DC2020/03/02

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2.3.9 Stiffness and mass matrices for 1D elastostatics

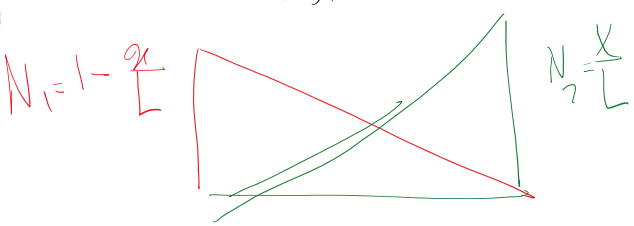
- Shape functions are given by,

$$N^e = [N_1^e \; N_2^e] \quad \text{where}$$
$$N_1^e = \frac{x_{i+1} - x}{L}, \quad N_2^e = \frac{x - x_i}{L}, \quad L = x_{i+1} - x_i$$

1D, p=1 element

$B = \frac{d}{dx} N = \frac{1}{L}[-1, 1]$

$K = \int B^T E \, 3(N) \, dV = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$



Mass matrix $M = \int_0^L \begin{bmatrix} 1 - x/L \\ x/L \end{bmatrix} \begin{bmatrix} 1 - x/L & x/L \end{bmatrix} \rho A \, dx = \frac{L\rho}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$= m \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}$

m: mass of element

the sum of components of mass matrix = Mass of element

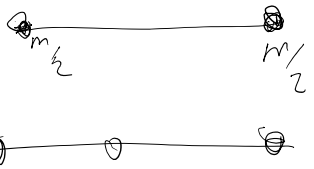
where ρ is mass density and $m^e = AL\rho$ is the mass of element. Note that,

- The sum of components of $\mathbf{M}^e$ is equal to m^e.
- The mass matrix is NOT diagonal.
- As we will see this results in NON-DIAGONAL global level mass matrix.

2.3.10 Lumped mass matrix

p=1, 1D constant $\rho(x)$, $A(x)$

$M^e = \begin{bmatrix} m/2 & 0 \\ 0 & m/2 \end{bmatrix}$



$M^e = \begin{bmatrix} m/3 & m/3 \\ m/3 & m/3 \end{bmatrix}$?

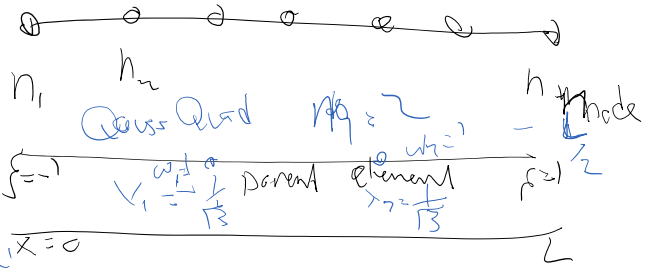
There is a systematic way to come up with diagonal mass matrices

What is we use a quadrature scheme that only samples on the nodes of the element?

$M_{ij}^e = \int_e N_i^e(x) \rho \, N_j^e(x) \, dV$

#quad pts $= \sum_{k=1}^{nq} w_k N_i^e(x_k) \rho(x_k) N_j^e(x_k) J(x_k)$ ← Jacobian

x_k's are element nodes if we use Newton Cotes scheme (sampling @ equidistance points)



nq

points
$n_q = n_{node}$

$$\sum_{k=1}^{nq} w_k N_i^e(x_k) \rho(x_k) N_j(x_k) J(x_k)$$

$N_i^e(x_k) = \delta_{ik}$
shape fnctn δ property for Nodes & quad points

$$M_{ij}^e = \sum_k w_k \delta_{ik} \delta_{jk} \rho(x_k) J(x_k) \longrightarrow$$

$$M_{ii}^e = w_i \rho(x_i) J(x_i)$$
$$M_{ij}^e = 0 \quad i \neq j$$

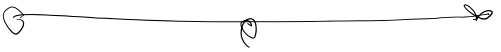
NC scheme $\int_0^L f(x) = \frac{L}{2}(f(0)+f(L))$ $p=1$

trapezoidal rule

$$\longrightarrow M = \begin{bmatrix} m/2 & 0 \\ 0 & m/2 \end{bmatrix} \quad M_{11} = \underbrace{\left(\frac{L}{2}\right)}_{w_1} \rho \cdot 1 = \frac{m}{2}$$

