# DC2020/03/04

Wednesday, March 4, 2020 1:12 PM

From last time

$$\boxed{M\ddot{U} + C\dot{U} + KU = F} \quad (1)$$

$U = \phi \sin \omega(t - t_0) \longrightarrow K\phi = \omega M \phi$

$\rightarrow$ solved $(\omega_i, \phi_i)$ $\quad \Phi = [\phi_1 | \cdots | \phi_N] \quad \Omega = \begin{bmatrix} \omega_1 & & \\ & \ddots & \\ & & \omega_n \end{bmatrix}$

modal analysis without C

$\Phi^T M \Phi = I \quad \Phi^T K \phi = \Omega^2$

$U = \phi X \rightarrow$ new variable

$\Phi^T \times \quad M\phi\ddot{X} + \underbrace{C\phi\dot{X}}_{\text{if we add it now}} + K\phi X = F$

$$\underbrace{\Phi^T M \phi}_{I} \ddot{X} + \phi^T C \phi \dot{X} + \underbrace{\Phi^T K \phi}_{\Omega^2} X = \phi^T F$$

$X = \Phi^{-1} U = \phi^T M U$

$$\boxed{\begin{aligned} &\text{In terms of } X: \\ &\ddot{X} + \phi^T C \phi \dot{X} + \Omega^2 X = \phi^T F \\ &X(t=0) = \phi^T M U(0) = \phi^T M U_0 \\ &\dot{X}(t=0) = \phi^T M \dot{U}_0 \end{aligned}}$$

decoupled if $C = 0$

$$\boxed{\begin{aligned} &\ddot{x}_i + \omega_i^2 x_i = r_i \\ &x_i^0 = \phi_i M U_0 = \\ &\dot{x}_i^0 = \phi_i M \dot{U}_0 = \end{aligned}}$$

need to solve this

need to solve this

to need to solve this

$$\boxed{\begin{aligned} &\ddot{x} + \omega^2 x = r \\ &x(0) = x_0 \\ &\dot{x}(0) = \dot{x}_0 \end{aligned}} \longrightarrow x(t) = \frac{1}{\omega}\int_0^t r \sin\omega(t-\tau)\,d\tau + \alpha\sin\omega t + \beta\cos\omega t$$

~~particular sln~~

$$\dot{x} = \int_0^t r\cos\omega(t-\tau) + \alpha\omega\cos\omega t - \beta\omega\sin\omega t$$

$$\ddot{x} = r - \omega\int_0^t r\sin\omega(t-\tau) - \alpha\omega^2\sin\omega t - \beta\omega^2\cos\omega t$$

$$\boxed{\alpha = \frac{\dot{x}^0}{\omega} \quad \beta = x_0}$$

Finally $\quad U = \sum_{i=1}^{n} \phi_i x_i(t)$

***EXAMPLE 9.6:*** **Calculate the transformation matrix $\Phi$ for the problem considered in Examples 9.1 to 9.4 and thus establish the decoupled equations of equilibrium in the basis of mode shape vectors.**

**For the system under consideration we have**

$$\mathbf{K} = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}; \qquad \mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}; \qquad \mathbf{R} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

$U_0 \quad \dot{U}_0$ are given

(a) Eigen problem

$$K\phi = \omega^2 M\phi \longrightarrow \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}\phi = \omega^2\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\phi$$

$$\omega_1^2 = 2; \qquad \boldsymbol{\phi}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \qquad \omega_1 = \sqrt{2}$$

$$\omega_2^2 = 5; \qquad \boldsymbol{\phi}_2 = \begin{bmatrix} \frac{1}{2}\sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} \end{bmatrix} \qquad \omega_2 = \sqrt{5}$$

Free vibration

$$A\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}\sin\sqrt{2}(t-t_a) + B\begin{bmatrix} \frac{1}{2}\sqrt{2/3} \\ -\sqrt{2/3} \end{bmatrix}\sin\sqrt{5}(t-t_b)$$

