DC2020/03/09

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3.1.4.4 Modal analysis vs. Direct numerical integration of $\mathbf{M\ddot{U}} + \mathbf{C\dot{U}} + \mathbf{KU} = \mathbf{R}$

The choice between computing natural frequencies / modes vs. direct temporal integration of $\mathbf{M\ddot{U}} + \mathbf{C\dot{U}} + \mathbf{KU} = \mathbf{R}$ depends on various aspects.

- **Need for natural frequencies / modes**: In Many applications, regardless of the need to solve $\mathbf{M\ddot{U}} + \mathbf{C\dot{U}} + \mathbf{KU} = \mathbf{R}$, we need to obtain natural frequencies and modes which warrants a modal analysis.
- **Load frequency band** The frequency band of the loadings (BCs, ICs, body force) to a large extend determine how many modes (p) should be included in a modal analysis. We can define two classes of problems:
  1. Structural dynamic problems: Only the first few terms are sufficient for an accurate solution with modal analysis. For example, for earthquake loading in some cases only the 10 lowest modes need to be considered[Bathe, 2006]. If instead of using modal analysis, we directly want to integrate $\mathbf{M\ddot{U}} + \mathbf{C\dot{U}} + \mathbf{KU} = \mathbf{R}$ in time, an implicit scheme is preferred because from accuracy perspective large time steps can be taken without affecting the solution much. Thus, the very small time step restriction of explicit methods can render them inefficient.
  2. Wave propagation problems: The loading frequency is very broadband. For example, in blast of shock loading p can be as high as $2/3n$ [Bathe, 2006]. Often, for wave propagation problems explicit numerical integration schemes are used

3.1.5 Effect of damping matrix

3.1.5.1 Damping in a SDOF problem

- To better understand the behavior of $\mathbf{C}$ in the modal analysis and solution of (174) ($\mathbf{M\ddot{U}} + \mathbf{C\dot{U}} + \mathbf{KU} = \mathbf{R}$ we first discuss the response of a damped SDOF problem.
- Consider the damped SDOF problem (188),

$$\boxed{\ddot{x} + \overbrace{2\xi\omega}^{d}\dot{x} + \omega^2 x = f(t)} \tag{202}$$

$d = 2\xi\omega \longrightarrow \xi = \frac{d}{2\omega}$

- It is easier to discuss the response of the system and the dynamic amplification factor in frequency domain.
- The Fourier transform and the inverse Fourier transform in 1D are defined as,

$$\check{f}(\hat{\omega}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\imath\hat{\omega}t}\, \mathrm{d}t \quad \Leftrightarrow \tag{203a}$$

$$\underbrace{\check{(\dot{f})} = I\omega \check{f}}$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \check{f}(\hat{\omega}) e^{\imath\hat{\omega}t} \mathrm{d}\hat{\omega} \tag{203b}$$

provided that the integrals are defined in their domains.

$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = 0$ look at "resonance" modes

$x = e^{i\omega_r t} \longrightarrow \dot{x} = I\omega_r e^{I\omega_r t} \longrightarrow \ddot{x} = (I\omega_r)^2 e^{i\omega_r t}$

$\boxed{\omega = \text{undamped natural frequency}}$

plug in $\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = 0$

$\left[(i\omega_r)^2 + 2\xi\omega(i\omega_r) + \omega^2\right] e^{i\omega_r t} = 0 \longrightarrow$

$-\omega_r^2 + (2\xi\omega i)\omega_r + \omega^2 = 0 \longrightarrow$ Solve for ω_r

$$\boxed{\omega_r = \omega\left(\xi i \pm \sqrt{1-\xi^2}\right)}$$

3 cases $\xi < 1$ underdamped

$e^{i\omega_r t}$ $e^{-\omega\xi t}\, e^{\pm i\sqrt{1-\xi^2}\,\omega t}$

$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{}$

3 cases $\quad \xi < 1$ underdamped

$$e^{i\omega_r t} = e^{-\omega\xi t}\underbrace{e^{\pm i\sqrt{1-\xi^2}\,\omega t}}$$

