitrary constants. As $w^+(\xi_1, s)$ cannot grow exponentially for $\xi_1 \to \infty$, c_2^+ has to be zero. Likewise, c_1^- has to be zero. The other two constants c_1^+ and c_2^- are determined by the conditions

$$\begin{aligned} &w^+(0,s) = w^-(0,s), \\ &\mu^+ \tfrac{\partial w^+}{\partial \xi_1}(0,s) = \mu^- \tfrac{\partial w^-}{\partial \xi_1}(0,s), \end{aligned} \tag{A.4}$$

above

D

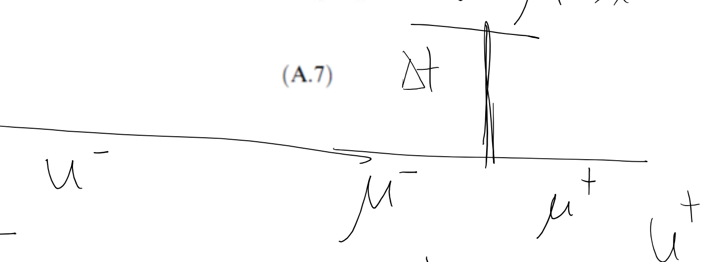
↑

$u - \mu u_{xx} = 0$

$$f(0,t) = \mu^+ \frac{\partial w^+}{\partial \xi_1}(0,t) = \frac{[\![u]\!]\sqrt{\mu^+\mu^-}}{\sqrt{\pi t}(\sqrt{\mu^+} + \sqrt{\mu^-})} + \frac{\sqrt{\mu^+} f_{\vec{n}}^- + \sqrt{\mu^-} f_{\vec{n}}^+}{\sqrt{\mu^+} + \sqrt{\mu^-}}. \tag{A.7}$$

spatial flux

$-\mu u_{,x}$



$u_{,x}$

$$\boxed{g_{\vec{n}} := \frac{1}{\Delta t}\int_0^{\Delta t} f(0,t)\,dt} = \boxed{\frac{2[\![u]\!]\sqrt{\mu^+\mu^-}}{\sqrt{\Delta t}\sqrt{\pi}(\sqrt{\mu^+} + \sqrt{\mu^-})} + \frac{\sqrt{\mu^+} f_{\vec{n}}^- + \sqrt{\mu^-} f_{\vec{n}}^+}{\sqrt{\mu^+} + \sqrt{\mu^-}}}.$$

$$F = \frac{1}{\Delta t}\int_0^{\Delta t} f(u_x)$$

better flux

$$= \frac{2\,[u^+ - u^-]\sqrt{\mu^-\mu^+}}{\sqrt{\mu^-} + \sqrt{\mu^+}} + \frac{\sqrt{\mu^+}(u^-_{,x}) + \sqrt{\mu^-}(u^+_{,x})}{\sqrt{\mu^-} + \sqrt{\mu^+}}$$

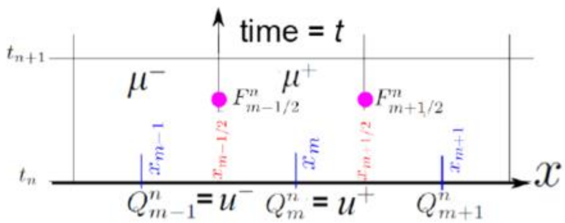
$\tilde{\mu} = \mu^+ \quad F = 2[u^+ - u^-]\sqrt{\mu} + \left(\frac{u^-_{,x} + u^+_{,x}}{2}\right)$

From DG community

$(u_{,x})^* = \{u_{,x}\} + \alpha(u^+ - u^-)$

penalty term needed for stability

2.2.5.3 FV example for Diffusion equation: 2. Exact flux



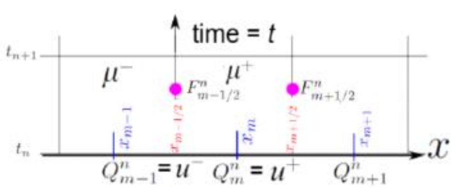
- The figure shows Riemann-like problem set up where the two values Q^n_{m-1} and Q^n_m from time step t_n are used as initial conditions for the diffusion equation:

  $$u_t - \mu u_{,xx} = 0$$

  where μ is the diffusion coefficient.
- To model a material interface with two distinct μ, μ^- and μ^+ left and right sides of an interface, respectively.