Sln 7— Dus T —

Sln 2 = RHS IC

$$\mathbf{K} = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}; \quad \mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}; \quad \mathbf{R} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

$U_0$ $\dot{U}_0$ are given

$$\ddot{X} + \omega^2 X = \phi^t R$$

$$\phi = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{2}\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \end{bmatrix}$$

$\phi_1$ $\phi_2$

$$\begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{2}\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} \end{bmatrix}\begin{bmatrix} 0 \\ 10 \end{bmatrix} \rightarrow \ddot{X}_1 + 2X_1 = \frac{10}{\sqrt{3}}$$

$$\ddot{x}_2 + 5x_2 = -10\sqrt{\frac{2}{3}}$$

$$\begin{bmatrix} X_1^0 \\ X_2^0 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{2}\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} \end{bmatrix}}_{\phi^t}\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}}_{M}\begin{bmatrix} U_1^0 \\ U_2^0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{X}_1^0 \\ \dot{X}_2^0 \end{bmatrix} = \quad // \quad // \quad \begin{bmatrix} \dot{U}_1^0 \\ \dot{U}_2^0 \end{bmatrix}$$

If IC is zero

$$X_1 = \frac{5}{\sqrt{3}} + a\cos\sqrt{2}t + b\sin\sqrt{2}t$$

$a = -\frac{5}{\sqrt{3}}$

$$X_2 = -2\sqrt{\tfrac{2}{3}} + c\cos\sqrt{5}t + d\sin\sqrt{5}t$$

$c = 2\sqrt{\tfrac{2}{3}}$

zero IC

$$\boxed{\begin{aligned} X_1(t) &= \frac{5}{\sqrt{3}}\left(1 - \cos\sqrt{2}t\right) \\ X_2(t) &= 2\sqrt{\tfrac{2}{3}}\left(-1 + \cos\sqrt{5}t\right) \end{aligned}}$$

sln to U?

$$U = \phi X = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{2}\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = X_1(t)\underbrace{\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}}_{\phi_1} + X_2(t)\underbrace{\begin{bmatrix} \frac{1}{2}\sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} \end{bmatrix}}_{\phi_2}$$

Hence, using (c), we have

$$\mathbf{U}(t) = \begin{bmatrix} \dfrac{1}{\sqrt{3}} & \dfrac{1}{2}\sqrt{\dfrac{2}{3}} \\ \dfrac{1}{\sqrt{3}} & -\sqrt{\dfrac{2}{3}} \end{bmatrix} \begin{bmatrix} \dfrac{5}{\sqrt{3}}(1 - \cos\sqrt{2}t) \\ 2\sqrt{\dfrac{2}{3}}(-1 + \cos\sqrt{5}t) \end{bmatrix} \qquad \text{(e)}$$

Evaluating the displacements from (e) for times $\Delta t, 2\Delta t, \ldots, 12\Delta t$, where $\Delta t = 0.28$, we obtain

| *Time* | $\Delta t$ | $2\Delta t$ | $3\Delta t$ | $4\Delta t$ | $5\Delta t$ | $6\Delta t$ | $7\Delta t$ | $8\Delta t$ | $9\Delta t$ | $10\Delta t$ | $11\Delta t$ | $12\Delta t$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $^t\mathbf{U}$ | 0.003<br>0.382 | 0.038<br>1.41 | 0.176<br>2.78 | 0.486<br>4.09 | 0.996<br>5.00 | 1.66<br>5.29 | 2.338<br>4.986 | 2.861<br>4.277 | 3.052<br>3.457 | 2.806<br>2.806 | 2.131<br>2.484 | 1.157<br>2.489 |

### 3.1.4 Use of the first few modes in the analysis

- To obtain the nodal solution U we need to solve for all $x_i(t), i \leq n$ to form (*cf.* (191)),

$$\mathrm{U} = \sum_{i=1}^{n} \boldsymbol{\Phi}_i x_i(t)$$

- where each $x_i(t)$ is obtained by solving an ODE (187a)

$$\ddot{x}_(t) + \omega_i^2 x_i(t) = r_i(t) = \boldsymbol{\Phi}_i^{\mathrm{T}} \mathbf{R}(t)$$

  or even one with the damping term as in (188) $(\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = f(t))$ in general.

