DC20200226

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From last time

2.3.4.1 Continuum weak statement (WK)

The weak statement for elastodynamics and the boundary conditions are:

$$\text{Find } \mathbf{u} \in \mathcal{V} = \{v \in C^1(\mathcal{D}^t) \mid \forall \mathbf{x} \in \partial\mathcal{D}_u^t \ \mathbf{v}(\mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x})\}, \text{ such that,} \tag{138a}$$

$$\forall \mathbf{w} \in \mathcal{W} = \{v \in C^1(\mathcal{D}^t) \mid \forall \mathbf{x} \in \partial\mathcal{D}_u^t \ \mathbf{v}(\mathbf{x}) = \mathbf{0}\}, \ \forall t \in \mathcal{I}^t \tag{138b}$$

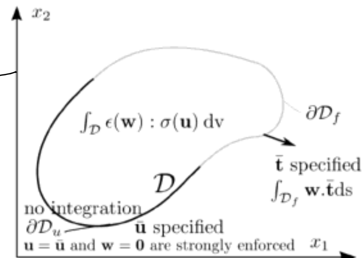
$$\int_{\mathcal{D}} [\rho \mathbf{w}.\ddot{\mathbf{u}} + \alpha \mathbf{w}.\dot{\mathbf{u}} + \epsilon(\mathbf{w}) : \sigma(\mathbf{u})] \, \mathrm{dv} = \int_{\mathcal{D}} \mathbf{w}.\rho \mathbf{b} \, \mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}.\bar{\mathbf{t}} \, \mathrm{ds} \tag{138c}$$

- Both $\mathcal{V}$ and $\mathcal{W}$ have the same regularity ($C^m(\mathcal{D})$): $m = M/2$, $M = 2$ is the order of the differential equation.
- The less demanding regularity conditions for the solution compared to the weighted residual statement ($C^M(\mathcal{D}) \to C^m(\mathcal{D})$) takes us to the same function space needed for the balance law (highest derivative is for $\sigma(\mathbf{u}) = C_{ijkl}\, u_{k,l}$ is 1).
- Both $\mathcal{V}$ and $\mathcal{W}$ exactly enforce the essential boundary conditions, with the difference that $\mathcal{W}$ satisfies the homogeneous version.

Discretize the problem

$$u^h(x,t) = \sum_{i=1}^{n_f} a_i^f(t)\, \phi_i(x) + u^{ph}(x,t)$$

$$\forall x \in \partial D_u \quad \phi_i(x) = 0 \qquad u^{ph}(x,t) = \bar{u}(x,t)$$



$$\forall x \in \partial D_u \quad u^h(x,t) = \bar{u}(x,t)$$

how we satisfy essential BC

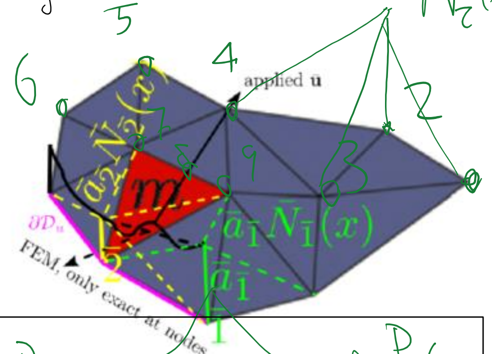
we use shape functions for ϕ's

$n_f = 9$

$n_p = 3$

$$\underline{u_p(x,t) = \sum_{i=1}^{n_p} a_i^p(t)\, N_i^p(x)}$$

$\phi_2(x) = N_2(x)$



$a_1^p(t)$ $N_1^p(x)$

unknown known

$$u = \sum_{i=1}^{n_f} a_i^f(t)\, N_i^f(x) + \sum a_i^p(t)\, N_i^p(x)$$

$$= Na, \quad N = [N^f \mid N^p] = [\underbrace{N_1^f \cdots N_{n_f}^f}_{n_f} \mid N_1^p \cdots N_{n_p}^p](x)$$

unknown

$$a = \begin{bmatrix} a^f \\ \hline a^p \end{bmatrix} = \begin{bmatrix} a_1^f \\ \vdots \\ a_{n_f}^f \\ \hline a_1^p \\ \vdots \\ a_{n_p}^p \end{bmatrix}$$