- In [Lorcher et al., 2008] a solution for this initial condition is obtained,

$$f(0,t) = \mu^+ \frac{\partial w^+}{\partial \xi_1}(0,t) = \frac{[\![u]\!]\sqrt{\mu^+\mu^-}}{\sqrt{\pi t}\left(\sqrt{\mu^+} + \sqrt{\mu^-}\right)} + \frac{\sqrt{\mu^+}f^-_{\vec{n}} + \sqrt{\mu^-}f^+_{\vec{n}}}{\sqrt{\mu^+} + \sqrt{\mu^-}}$$

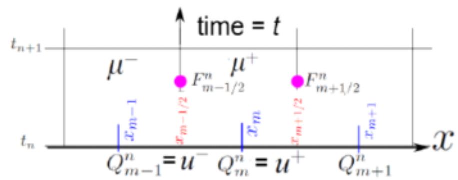


where $f(0,t)$ is the spatial flux function evaluated at the material interface, *e.g.*, $x_{m-1/2}$ in the figure and $f^\pm_{\vec{n}}$ are spatial fluxes from the two sides $f^\pm_{\vec{n}} = -\mu^\pm u^\pm_{,x}$.

- After integration of $f(0,t)$ in time (that is in the form of equation $F^n_{m\pm 1/2} = \frac{1}{k}\int_{t_n}^{t_{n+1}} f(q(x_{m\pm 1/2}, t)$ [Lorcher et al., 2008] obtains ($g_{\vec{n}} = F^n_{m\pm 1/2}$),

$$g_{\vec{n}} := \frac{1}{\Delta t}\int_0^{\Delta t} f(0,t)\mathrm{d}t = \frac{2[\![u]\!]\sqrt{\mu^+\mu^-}}{\sqrt{\Delta t}\sqrt{\pi}\left(\sqrt{\mu^+} + \sqrt{\mu^-}\right)} + \frac{\sqrt{\mu^+}f^-_{\vec{n}} + \sqrt{\mu^-}f^+_{\vec{n}}}{\sqrt{\mu^+} + \sqrt{\mu^-}}$$

- This flux can be used in the context of a FV or discontinuous Galerkin method.



- There are few points to observe,
  1. Unlike hyperbolic problems where there is a finite wave speed and for small time steps $k = t_{n+1} - t_n$ solution and flux on $x_{m-1/2}$ only depends on the two side solutions Q^n_{m-1} and Q^n_m ($u^\pm$ in the figure) in parabolic equations solution depends on the values of ALL cells (*e.g.*, $\mathcal{C}_{m+1}$ in the figure) and the assumption of having constant IC on either side of the interface no longer makes sense for the duration of the solution k.
  2. However, when the stable time step is used ($k_{\max} \propto \frac{h^2}{D}$) the error in ignoring nonadjacent cells does not cause numerical instability.
  3. In fact the fluxes based on the exact solution of the diffusion equation is a much better option than the previous FD type flux approximation (95) $F^n_{m-1/2} = \mathcal{F}(Q^n_{m-1}, Q^n_m) = -D\frac{Q^n_m - Q^n_{m-1}}{h}$.

2.2.6 FV example for 2nd order PDEs: Hyperbolic equation

- Consider the balance of linear momentum in 1D,

$$p_{,t} + (-s)_{,x} = \rho b \tag{98}$$

balance of linear momentum $\quad p_{,t} - s_{,x} = \rho b \quad$ s = stress

$p = \rho v$

close the equations $\quad s_{,t} = (E\varepsilon)_{,t} = E(u_{,x})_{,t} = E\, v_{,x} = \frac{E}{\rho} p_{,x} \quad v = \dot{u}$

$$\left\{\begin{array}{l} p_{,t} + (-s)_{,x} = \rho b \\ s_{,t} - \frac{E}{\rho} p_{,x} = 0 \end{array}\right.$$