- In practice not the coefficients of all modes $x_i$ are significant and in some application only a first few modes are considered in the solution,

$$\boxed{\mathrm{U} \approx \sum_{i=1}^{p} \boldsymbol{\Phi}_i x_i(t)}$$

only use p out of n natural modes

- Then instead of solving $n$ SDOF ODEs for $x_i(t)$, only $p$ ODEs are solved. In some applications $p \ll n$.

  - If we choose $p = n$ there is no difference in directly solving $\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$ or using modal dofs $x_i$ and then forming U by $\mathrm{U} = \sum_{i=1}^{n} \boldsymbol{\Phi}_i x_i(t)$. That is the solution of these two systems analytically is the same!
  - Obviously, once each scheme is numerically integrated, that is $n$ SDOF ODEs with modal analysis and 1 size $n$ U the source of error is the numerical integration of each system.
  - Once we replace $\mathrm{U} = \sum_{i=1}^{n} \boldsymbol{\Phi}_i x_i(t)$ by $\mathrm{U} = \sum_{i=1}^{p} \boldsymbol{\Phi}_i x_i(t)$ in the modal analysis there is another source of error: Truncation of modes.
  - The important question is when and how we can reduce the number of modes considered in the analysis. That is, how to decide $p$.
  - To answer this question, we analyze the SDOF ODE (188), with and without $c$.

#### 3.1.4.1 Analysis of a SDOF

For a SDOF the input the system is through source term $f(t)$ → Force frequency

$$\ddot{x} + \omega^2 x = R \sin \hat{\omega} t$$

$x_0 = \ldots$ given    example $x_0 = 0$

$\dot{x}_0 = \ldots$    $\dot{x}_0 = 1$

Duhamel's solution:

Duhamel's solution:

$$x(t) = \frac{1}{\omega}\int_0^t \underbrace{R\sin\hat{\omega}\tau}_{f(\tau)} \sin\omega(t-\tau)\,d\tau \;+\alpha\sin\omega t+\beta\cos\omega t$$

$$x(t) = \frac{R/\omega^2}{1-\hat{\omega}^2/\omega^2}\sin\hat{\omega}t + \alpha\sin\omega t+\beta\cos\omega t$$

if we use $x_0 = 0 \quad \dot{x}_0 = 1$

Sln is

$$x(t) = \underbrace{\frac{R/\omega^2}{1-\hat{\omega}^2/\omega^2}\sin\hat{\omega}t}_{\text{forced response}} + \overbrace{\frac{1}{\omega}\left(1 - \frac{R\,\hat{\omega}/\omega^2}{1-\hat{\omega}^2/\omega^2}\right)\sin\omega t}^{\text{IC \& can change}}$$

$x_{trans}$

if damping is present & long observation is sought this can be ignored

$$x(t) = D\,x_{stat} + x_{trans}$$

$$x_{stat}(t) = \frac{R}{\omega^2}\sin\hat{\omega}t$$

$$\cancel{\ddot{x}} + \omega x = R\sin\hat{\omega}t$$

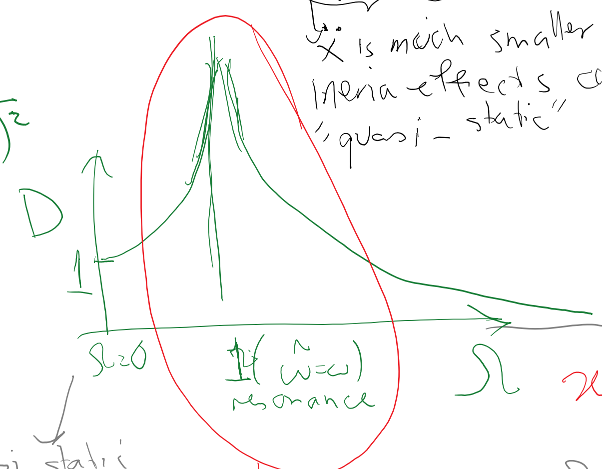
$\ddot{x}$ is much smaller than $\omega x$