Simpsons rule

$$\int_0^L f(x)dx = \frac{L}{6} f(0) + \frac{4L}{6} f\left(\frac{L}{2}\right) + \frac{L}{6} f(L)$$



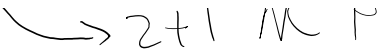
$$M = \begin{bmatrix} m/6 & 0 & 0 \\ 0 & 4m/6 & 0 \\ 0 & 0 & m/6 \end{bmatrix}$$

$$M = \int_0^L \left[\frac{1-x}{L} \quad \frac{x}{L}\right] \rho \begin{bmatrix} 1-\frac{x}{L} \\ \frac{x}{L} \end{bmatrix} dx$$

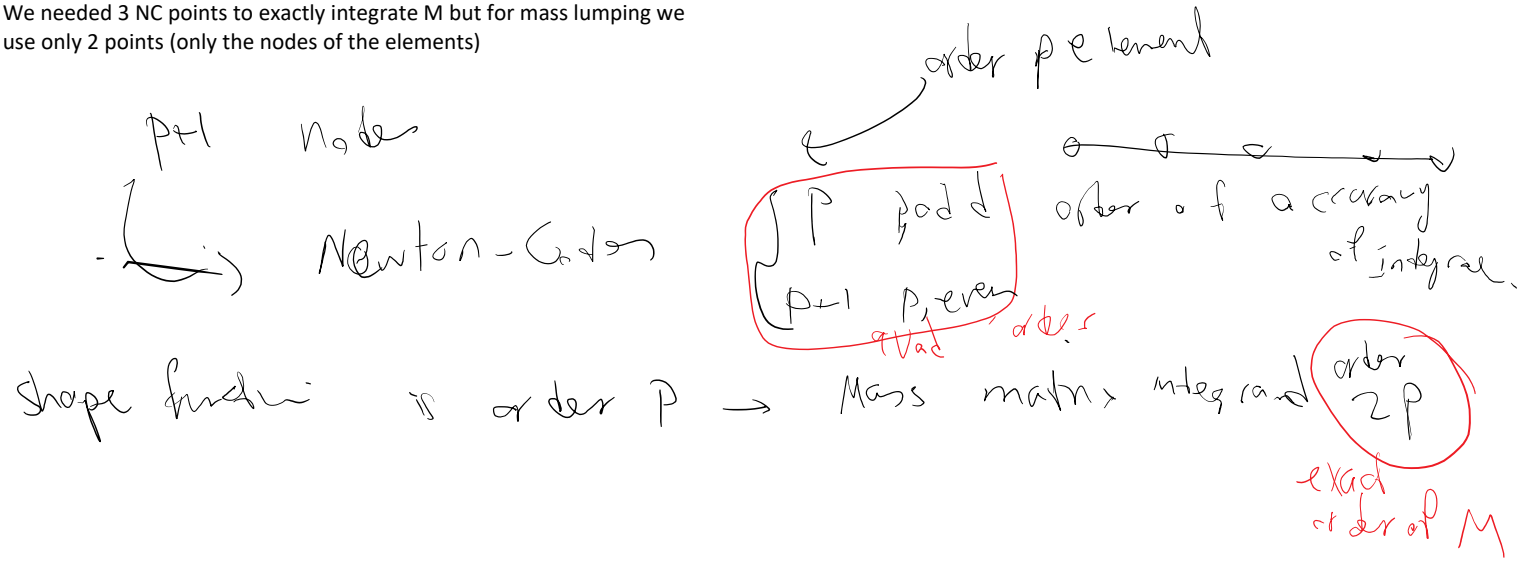
$p=1$ element

2nd order → 2 Gauss pts
→ 2+1 NC points

We needed 3 NC points to exactly integrate M but for mass lumping we

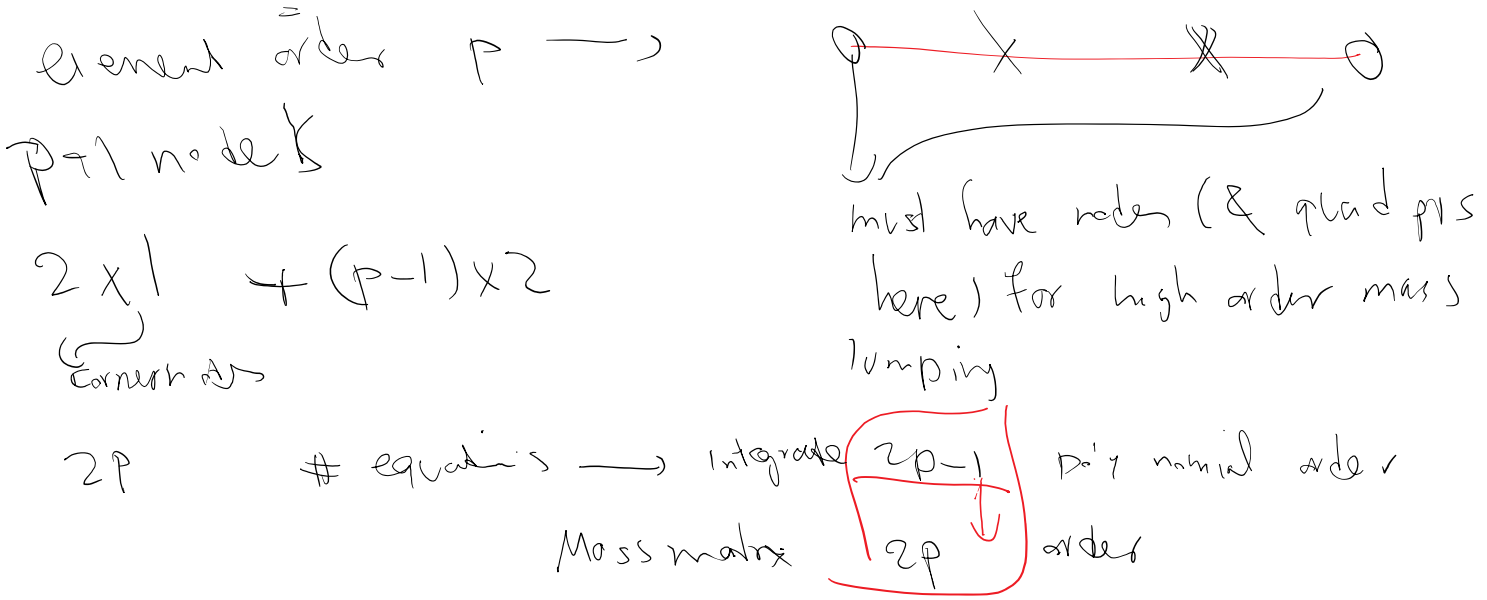


We needed 3 NC points to exactly integrate M but for mass lumping we use only 2 points (only the nodes of the elements)



In mass lumping we basically integrate M with about half the order of accuracy needed for its integration and this enables obtaining a lumped-mass matrix.

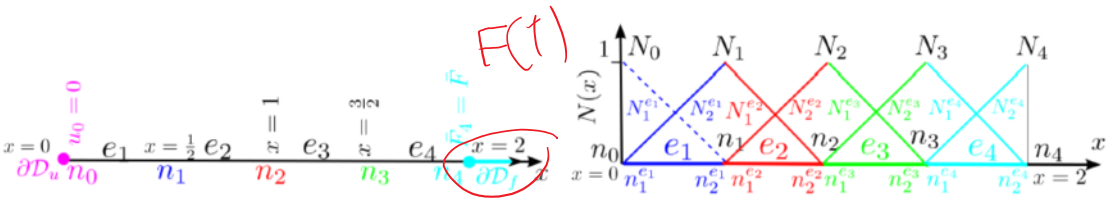
- For even higher order (p) bar elements, unfortunately using uniform distant internal element nodes at L/p positions and quadrature points corresponding to those points limits the order of accuracy in which the mass matrix is integrated.
- This in turn affects the FEM solutions convergence rates for the nodal solution $\mathbf{U}$ and other solution features such as modal quantities; *cf.* §3.1.7.
- To not sacrifice the order in which the element mass matrix is integrated we do two things:
  1. Choose the two end point of the element as two of the quadrature points because we have no freedom in changing the position of the element end nodes.
  2. Similar to Gauss quadrature formulation we optimize the position of high order element nodal (*i.e.*, quadrature) positions.
- As a result the corresponding quadrature scheme with these optimized points will have sufficient order of accuracy and the FEM solution convergence rates are not affected.