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

natural solutions

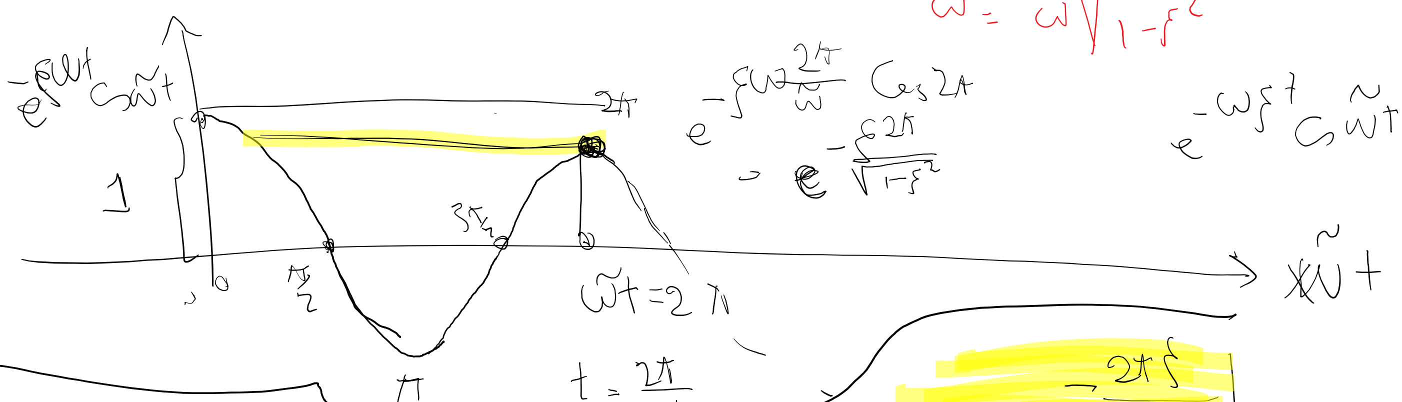
$$\boxed{\begin{array}{l} \xi = 0 \\ \cos\omega t \\ \sin\omega t \end{array}}$$

undamped natural s'lns

$$\boxed{\begin{array}{l} e^{-\omega\xi t}\cos(\omega\sqrt{1-\xi^2}\,t) \\ e^{-\omega\xi t}\sin(\underbrace{\omega\sqrt{1-\xi^2}}\,t) \\ \tilde{\omega} = \omega\sqrt{1-\xi^2} \end{array}}$$

natural frequency

$\tilde{\omega} = \omega\sqrt{1-\xi^2}$



$e^{-\xi\omega t}\cos\tilde{\omega}t$

$e^{-\xi\omega\frac{2\pi}{\tilde{\omega}}}\cos 2\pi$

$= e^{-\frac{\xi 2\pi}{\sqrt{1-\xi^2}}}$

$e^{-\omega\xi t}\cos\tilde{\omega}t$

$t = \frac{2\pi}{\tilde{\omega}}$

$$\boxed{AD = \frac{\text{amplitude of once cycle ahead}}{\text{current amplitude}} = e^{-\frac{2\pi\xi}{\sqrt{1-\xi^2}}}}$$