prescribed

known

- Discrete solution function can be written as (*cf.* (139), (140), (141)),

$$
\begin{aligned}
\mathbf{u}^h(\mathbf{x},t) &= \mathbf{u}^{f^h} + \mathbf{u}^{p^h}(\mathbf{x},t) && \text{(142a)}\\
&= \sum_{i=1}^{n_f} a_i^f(t)\mathbf{N}_i^f(\mathbf{x}) + \sum_{i=1}^{n_p} a_i^p(t)\mathbf{N}_i^p(\mathbf{x}) && \text{(142b)}\\
&= \mathbf{N}.\mathbf{a} && \text{(142c)}
\end{aligned}
$$

where

$$\mathbf{N} = [\mathbf{N}^f \quad \mathbf{N}^p] = \left[\mathbf{N}_1^f \quad \cdots \quad \mathbf{N}_{n_f}^f \quad |\mathbf{N}_1^p \quad \cdots \quad \mathbf{N}_{n_p}^p\right] \qquad 3\times n \text{ array} \qquad \text{(143a)}$$

$$\mathbf{a} = \begin{bmatrix}\mathbf{a}^f\\ \mathbf{a}^p\end{bmatrix} = \begin{bmatrix} a_1^f \\ \vdots \\ a_1^f \\ -- \\ a_1^p \\ \vdots \\ a_{n_p}^p \end{bmatrix} \qquad n\times 1 \text{ array} \qquad \text{(143b)}$$

$$\mathbf{u}^h(\mathbf{x},t) \qquad 3\times 1 \text{ array} \qquad \text{(143c)}$$

$$n = n_f + n_p \quad \text{total number of dofs} \qquad \text{(143d)}$$

Notes:

- Unknown quantities, $\mathbf{a}^f$ are shown in this color. There are n_f unknowns in $\mathbf{a}$.

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$$
\begin{aligned}
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&\forall \mathbf{w} \in \mathcal{W} = \{v \in C^1(\mathcal{D}^t) \mid \forall \mathbf{x} \in \partial\mathcal{D}_u^t \ \mathbf{v}(\mathbf{x}) = 0\}, \ \forall t \in \mathcal{I}^t && \text{(138b)}\\
&\int_{\mathcal{D}} [\rho\mathbf{w}.\ddot{\mathbf{u}} + \alpha\mathbf{w}.\dot{\mathbf{u}} + \epsilon(\mathbf{w}) : \sigma(\mathbf{u})]\, \mathrm{dv} = \int_{\mathcal{D}} \mathbf{w}.\rho\mathbf{b}\, \mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}.\bar{\mathbf{t}}\, \mathrm{ds} && \text{(138c)}
\end{aligned}
$$

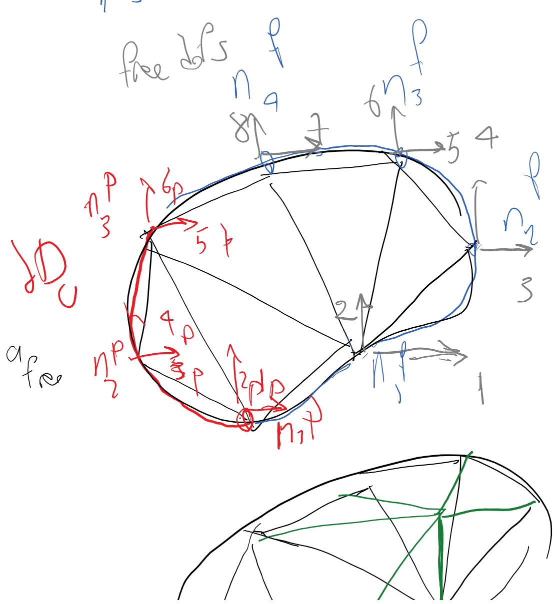
$u = Na \qquad \dot{u} = N\dot{a} \qquad \ddot{u} = N\ddot{a} \qquad N = [N_f \mid N_p]$

$w = ?$ $\qquad N^t \qquad N_f^t \qquad N_p^t \qquad \begin{bmatrix} N_1^f \\ \vdots \\ N_{n_f}^f \end{bmatrix}$

Need to form $\varepsilon(w)$ & $\sigma(u)$

$\sigma(u) = C\,\varepsilon(u)$
↳ stiffness

$n_f = 8$
$n_p = 6$
$n = 14$



Vector equation : 2D

$$\vec{u}(x,y) = \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix} = \underbrace{\left[\ \overbrace{\quad\quad\quad}^{N_f} \ \middle|\ \overbrace{\quad\quad}^{N_p}\ \right]}_{N \quad 2\times 14} \begin{bmatrix} a_1^f \\ \vdots \\ a_8^f \\ \hline a_1^p \\ \vdots \\ a_6^p \end{bmatrix} a_{free}$$