- The optimized scheme of quadrature points that include the end points is called Lobatto quadrature
- For more information refer to Section 7.3.2 [Hughes, 2012].

For high order CFEM mass lumping put node locations (in parent element) on Lobatto points.

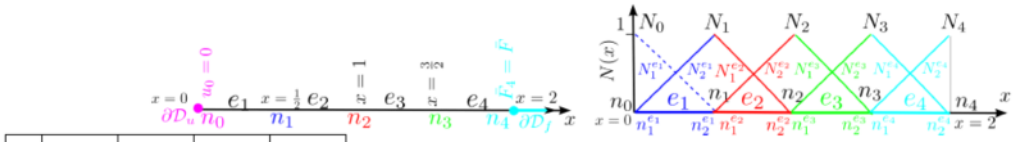
2.3.11 Example for the assembly of global matrix systems



- We recall that the element matrices for first order 1D bar elements were

$$\mathbf{k}^e = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \text{Stiffness matrix} \qquad (170a)$$

$$\mathbf{M}^e = \frac{m^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \text{consistent mass matrix} \qquad (170b)$$



e	e_1	e_2	e_3	e_4
$\mathbf{k}^e$	$\bar{1}\ 1$ $\bar{1}$ 2 -2 1 -2 2	1 1 1 2 -2 1 -2 2	2 3 2 2 -2 3 -2 2	3 4 3 2 -2 4 -2 2
$\mathbf{M}^e$	$\bar{1}\ 1$ $\bar{1}$ $\frac{1}{3}$ $\frac{1}{6}$ 1 $\frac{1}{6}$ $\frac{1}{3}$	1 1 1 $\frac{1}{3}$ $\frac{1}{6}$ 1 $\frac{1}{6}$ $\frac{1}{3}$	2 3 2 $\frac{1}{3}$ $\frac{1}{6}$ 3 $\frac{1}{6}$ $\frac{1}{3}$	3 4 3 $\frac{1}{3}$ $\frac{1}{6}$ 4 $\frac{1}{6}$ $\frac{1}{3}$

e	e_1	e_2	e_3	e_4
$\mathbf{f}_D^e$	$\mathbf{k}^{e1}\mathbf{a}_1^e + \mathbf{M}f^{e1}\ddot{\mathbf{a}}_1^e =$ $\bar{1}$ 0 1 0	$\mathbf{k}^{e2}\mathbf{a}_2^e + \mathbf{M}f^{e2}\ddot{\mathbf{a}}_2^e =$ 1 0 2 0	2 0 3 0	3 0 4 0
$\mathbf{f}_e^e$	$\mathbf{f}_e^{e1} = \mathbf{f}_r^{e1} + \mathbf{f}_N^{e1} - \mathbf{f}_D^{e1} =$ $\bar{1}$ 0 1 0	$\mathbf{f}_e^{e2} = \mathbf{f}_r^{e2} + \mathbf{f}_N^{e2} - \mathbf{f}_D^{e2} =$ 1 0 2 0	$\mathbf{f}_e^{e3} =$ 2 0 3 0	$\mathbf{f}_e^{e4} =$ 3 0 4 0

$$\mathbf{K} = \begin{bmatrix} 2+2 & -2 & 0 & 0 \\ -2 & 2+2 & -2 & 0 \\ 0 & -2 & 2+2 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} \frac{1}{3}+\frac{1}{3} & \frac{1}{6} & 0 & 0 \\ \frac{1}{6} & \frac{1}{3}+\frac{1}{3} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & \frac{1}{3}+\frac{1}{3} & \frac{1}{6} \\ 0 & 0 & \frac{1}{6} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & 0 & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & 0 & \frac{1}{6} & \frac{1}{3} \end{bmatrix}, \quad \mathbf{F} = \mathbf{F}_N + \mathbf{F}_e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ F(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ F(t) \end{bmatrix} \Rightarrow$$