if we experimentally measure AD $\longrightarrow \xi$

2. $\xi = 1$ $\qquad \omega_r = \omega\xi i = \omega i$

critical damping $\qquad$ s'ln

$$\boxed{\begin{array}{l} e^{-\omega t} \\ t e^{-\omega t} \end{array}}$$

s'ln

$$\boxed{Ae^{-\omega t} + Bte^{-\omega t}}$$

find A & B from IC

3. $\xi > 1$ $\quad$ overdamped

$\omega_r = i(\omega\xi \pm \omega\sqrt{\xi^2-1})$

$$\omega_r = L(\omega\zeta \pm \omega\sqrt{\zeta^2-1})$$

$$e^{i\omega_1 t} \longrightarrow$$

$$A e^{-\omega_1 t} + B e^{-\omega_2 t}$$

dominate

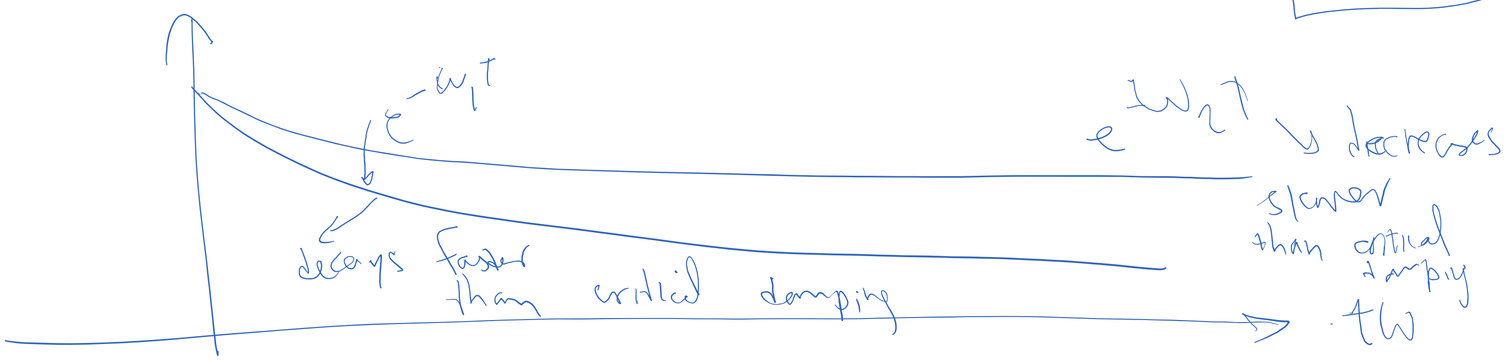
$$\omega_{1,2} = \omega\left(\zeta \pm \sqrt{\zeta^2-1}\right)$$