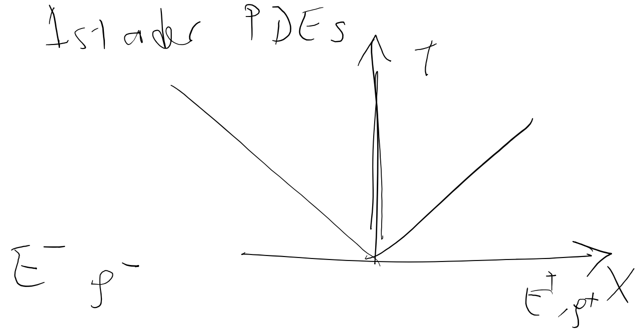
system of 1st order PDEs

kinematic flds $\quad$ force-like fields

$v \longleftrightarrow p$

$\varepsilon \longleftrightarrow \sigma$

Continuous quantities at the interface

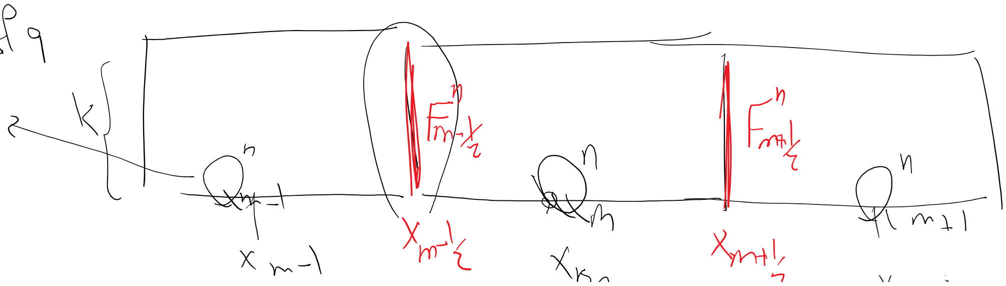


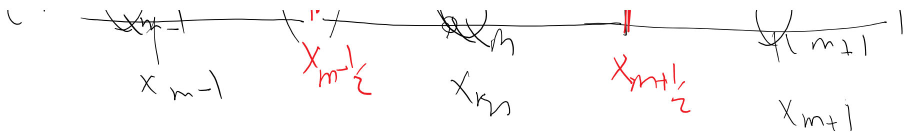
$$\boxed{\begin{array}{l} v_{,t} + \frac{-1}{\rho} s_{,x} = b \\ s_{,t} - E\, v_{,x} = 0 \end{array}}$$

$q = \begin{bmatrix} v \\ s \end{bmatrix} \qquad \dot{q} + f(q)_{,x} = r \quad$ source term

$f(q) = Aq, \; A = \begin{bmatrix} 0 & -\frac{1}{\rho} \\ -E & 0 \end{bmatrix}, \quad \boxed{f(q) = \begin{bmatrix} -\frac{1}{\rho} s \\ -Ev \end{bmatrix}}$

approximate of q





$$Q_m^{n+1} = Q_m^n - \frac{k}{h}\left[F_{m+\frac{1}{2}}^n - F_{m-\frac{1}{2}}^n\right] \qquad Q_m^n = \begin{bmatrix} V_m^n \\ S_m^n \end{bmatrix}$$

$$\vec{F}_{m-\frac{1}{2}}^n = \frac{1}{k}\int_0^k f(Q)\,dt \quad @ \quad x_{m-\frac{1}{2}}$$

$$= \frac{1}{k}\int_0^k \begin{bmatrix} -\frac{1}{\rho}\, S(x_{m-\frac{1}{2}}, t) \\ -E\, V(x_{m-\frac{1}{2}}, t) \end{bmatrix} dt \qquad f(Q) = \begin{bmatrix} -\frac{1}{\rho} & S \\ -E & V \end{bmatrix}$$

$$F_{m-\frac{1}{2}}^n = \begin{bmatrix} -\frac{1}{\rho} & \hat{S}_{m-\frac{1}{2}} \\ -E & \hat{V}_{m-\frac{1}{2}} \end{bmatrix} \qquad \hat{S}_{m-\frac{1}{2}} = \frac{1}{k}\int_0^k S(x_{m-\frac{1}{2}}, t)\,dt$$

$$\hat{V}_{m-\frac{1}{2}} = \frac{1}{k}\int_0^k V(x_{m-\frac{1}{2}}, t)\,dt$$