2×14 14×1 $n\times 1$

$$\begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix} = \underbrace{\left[\begin{array}{cccccccc|cccccc} N_1^f(x) & 0 & N_2^f & 0 & N_3^f & 0 & N_4^f & 0 & N_1^p & 0 & N_2^p & 0 & N_3^p & 0 \\ 0 & N_1^f(x) & 0 & N_2^f & 0 & N_3^f & 0 & N_4^f & 0 & N_1^p & 0 & N_2^p & 0 & N_3^p \end{array}\right]}_{\mathbf{N}^f_{n_f} \quad \mathbf{N}^p_{n_p}} \begin{bmatrix} a_1^f \\ \vdots \\ a_8^f \\ \hline a_1^p \\ \vdots \\ a_6^p \end{bmatrix}$$

$\bar{N}_1^p(x)$

$n_f = 2\times n_{\text{free nodes}}$ $\quad \mathbf{N}^f_{n_f} \to 2\times(2n_f)$ $\quad \mathbf{N}^p_{n_p}$ $\quad n_p = 2\times n_p$ nodes

$$N^f = [N_1^f | N_2^f | \dots | N_4^f]_{1\times 4 n_f}$$

Voigt notation for strain & stress

2D

$$\epsilon = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{bmatrix} = \begin{bmatrix} u_{1,1} & \frac{1}{2}(u_{1,2}+u_{2,1}) \\ \swarrow & u_{2,2} \end{bmatrix}$$

sym

$1 \to x$

$2 \to y$

4th order elasticity

(a) $\sigma = C_{2\times2\times2\times2}\, \epsilon_{2\times2} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$

sym: $C_{ijkl} = C_{jikl}$ $\quad$ general sym. σ, ϵ: $C_{ijkl} = C_{ijlk}$

$C_{ijkl} = C_{klij}$ hyperelastic

$$\varepsilon(w) : \sigma(u) = \varepsilon_{11}(w)\,\sigma_{11}(u) + \varepsilon_{22}(w)\,\sigma_{22}(u) + \underline{2\varepsilon_{12}(w)\,\sigma_{12}(u)}$$

Voigt notation

$$\gamma = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix}$$

$2\epsilon_{12} = \gamma_{12}$

$$S = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

x_2, $\gamma_{12} = 2\epsilon_{12}$, x_1

$$\underline{S(u)\cdot\gamma(w)} = \sigma_{11}(u)\,\epsilon_{11}(w) + \sigma_{22}(u)\,\epsilon_{22}(w) + 2\varepsilon_{12}(w)\,\sigma_{12}(u)$$

they match

$$S_{3\times1} = C_{3\times3}\,\gamma_{3\times1}$$

C: Voigt stiffness

if hyperelastic $C_{3\times3}$ is symmetric

The factor of 2 in Voigt shear "vector" is for a) s.gamma being 2 times strain energy density b) getting a symmetric C Voigt when material is hyperelastic

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$$\begin{bmatrix} u_1(x_1,x_2) \\ u_2(x_1,x_2) \end{bmatrix} = \underbrace{\left[\begin{array}{cc|c|cc|cc|c|cc} N_1^f & 0 & \cdots & N_{n_f}^f(x_1,x_2) & & N_1^p(x_1,x_2) & 0 & \cdots & N_{n_p}^p(x_1,x_2) & 0 \\ 0 & N_1^f & & & N_{n_f}^f(x_1,x_2) & 0 & N_1^p(x_1,x_2) & \cdots & 0 & N_{n_p}^p(x_1,x_2) \end{array}\right]}_{N} \begin{bmatrix} a_1^f \\ \vdots \\ a_{n_f}^f \\ \hline a_1^p \\ \vdots \\ a_{n_p}^p \end{bmatrix}$$

$$\begin{pmatrix} \gamma_1(x_1,x_2) \\ \gamma_2(x_1,x_2) \\ \gamma_3(x_1,x_2) \end{pmatrix}, \quad \begin{pmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{12} \end{pmatrix} = \begin{bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2}+u_{2,1} \end{bmatrix} = \underbrace{\begin{bmatrix} \partial/\partial x_1 & 0 \\ 0 & \partial/\partial x_2 \\ \partial/\partial x_2 & \partial/\partial x_1 \end{bmatrix}}_{L}{}_{3\times 2} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$L \longrightarrow$ going from displacement to strain