$$\omega_1 = \zeta + \sqrt{\zeta^2-1}$$

$$\boxed{\omega_2 = \zeta - \sqrt{\zeta^2-1}}$$

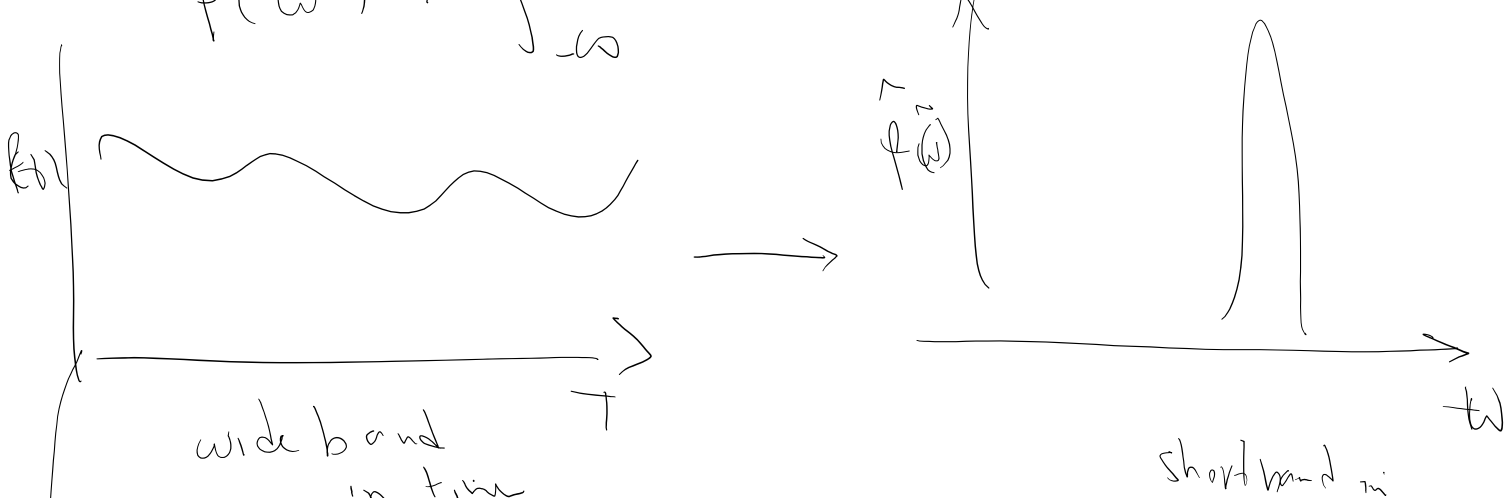
if $\zeta \gtrsim 1$

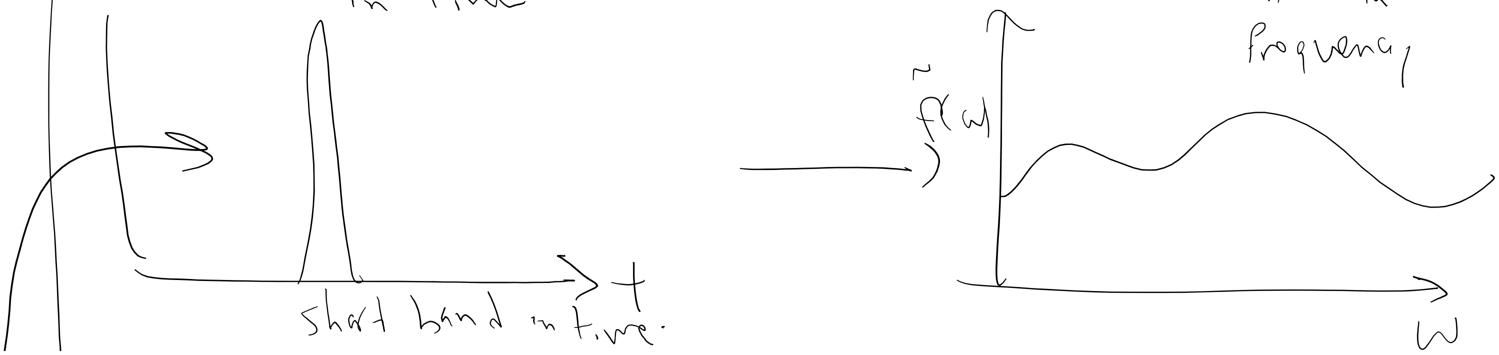
$$\boxed{\begin{array}{l}\omega_1 > \zeta \\ \omega_2 < \zeta\end{array}}$$



Above was the natural mode analysis for a damped SDOF

$$\hat{f}(\hat{\omega}) = \int_{-\infty}^{+\infty} f(x)\, e^{-i\hat{\omega} t}\, dx$$





$\longrightarrow t$ short band in time. $\longrightarrow \omega$

$\boxed{e^{i\hat{\omega}_0 t}} \longrightarrow \ ? \ \boxed{\delta(\hat{\omega} - \omega_0)}$

$\boxed{\delta(t - t_0)} \longrightarrow \ ? \ e^{i\hat{\omega} t_0}$

$$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = f$$

take the Fourier transform

$$(i\hat{\omega})^2 \hat{x} + 2\xi\omega(i\hat{\omega})\hat{x} + \omega^2 \hat{x}(\hat{\omega}) = \hat{f}(\hat{\omega})$$

$$\longrightarrow \boxed{\hat{x}_{dyn}(\hat{\omega}) = \frac{\hat{f}(\hat{\omega})}{(\omega^2 - \hat{\omega}^2) + 2i\xi\omega\hat{\omega}}} \quad (a)$$