Inertia effects can be neglected

"quasi-static" sln

$$D = \frac{1}{1-\left(\frac{\hat{\omega}}{\omega}\right)^2}$$

dimensionless forced frequency $\Omega = \frac{\hat{\omega}}{\omega}$




$x(t) \to 0$

$\Omega \to 0$ (quasi-static regime)

truly dynamic response

$\Omega \to \infty$

$\hat{\omega} \gg \omega$

$x(t) \approx x_{static}(t)$

very slowly varying load

very fast varying load

- and by using ICs,

$$x|_{t=0} = \beta, \dot{x}|_{t=0} = \frac{R\hat{\omega}/\omega^2}{1-\hat{\omega}^2/\omega^2} + \alpha \sin \omega t$$

- which gives,

$$\beta = 0, \qquad \alpha = \frac{1}{\omega} - \frac{R\hat{\omega}/\omega^3}{1-\hat{\omega}^2/\omega^2}$$

- Substituting $\alpha$ and $\beta$ in (193) we obtain,

$$x(t) = \frac{R/\omega^2}{1-\hat{\omega}^2/\omega^2}\sin\hat{\omega}t + \frac{1}{\omega}\left(1 - \frac{R\hat{\omega}/\omega^2}{1-\hat{\omega}^2/\omega^2}\right)\sin\omega t \tag{194}$$

- This can be written as,

$$x(t) = Dx_{\text{stat}} + x_{\text{trans}} \tag{195a}$$

$$x_{\text{stat}} = \frac{R}{\omega^2}\sin\hat{\omega}t \qquad \text{static response} \tag{195b}$$

$$x_{\text{trans}} = \left(\frac{1}{\omega} - \frac{R\hat{\omega}/\omega^3}{1-\hat{\omega}^2/\omega^2}\right)\sin\omega t \qquad \text{transient response} \tag{195c}$$

$$D = \frac{1}{1-\hat{\omega}^2/\omega^2} \qquad \text{dynamic load factor} \tag{195d}$$

- The static response is obtained by ignoring inertia terms $\ddot{x}$ in (192a),

$$\omega^2 x_{\text{stat}} = R\sin\hat{\omega}t \qquad \Rightarrow \qquad x_{\text{stat}} = \frac{R\sin\hat{\omega}t}{\omega^2}$$

- $x_{\text{trans}}$ specifically depends on the choice of IC ( $x_i|_{t=0} = 0$ $\dot{x}_{|_{t=0}} = 1$ herein or for example $x_i|_{t=0} = 1$ $\dot{x}_{|_{t=0}} = 0$).
- In addition if the damping is nonzero $c \neq 0$ ($\xi \neq 0$) the transient response damps out.
- Given the dependence of $x_{\text{trans}}$ on the form of IC and the fact that it damps out when damping is nonzero (as opposed to $x_{\text{stat}}$), makes $x_{\text{stat}}$ the main term to consider.
- Dynamic load factor $D$ shows how larger the dynamic response is relative to a static response (when inertia effects are ignored).

- The ratio of dynamic solution (total solution minus transient solution) to static solution (what we would have obtained by inertia effects) is dynamic amplification factor which took the value:

$$D = \frac{1}{1-\hat{\omega}^2/\omega^2} \qquad \text{undamped oscillator}$$

- There are three important ranges of $\hat{\omega}$ that we observe from this equation