$$\underset{3\times 2n}{B} = \underset{3\times 2}{L}\,\underset{2\times 2n}{N}$$

$\downarrow$ displacement-to-strain matrix

$$B = \left[\begin{array}{cc|c|c|c|c} \frac{\partial N_1^f}{\partial x_1} & 0 & \cdots & & \cdots & \\ 0 & \partial N_1^f/\partial x_2 & & & & \\ \frac{\partial N_1^f}{\partial x_2} & \partial N_1^f/\partial x_1 & & & & \end{array}\right]$$

$n_f \quad i=1\,p \quad i=n_p\,p$

$$B_{3\times 2n} = \left[B_f \mid B_p\right] \qquad w = \left(N_f^t\right)_{n_f\times 2}$$

$\downarrow$ # free nodes

(2) $\dot{u} = N_{2\times n}\,\dot{a}_{n\times 1}$

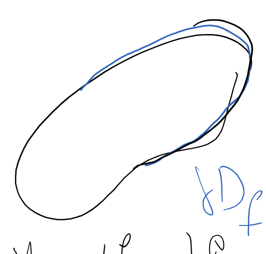
(3) $\ddot{u} = N_{2\times n}\,\ddot{a}_{n\times 1}$

$$N_{2\times n} = \left[\underset{2\times n_f}{(N_f)} \mid N_p\right]$$

(4) $\varepsilon(w) = \left(B_f^t\right)_{n_f\times 3}$

$\varepsilon(u) = B_{3\times n}\, a_{n\times 1}$

$$\iint_D \left[\rho \underset{(1)}{w} \underset{(3)}{\ddot{u}} + \alpha \underset{(1)}{w} \underset{(2)}{\dot{u}} + \underset{(4)}{\varepsilon(w)} \cdot \underset{(5)}{\sigma(u)}\right] dv = \int_D \underset{(1)}{w} \rho b\, dv + \int_{\partial D_f} \underset{(1)}{w}\, \bar{t}\, ds \qquad \sigma(u) = C_{3\times 3}\,\varepsilon(u)$$



∂D_f

n_f = # free dofs = 2 × n # nodes

$$\iint \Big\{ \rho N_f^t \left[\underset{(a)}{N_f} \mid \underset{(a')}{N_p}\right] \begin{bmatrix} \ddot{a}_f \\ \ddot{a}_p \end{bmatrix} + \alpha N_f^t \left[\underset{(b)}{N_f} \mid \underset{(b')}{N_p}\right] \begin{bmatrix} \dot{a}_f \\ \dot{a}_p \end{bmatrix}$$

$$+ B_f^t\, \bar{C}_{3\times 3} \left[\underset{(c)}{B_f}\ \ \underset{(c')}{B_p}\right] \begin{bmatrix} a_f \\ a_p \end{bmatrix} \Big\}\, dv = \int N_f^t \rho b\, dv + \int N_f^t\, \bar{t}\, ds$$

$$+ B^t \bar{C}_{3\times3} \tilde{B}_p B_p \, [a_p] \} dv = \int_D N_p^t \rho b \, dv + \int_{\partial D_f} N_f^t t \, ds$$

$$\underbrace{\left(\int_D \rho N_f^t N_f \, dv\right)}_{M_{ff}} \ddot a_f + \underbrace{\left(\int_D \alpha N_f^t N_f \, dv\right)}_{C_{ff}} \dot a_f + \underbrace{\left(\int_D B_f^t C_{3\times3} B_f \, dv\right)}_{K_{ff}} a_f$$

(a) (b) (c)

$$= \underbrace{\int_D N_f^t \rho b \, dv}_{F_r} + \underbrace{\int_{\partial D_f} N_f^t \bar t \, ds}_{F_N} - \underbrace{\left(\left(\int \rho N_f^t N_p \, dv\right)\ddot a_p + \left(\int \alpha N_f^t N_p\right)\dot a_p + \left(\int B_f^t C B_p\right) a_p\right)}_{a' \; M_{fp}\ddot a_p \quad b' + C_{fp}\dot a_p \quad c' + K_{fp} a_p}$$

F_D (Dirichlet force)

F_r → Source term force

F_N (Neuman force)