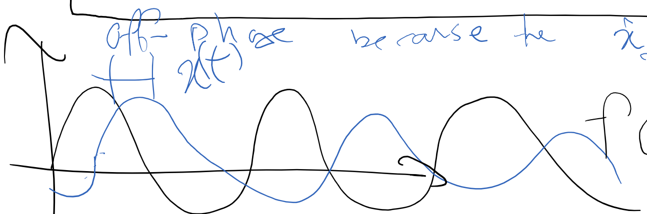
static solution

ignore these $\cancel{\ddot{x} + 2\xi\omega\dot{x}} + \omega^2 x = f(t) \longrightarrow$ (b) $\boxed{\hat{X}_{static}(\hat{\omega}) = \frac{\hat{f}(\hat{\omega})}{\omega^2}}$

$$\boxed{H(\Omega, \xi) = \frac{\hat{X}_{dyn}(\hat{\omega})}{\hat{X}_{static}(\hat{\omega})} = \frac{1}{(1 - \Omega^2) + i2\xi\Omega} \qquad \Omega = \frac{\hat{\omega}}{\omega}}$$

normalized frequency

off-phase because the $\hat{x}_{dyn}(\hat{\omega}) = (\text{complex number}) \times \hat{f}(\hat{\omega})$



$f(t) = \sin\hat{\omega}t$

$$\boxed{|H| = \left|\frac{X_{dyn}}{X_{stic}}(\hat{\omega})\right| = \frac{1}{\sqrt{(1-\Omega^2)^2 + 4\xi^2\Omega^2}}}$$

$$|H| = |x_{stat}(\omega)| = \frac{1}{\sqrt{(1-\Omega^2)^2 + 4\xi^2\Omega^2}}$$

$\xi = 0 \rightarrow H = \frac{1}{1-\Omega^2} \rightarrow \infty$ @ $\Omega = \frac{\hat\omega}{\omega} = 1$ at resonance

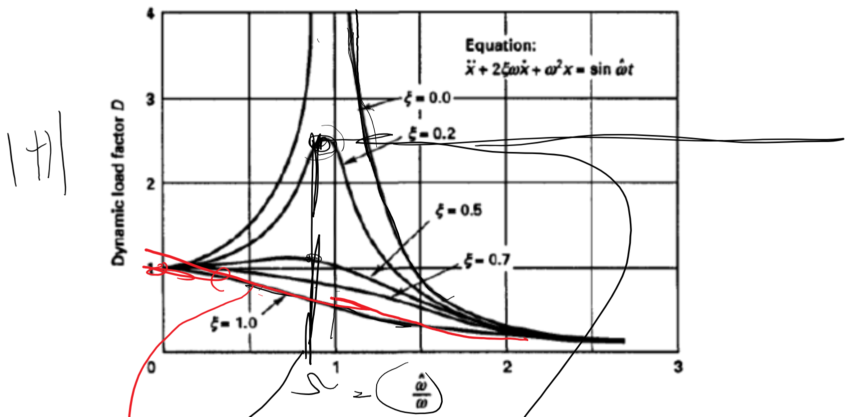


Figure 9.3 The dynamic load factor

$$D_M|_\Omega(\Omega,\xi) = D(\Omega_M,\xi) = \begin{cases} \frac{1}{2\xi\sqrt{1-\xi^2}} & \xi \le \frac{\sqrt{2}}{2} \quad (\Omega_M = \sqrt{1-2\xi^2}) \\ 1 & \text{otherwise} \quad (\Omega_M = 0;\ i.e., \text{ static loading}) \end{cases}$$

$\xi > \frac{\sqrt{2}}{2}$

- There are many important identities in Fourier transform analysis. One of relevance here is,

$$\left(\widehat{\frac{d^n f}{dt^n}}\right)(\hat\omega) = (\imath\hat\omega)^n \check{f}(\hat\omega) \tag{204}$$