Mass lumping option

$$\mathbf{M}\ddot{\mathbf{a}} + \mathbf{K}\mathbf{a} = \mathbf{F} \quad \text{that is} \quad \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & 0 & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & 0 & \frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \ddot{a}_1 \\ \ddot{a}_2 \\ \ddot{a}_3 \\ \ddot{a}_4 \end{bmatrix} + \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ F(t) \end{bmatrix} \quad \begin{matrix} \text{consistent} \\ \text{mass} \\ \text{matrix} \end{matrix} \tag{172}$$

more appropriate for implicit solve

with lumped mass matrix (171b) we would have got

$$\mathbf{M}\ddot{\mathbf{a}} + \mathbf{K}\mathbf{a} = \mathbf{F} \quad \text{that is} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \ddot{a}_1 \\ \ddot{a}_2 \\ \ddot{a}_3 \\ \ddot{a}_4 \end{bmatrix} + \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ F(t) \end{bmatrix} \quad \begin{matrix} \text{lumped} \\ \text{mass} \\ \text{matrix} \end{matrix} \tag{173}$$

explicit solve

M is on LHS & M^{-1} is trivial

Solution methods for (semi) discrete problem:

$M\ddot{a} + Ka = F$

3.1.1 Modal analysis: Motivation

- We are interested in solving the system,

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R} \tag{174}$$

change of "coordinate" $U(t) = PX(t)$

$MP\ddot{X} + CP\dot{X} + KPX = R$

premultiply P^T

$(P^TMP)\ddot{X} + (P^TCP)\dot{X} + (P^TKP) = P^TR$

$\tilde{M}$ $\tilde{C}$ $\tilde{K}$

look for P's that make $\tilde{M}$, $\tilde{C}$, $\tilde{K}$ nice enough (e.g. diagonal)

once we find $X(t)$ $U(t) = PX(t)$

3.1.2 Modal analysis: Natural modes and frequencies

- Modal analysis facilitate an finding appropriate choice for **P**.
- Consider (174) but without the damping term,

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{0} \quad \text{Matrix equations without damping term} \tag{177}$$

M

$U = \phi \sin(\omega t - t_0)$

ϕ: natural mode, $n \times 1$ vector

assume this is the solution

$M_{n \times n}$ — # of free dofs

$\ddot{U} = -\phi \omega^2 \sin(\omega t - t_0)$

$(-M\omega^2\phi + K\phi)\sin(\omega t - t_0) = 0$

$$K\phi = \omega^2 M\phi$$

→ generalized eigenvalue problem

ω^2: natural frequency; ϕ: natural mode

K, M are symmetric & $K > 0$ positive $M > 0$ positive definite

$K\phi_i = \lambda_i M \phi_i$ λ_i & ϕ_i eigenvalue, vector # i

(1) $\lambda_i > 0$ ($\lambda_i = \omega_i^2 \rightarrow \omega_i = \sqrt{\lambda_i}$) $(u, v) = u \cdot Mv$

(2) for $i \neq j$ choose ϕ_i & ϕ_j ∋ $\phi_i M \phi_j = \delta_{ij}$ →

"Mass normal"

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(1) $\underbrace{\phi_i \cdot K\phi_i}_{>0} = \lambda_i \underbrace{\phi_i \cdot M\phi_i}_{>0}$ premultiply by ϕ_i

no sum on i

positive

$\rightarrow \lambda_i = \dfrac{\phi_i \cdot K\phi_i}{\phi_i \cdot M\phi_i} > 0 = \phi_i \cdot K\phi_j$ ($\phi_i M\phi_j$, $M = M^T$)

(2) $\phi_j \cdot (K\phi_i = \lambda_i M\phi_i)$ $\int \phi_j \cdot K\phi_i = \lambda_i \phi_j \cdot M\phi_i$

$\phi_j \cdot (K\phi_i = \lambda_i M\phi_i)$

$\phi_i \cdot (K\phi_j = \lambda_j M\phi_j)$

$\phi_j \cdot K\phi_i - \phi_i \cdot K\phi_j = \lambda_i \phi_j \cdot M\phi_i - \lambda_j \phi_i \cdot M\phi_j$

subtract

$0 = (\lambda_i - \lambda_j)\, \phi_i \cdot M\phi_j$

$(\lambda_i - \lambda_j)\phi_i \cdot M\phi_j = 0$

$\lambda_i \neq \lambda_j \longrightarrow \phi_i \cdot M\phi_j = 0$

if λ_i is repeated we can find distinct M normal eigenvalues for it

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- $\boldsymbol{\Phi}$ is an $n \times 1$ vector called mode shape or natural mode which captures the shape of vibration.
- ω is the natural frequency.
- Substituting (178) in (177) we obtain a generalized eigenproblem,

$$\mathbf{K}\boldsymbol{\Phi} = \omega^2 \mathbf{M}\boldsymbol{\Phi} \tag{179}$$

- The eigenproblem (179) yields n eigen solutions,

$$(\omega_1^2, \boldsymbol{\Phi}_1), (\omega_2^2, \boldsymbol{\Phi}_2), \cdots, (\omega_n^2, \boldsymbol{\Phi}_n) \tag{180}$$