$\cancel{\mathbf{M}^{ff}\ddot{\mathbf{a}}^f} + \cancel{\mathbf{C}^{ff}\dot{\mathbf{a}}^f} + \cancel{\mathbf{K}^{ff}\mathbf{a}^f} =$	$\mathbf{F}_r + \mathbf{F}_N - \mathbf{F}_D$ ODE where	(153a)
$\mathbf{a}^f(t=0) = \mathbf{a}_0^f, \dot{\mathbf{a}}^f(t=0) = \dot{\mathbf{a}}_0^f$	Initial condition (IC)	(153b)
$\mathbf{F}_r = \int_{\mathcal{D}} \mathbf{N}^{f^{\mathrm{T}}} \rho \mathbf{b} \, \mathrm{dv}$	Source term (body force) force vector	(153c)
$\mathbf{F}_N = \int_{\partial\mathcal{D}_f} \mathbf{N}^{f^{\mathrm{T}}} \bar{\mathbf{t}} \, \mathrm{ds}$	Natural (Neumann) BC force vector	(153d)
$\boxed{\mathbf{F}_D = \mathbf{M}^{fp}\ddot{\mathbf{a}}^p + \mathbf{C}^{fp}\dot{\mathbf{a}}^p + \mathbf{K}^{fp}\mathbf{a}^p}$	Essential (Dirichlet) BC force vector	(153e)
$\mathbf{M}^{ff} = \int_{\mathcal{D}} \rho \mathbf{N}^{f^{\mathrm{T}}} \mathbf{N}^f \, \mathrm{dv},$		
$\mathbf{M}^{fp} = \int_{\mathcal{D}} \rho \mathbf{N}^{f^{\mathrm{T}}} \mathbf{N}^p \, \mathrm{dv}$	Mass matrices	(153f)

→ static part

$\mathbf{C}^{ff} = \int_{\mathcal{D}} \alpha \mathbf{N}^{f^{\mathrm{T}}} \mathbf{N}^f \, \mathrm{dv},$		
$\mathbf{C}^{fp} = \int_{\mathcal{D}} \alpha \mathbf{N}^{f^{\mathrm{T}}} \mathbf{N}^p \, \mathrm{dv}$	Damping matrices	(153g)
$\mathbf{K}^{ff} = \int_{\mathcal{D}} \mathbf{B}^{f^{\mathrm{T}}} \bar{\mathbf{C}} \mathbf{B}^f \, \mathrm{dv},$		
$\mathbf{K}^{fp} = \int_{\mathcal{D}} \mathbf{B}^{f^{\mathrm{T}}} \bar{\mathbf{C}} \mathbf{B}^p \, \mathrm{dv}$	Stiffness matrices	(153h)

Often , (153b) are written in the short form,

$$\mathbf{M}\ddot{\mathbf{a}} + \mathbf{C}\dot{\mathbf{a}} + \mathbf{K}\mathbf{a} = \mathbf{F}_r + \mathbf{F}_N - \mathbf{F}_D \quad \text{ODE where} \tag{154a}$$

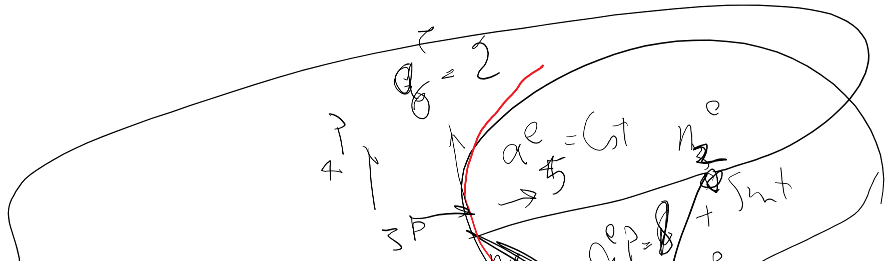
$$\mathbf{a}(t=0) = \mathbf{a}_0, \quad \dot{\mathbf{a}}(t=0) = \dot{\mathbf{a}}_0 \quad \text{Initial condition (IC)} \tag{154b}$$

where for short the superscripts for free dofs is dropped knowing that free and prescribed dofs are handles according to (153).

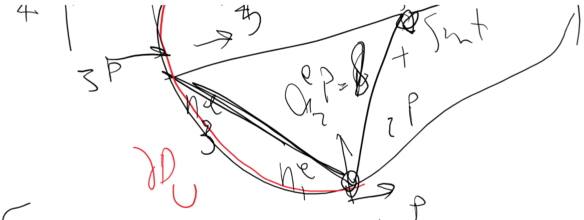
We already know how to do assembly. All terms (M, C, K, Fr, FN) can directly be formed with similar formulas at the element level and assembled to the global system.