1. $\hat{\omega} \ll \omega$ very slow varying load: $D \approx 1$: That is, we are in quasi-static regime and ignoring inertia effects $\ddot{x}$ (and damping as we ~~discuss later $\dot{x}$) is reasonable~~. Basically, loading rate is so slow that with any increment of loading the system has enough time to reach to a static equilibrium which is why we can ignore $\ddot{x}$ (and $\dot{x}$). In fact, for quasi-static loading regime, we can solve the solution by ignoring $\mathbf{M}$ (and $\mathbf{C}$) in (174) and have $\boxed{\mathbf{K}\Delta\mathbf{U} = \Delta\mathbf{R}}$ between time steps.
2. $\hat{\omega} \approx \omega$ which is at or near resonance: We have the largest $D$. For an undamped oscillator $D \to \infty$ as $\hat{\omega} \to \omega$, *i.e.*, when the loading resonance occurs. Later, we show that $D$ remains bounded when damping is added. Still $D$ can get larger than unity for $\hat{\omega}$ near the undamped resonance frequency.
3. $\hat{\omega} \gg \omega$ very fast varying/oscillating load: In this case the load oscillates so fast that the SDOF system does not have time to respond and basically dynamic respond would be close to zero. That is $D \to 0$ when $\hat{\omega}/\omega \to \infty$.

$M\ddot{U} + KU = R$

- Summary:

1. $\Omega \to 0$ very slow varying load: $D \approx 1$: A quasistatic solution (using $\mathbf{K}$ suffices): $\mathbf{K}\Delta\mathbf{U} = \Delta\mathbf{R}$.
2. $\omega = \mathcal{O}(1)$ (or $0 \ll \Omega \ll \infty$) near resonance (or nontrivial solution): Need to find the dynamic solution and $D$ can get very large.
3. $\Omega \to \infty$ very fast varying load: $D \approx 0$: System response is almost zero and do not need to be considered.

- So far, we considered one SDOF oscillator but varied loading frequency $\hat{\omega}$.
- Conversely that we have many SDOF oscillators but one or a short band of loading frequencies $\hat{\omega}$.

MDOF

- The latter is of more importance for the solution of (174) $\mathbf{M}\ddot{\mathbf{U}}+\mathbf{C}\dot{\mathbf{U}}+\mathbf{K}\mathbf{U}=\mathbf{R}$ where when the system could be diagonalized by modal decomposition we could get an equation of the form (187a) (in this equation $\mathbf{C}$ is assumed to be zero):

$$\ddot{x}_i(t)+\omega_i^2 x_i(t)=r_i(t)=\boldsymbol{\Phi}_i^{\mathrm{T}}\mathbf{R}(t) \tag{197}$$

- For the moment assume $\mathbf{R}(t)=\mathbf{R}_0\sin(\hat{\omega}t)$ (we will generalize the discussion to when there is a frequency band in the Fourier transform of the loading).

1 frequency force

$\omega_i \rightarrow$

$\Omega_i = \dfrac{\hat{\omega}}{\omega_i}$ → 0 ; → O(1) ; → ∞

$\omega_i$ is high (loading is slow) $D=1$ (quasi-static)

$\hat{\omega} \simeq \omega_i$ rel. close ≈ resonance Dynamic

$\omega_i$ is very low (loading is fast) $D=0$ (ignore it)