- The eigenvectors (mode shapes / natural modes) are M-orthonormalized,

$$\boldsymbol{\Phi}_i^{\mathrm{T}} \mathbf{M} \boldsymbol{\Phi}_j = \delta_{ij}, \qquad \boldsymbol{\Phi}_i^{\mathrm{T}} \mathbf{M} \boldsymbol{\Phi}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{181a}$$

$$0 \leq \omega_1^2 \leq \omega_2^2 \leq \cdots \leq \omega_n^2 \tag{181b}$$

$\phi_i \left[\begin{array}{c} \phi_1^T \\ \hline \phi_2^T \\ \hline \vdots \\ \hline \phi_n^T \end{array} \right] M \left[\phi_1 \mid \phi_2 \mid \phi_3 \mid \cdots \mid \phi_n \right] = I_{n \times n}$

Φ^T $\qquad$ Φ

matrix of eigen vectors natural modes

$$\Phi^T M \Phi = I$$

$$\Phi^T K \Phi = \Omega^2 \qquad K\Phi = \omega^2 M$$

$$\Omega = \begin{bmatrix} \omega_1 & & \\ & \ddots & \\ & & \omega_n \end{bmatrix}$$

$$\Phi^{-1} = \Phi^T M \qquad ((\Phi^T M)\Phi = I)$$

Beyond finding natural modes and frequencies, modal analysis can also be used to solve

$$M\ddot{U} + KU = R \qquad U = \Phi X$$

Φ matrix

Φ^T

$$M\Phi\ddot{X} + K\Phi X = R$$

$$(\Phi^T M \Phi)\ddot{X} + \Phi^T K \Phi X = \Phi^T R$$

$(\Phi^T M \Phi) \rightarrow I$ $\qquad \Phi^T K \Phi \rightarrow \Omega^2$

$$\ddot{X} + \Omega^2 X = \Phi^T R$$

$$R_x = \Phi^T R$$

$$\ddot{X}_1 + \omega_1^2 X_1 = R_{x_1}(t)$$

$$\ddot{X}_2 + \omega_2^2 X_2 = R_{x_2}(t)$$

$$\vdots$$

$$\ddot{X}_n + \omega_n^2 X_n = R_{x_n}(t)$$

Single-degree-of freedom (SDOF) oscillators where we know the solution.

We also need IC

$$U(0) = U_0 \qquad \dot{U}(0) = \dot{U}_0$$

$U(0) = U_0 \qquad \dot{U}(0) = \dot{U}_0$

need $X(0)$ & $\dot{X}(0)$

$U = \phi X \rightarrow$

$X = \phi^{-1} U$

$\phi^T M U \qquad (\phi^T M \phi = I)$

$X(0) = \phi^T M U_0$

& $\dot{X}(0) = \phi^T M \dot{U}_0$

IC's for X

$\ddot{X}_1 + \omega_1^2 X_1 = R_{x_1}(t) \qquad X_1(0) = X_1(0) \qquad \dot{X}_1(0) = \dot{X}_1(0)$

$\ddots$

$\ddot{x}_n + \omega_n^2 x_n = R_{x_n}(t) \qquad x_n(0) \checkmark \qquad \dot{x}_n(0) \checkmark$

we need to solve N SDoF oscillators

they look like

$$\ddot{x} + \omega^2 x = f(t) \qquad x(0) = x_0 \qquad \dot{x}(0) = \dot{x}_0$$

what is the solution of this

$$x(t) = \frac{1}{\omega}\int_0^t f(\tau)\sin(t-\tau)\,d\tau + \underbrace{\frac{\dot{x}_0}{\omega}}_{\alpha}\sin\omega t + \underbrace{x_0}_{\beta}\cos\omega t$$

general sln for homogeneous soln

Duhamel's sln

$x(0) = 0 \rightarrow \beta = x_0$

$\dot{x}(0) = 0 \qquad \dot{x}_0 = \omega\alpha \rightarrow \alpha = \frac{\dot{x}_0}{\omega}$