For FD we do the calculation at the element level

$$a_e = \begin{bmatrix} 6t + t^2 \\ 8 + \sin t \\ 0 \\ \vdots \end{bmatrix}$$



$$\tilde{a}_e = \begin{bmatrix} 0 \\ 4\sin t \\ 0 \\ 0 \\ 5\,t \\ 0 \end{bmatrix}$$



$a_1^{ep}(t) = 6t + t^2$

$$\dot{\tilde{a}}_e = \begin{bmatrix} 6+2t \\ 5t \\ 0 \\ 0 \\ -\sin t \\ 0 \end{bmatrix} \qquad \ddot{\tilde{a}}_e = \begin{bmatrix} 2 \\ -5\sin t \\ 0 \\ 0 \\ -\cos t \\ 0 \end{bmatrix}$$

$$K^e_{6\times 6} \qquad C^e_{6\times 6} \qquad M^e_{6\times 6}$$

$$F_D^e = K^e a^e + \boxed{C^e \dot{a}^e + M^e \ddot{a}^e}$$

things added to F_D in dynamic problem

$$F^e = F_r^e + F_N^e - F_D^e$$

assemble to global system

- As we will see through the following example, we actually DO NOT form $\mathbf{M}^{fp}$, $\mathbf{C}^{fp}$, $\mathbf{K}^{fp}$ directly, rather computing their corresponding values from elements and assemble their effects to global force $\mathbf{F}_D$. Local versions of $\mathbf{F}_D$ is

$$\mathbf{f}_D^e = \mathbf{M}^e\ddot{\mathbf{a}}^e + \mathbf{C}^e\dot{\mathbf{a}}^e + \mathbf{k}^e\mathbf{a}^e \tag{155}$$

where $\mathbf{a}^e$ is the local displacement vector of element formed by

1. Having zero values for free dofs.
2. Having prescribed values for prescribed dofs.

Damping matrix

$$M = \int_D \rho N^t N \, dv \qquad C = \int \alpha N^t N \, dv$$

α: damping

assume α, ρ are constant

$$M = \rho \int_D N^t N \, dv \qquad C = \alpha \int_D N^t N \, dv$$

$$\alpha(x) = \rho(x)\, \bar{\alpha} \quad \text{constant}$$

$$\boxed{C = \bar{\alpha} M}$$

because of resistance of material to velocity

$$F_D = -\alpha v$$

$C = \alpha M$ because of ... to velocity

$F_D = -\alpha V$

$$\sigma = E\varepsilon + \underbrace{E'\dot{\varepsilon}}_{\text{viscous material}}$$

Fourier transform

$$\check{\sigma} = E\check{\varepsilon} + E' j\omega \check{\varepsilon}$$

$$\check{\sigma} = (\underbrace{E}_{E_r} + \underbrace{E'\omega j}_{E_i})\check{\varepsilon}$$

E_r: real part of stiffness

E_i: imaginary part of stiffness

$\frac{E_i}{E_r} \approx 0.01$ in frequency domain

$E\varepsilon$

E

$E'\dot{\varepsilon}$

$$Ka \rightarrow Ka + \underbrace{\left(\frac{E'}{E}\right)}_{\beta} K\dot{a}$$

C from viscosity

$$C = \beta K$$

$$C = \bar{\alpha} M + \beta K$$

damping $\leftarrow \alpha(x) = \bar{\alpha}\rho(x)$

$E'(x) = \beta E(x)$ → viscous stiffness

- Rayleigh damping matrix, generalizes the formula for $\mathbf{C}$ from (153g) by basically adding a coefficient of stiffness matrix. That is,

$$\mathbf{C} = \alpha \mathbf{M} + \beta \mathbf{K} \tag{156}$$

  - Justification for α is as before by modeling the equation of motion as in (128), that is $\rho\mathbf{b} \rightarrow \rho\mathbf{b} - \alpha\mathbf{v}$ and getting (129) which is,

$$\dot{\mathbf{p}} - \nabla.\sigma + \alpha\mathbf{v} - \rho\mathbf{b} = \mathbf{0}$$

  - Justification for β is modifying equation of motion in the form,

$$\sigma = \mathcal{C}(\epsilon + \beta\dot{\epsilon}) \tag{157}$$