$K\Delta U = \Delta R$

$D \approx 1$

ignore

$\omega_1$ $\omega_2$ $\hat{\omega}$ $\omega_n$

around $\hat{\pi}$

$\Omega_i \to \infty$ $D \to 0$

$\omega_{min}$ $p_{min}$ modes

consider these modes in modal analysis

$\omega_{max}$ $p_{max}$

use quasi-static analysis

$R(t) = R_0 \sin(\hat{\omega} t)$
vector

$\ddot{x}_i + \omega_i^2 x_i = f_i$

$f_i = (\phi_i^t \cdot R_0)\sin(\hat{\omega} t)$
scalar harmonic force

$i = p_{min} - p_{max}$

$$U_{dyn} = \sum_{i=p_{min}}^{p_{max}} \phi_i \, x_i(t)$$

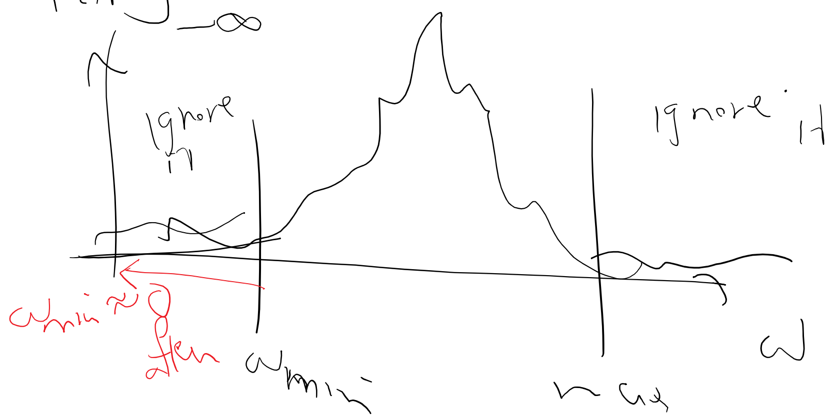
$\Delta U(t) = K^{-1}\Delta R(t)$
quasi-static sln

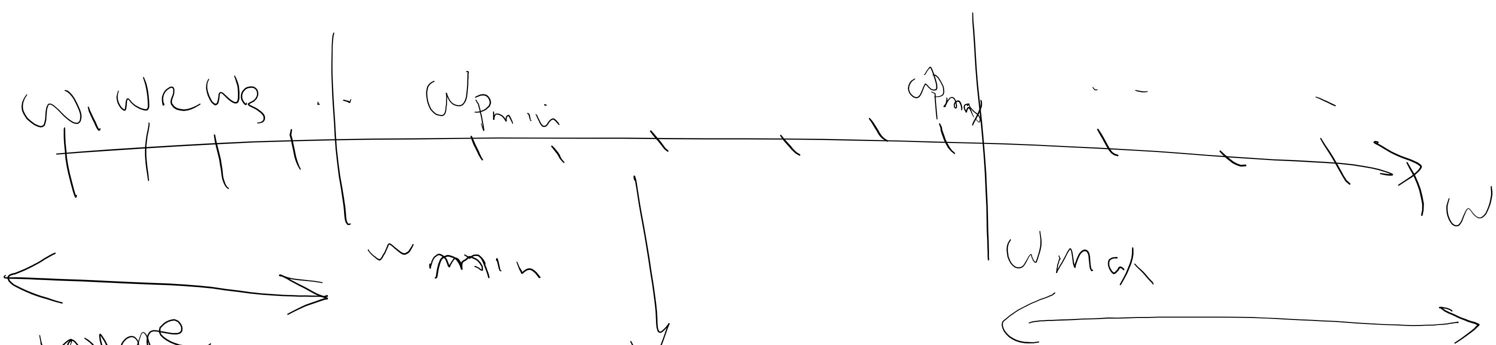
discrepancy between $R(t)$ & forces that have been active on few dynamic SDOF

$$U_{tot}(t) = U_{dyn}(t) + \underbrace{\Delta U(t)}_{\text{quasi-static}}$$

Complex loading

$$R(t) \longrightarrow \check{R}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} R(t)\, e^{-i\omega t}\, dt$$


$D \approx 0$

$$\ddot{x}_i + \omega_i^2 x_i = f_i$$

$$p_{min} \le i \le p_{max}$$

quasi-static analysis

$$K \Delta U = \Delta R$$

since mostly $\omega_{min} \approx 0$ (many applications)

is not much relevant

Most often we start from lowest frequency (because w_min is not much larger than w1 of the structure) all the way to maximum frequencies that the structure experiences. For these we do modal analysis

P is chosen such that $\omega_p > \omega_{max}$ loadings

sufficiently

instead of all n ← P

$$U_{dyn} = \sum_{i=1}^{P} x_i(t)\, \phi_i$$

$\ll NU$

$$U_{dyM} = \sum_{i=1} x_i(t)\,\varphi_i$$

$$U = U_{dyn} + \Delta U_{qs}$$

$$\& \Delta U_{q\,static} = K^{-1}\Delta R$$

#### 3.1.4.2 Use of first several natural modes

Now consider that loadings to the problem (ICs, BCs, and the source term) have a considerable frequency content between $[\hat{\omega}_m\ \hat{\omega}_M]$. That is their Fourier transform is almost zero or negligible outside this band. From the discussion above we conclude:

- Very high natural frequencies: $\omega_i \gg \hat{\omega}_M$: For these modes the loading frequency relative to them is very small. So, a quasi-static analysis would suffice. For example assume $\Delta\mathbf{R}^n$ is the part of force vector in time step $t_n$ that we do incorporate by only including the modal solutions of mode one to mode $p$. The response of the missing modes (high frequency modes) can be added by solving $\mathbf{K}\Delta\mathbf{U}^n = \Delta\mathbf{R}^n$.
- Medium natural frequencies: $\hat{\omega}_m \lesssim \omega_i \lesssim \hat{\omega}_M$: The SDOF solutions for these modes have nontrivial dynamic solution and must be solved. **These modes must be considered in the $p$ reduced set of modes.**
- Very low natural frequencies: $\omega_i \ll \hat{\omega}_m$: The loading frequency relative to these mode frequencies is very high and their effect is basically zero. Thus, these modes should not be considered in the $p$ reduced set of modes and their effect is basically zero.

In practice loadings often include lowest frequency modes and the structure can easily be vibrated in its lowest modes. That is $\hat{\omega}_m$ should often be considered zero and the case 3 above would be irrelevant in most practices. As discussed later, in certain vibration problems even some lowest modes can be omitted.

#### 3.1.4.3 Modal analysis: How to choose the reduced number of modes $p$

- Assume the highest relevant frequency content of the loading is $\hat{\omega}_M$. Choose $p$ natural modes / frequencies such that $\omega_p \gg \hat{\omega}_M$.
- In fact, the solution of natural modes and frequencies can be done one at a time, until the last frequency become irrelevant instead of solving for all natural frequencies / modes which can be very expensive for a large system.
- If the purpose is only obtaining the relevant natural modes / frequencies to the problem we are done.
- If we need to solve a transient problem of the form (174):

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$$

  we solve the SDOF ODEs similar to (187) (here damping term is added to SDOFs assuming that the damping term can also be diagonalized. This is discussed later):

$$\left.\begin{aligned} \ddot{x}_(t) + 2\xi_i\omega_i\dot{x}_(t) + \omega_i^2 x_i(t) = r_i(t) \\ r_i(t) = \boldsymbol{\Phi}_i^{\mathrm{T}}\mathbf{R}(t) \end{aligned}\right\} \qquad i = 1, \cdots, n \qquad \text{Uncoupled ODEs} \tag{198a}$$

$$\left.\begin{aligned} x_i{}^0 = x_i|_{t=0} \quad &= \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{U}^0 \\ \dot{x}_i{}^0 = s\dot{x}_{|t=0} \quad &= \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{U}}^0 \end{aligned}\right\} \qquad i = 1, \cdots, n \qquad \text{ICs} \tag{198b}$$

  ONLY for the first $p$ modes $(1 \leq p)$

- Compute the nodal solution vector $\mathbf{U}$ using the modal solutions,

$$\mathbf{U} \approx \sum_{i=1}^{p} \boldsymbol{\Phi}_i x_i(t) \tag{199}$$

- To add the quasistatic contribution of loading through the higher modes $(1 > p)$ that we did need include in the modal analysis ([199]) we compute the error in the load vector. Since $\mathbf{R} = \sum_{i=1}^{n} r_i \mathbf{M}\boldsymbol{\Phi}_i$ the error in $\mathbf{R}$ is,

$$\boxed{\Delta\mathbf{R} = \mathbf{R} - \sum_{i=1}^{p} r_i \mathbf{M}\boldsymbol{\Phi}_i} \tag{200}$$

error for not considering higher mode

The solution to the quasi-static problem, *i.e.*, it is time dependent but inertia and damping terms are ignored, corresponding to the modes that were not included in the modal analysis can be solved from,

$$\mathbf{K}\Delta\mathbf{U} = \Delta\mathbf{R} \tag{201}$$

- The total solution is the sum of the dynamic solution from ([199]) and the quasi-static solution from ([201]): $\mathbf{U}_{\text{tot}} = \sum_{i=1}^{p} \boldsymbol{\Phi}_i x_i(t) + \Delta\mathbf{U}$.

#### 3.1.4.4 Modal analysis vs. Direct numerical integration of $\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$

The choice between computing natural frequencies / modes vs. direct temporal integration of $\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$ depends on various aspects.

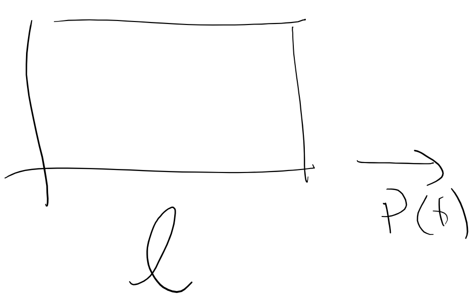
- **Need for natural frequencies / modes**: In Many applications, regardless of the need to solve $\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$, we need to obtain natural frequencies and modes which warrants a modal analysis.
- **Load frequency band** The frequency band of the loadings (BCs, ICs, body force) to a large extend determine how many modes ($p$) should be included in a modal analysis. We can define two classes of problems:
  1. Structural dynamic problems: Only the first few terms are sufficient for an accurate solution with modal analysis. For example, for earthquake loading in some cases only the 10 lowest modes need to be considered[Bathe, 2006]. If instead of using modal analysis, we directly want to integrate $\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$ in time, an implicit scheme is preferred because from accuracy perspective large time steps can be taken without affecting the solution much. Thus, the very small time step restriction of explicit methods can render them inefficient.
  2. Wave propagation problems: The loading frequency is very broadband. For example, in blast of shock loading $p$ can be as high as $2/3n$ [Bathe, 2006]. Often, for wave propagation problems explicit numerical integration schemes are used

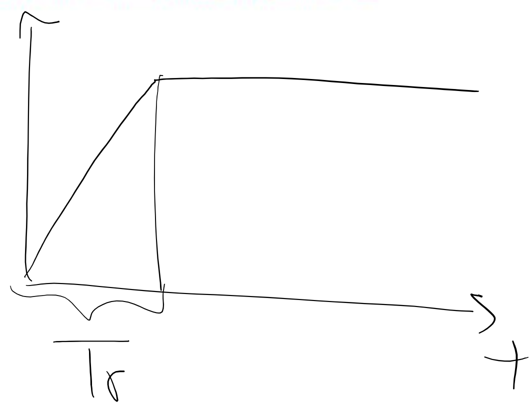


because they are inexpensive and their restrictive time step is not of major concern because from accuracy perspective small time steps should be taken.

Note: For certain vibration problems where loading has a narrow frequency band but the content is high frequency, *i.e.*, that is both $\hat{\omega}_m$ $\hat{\omega}_M$ are high but close to each other, we can omit the lowest natural modes whose frequencies are much smaller than $\hat{\omega}_m$ in the analysis. This reduced the number of modes that need to be considered.

- **Linearity of the problem**: Modal analysis is restricted to linear problems. Although, there may be cases that the nonlinear response can be linearized about the current state or approaches that can expand the applicability of such eigen mode analyses.
- **Influence of damping term**: If the damping term is nonzero AND nondiagonalizable with modal analysis we cannot directly use modal analysis for the solution of ([174]). Although, under structural dynamic loading we still can consider a much fewer modes $p \ll n$ but in this case $p$ $x_i$ terms will be coupled through the damping terms in their corresponding temporal ODEs. For further discussion refer to [Bathe, 2006] Example 9.11.




ramp time

$C = \sqrt{\frac{E}{\rho}}$

wave travel time

wave-travel time

$$T_w = \frac{L}{c}$$

$$T_r' = \frac{T_r}{T_w} \begin{cases} \to 0 & \text{wave propagation} \\ 1 \to (1) & \text{structural dynamics} \\ \infty & \text{quasi-static} \end{cases}$$