- Accordingly, by taking the Fourier transform of (202) and application of (204) we have,

$$(\imath\hat\omega)^2\check{x}(\hat\omega) + 2\xi\omega(\imath\hat\omega)\check{x}(\hat\omega) + \omega^2\check{x}(\hat\omega) = \check{f}(\hat\omega) \quad \Rightarrow \tag{205a}$$

$$\check{x}_{\text{dyn}}(\hat\omega) = \frac{\check{x}(\hat\omega)}{(\omega^2 - \hat\omega^2) + 2\imath\xi\omega\hat\omega} \tag{205b}$$

- The subscript dyn is added to the solution to emphasize that this is the full dynamic solution.
- Now, we consider a quasi-static solution that ignores the inertia term $\ddot{x}$ and the damping term $\dot{x}$. Clearly, the solution to this system is,

$$\omega^2\check{x}_{\text{stat}}(\hat\omega) = \check{f}(\hat\omega) \quad \Rightarrow \quad \check{x}_{\text{stat}}(\hat\omega) = \frac{\check{f}(\hat\omega)}{\omega^2} \tag{206}$$

- Recalling that $\Omega = \frac{\hat\omega}{\omega}$ we define ratio of dynamic to static solution,

$$H(\Omega,\xi) := \frac{\check{x}_{\text{dyn}}(\hat\omega)}{\check{x}_{\text{stat}}(\hat\omega)} = \frac{1}{(1-\Omega^2) + 2\imath\xi\Omega}, \qquad \Omega = \frac{\hat\omega}{\omega} \tag{207}$$

- The fact that the ratio of the solution is a complex number means that their solution has a phase difference when $\xi \neq 0$.
- The amplification factor then will be the magnitude of $H(\Omega,\xi)$:

$$\boxed{D(\Omega,\xi) = |H(\Omega,\xi)| = \frac{1}{\sqrt{(1-\Omega^2)^2 + (2\xi\Omega)^2}}} \tag{208}$$

- We observe that when $\xi > 0$ (is nonzero) unlike the undamped oscillator D never approaches infinity at a resonance frequency.
- In fact, the maximum amplification factor is,

$$D_M|_\Omega(\Omega,\xi) = D(\Omega_M,\xi) = \begin{cases} \frac{1}{2\xi\sqrt{1-\xi^2}} & \xi \le \frac{\sqrt{2}}{2} \quad (\Omega_M = \sqrt{1-2\xi^2}) \\ 1 & \text{otherwise} \quad (\Omega_M = 0;\ i.e., \text{ static loading}) \end{cases} \tag{209}$$

added term $\hat\omega$

$\quad 1 \qquad$ otherwise $(\Omega_M = 0$; *i.e.*, static loading)

added term $\quad \omega$

- Furthermore the dynamic solution to (202) $(\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = f(t))$ is obtained by the Duhamel integral:

$$x(t) = \frac{1}{\bar{\omega}}\int_0^t f(\tau) e^{-\xi\omega(t-\tau)} \sin\bar{\omega}(t-\tau)\mathrm{d}\tau + e^{-\xi\omega t}(\alpha \sin\bar{\omega}t + \beta\cos\bar{\omega}t), \quad \text{where} \quad \bar{\omega} := \omega\sqrt{1-\xi^2} \tag{210}$$

where α and β are obtained from the ICs.

- Thus, we can analyze a SDOF with damping term.
- We will discuss when these SDOFs become relevant to solve (174) $(\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R})$.

" $e^{-\xi\omega t} \cos\tilde{\omega}t$

$e^{-\xi\omega t} \sin\tilde{\omega}t$

are natural resonance modes

3.1.5.2 Modal analysis of $\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = 0$ with $\mathbf{C} \neq 0$

$M\phi = \omega^2 K\phi$

from undamped natural frequencies ω_i & modes ϕ_i

- In practice **C** is often **not** compiled from local element matrices.
- This is unlike **M** and **C** matrices.
- In many applications, it is reasonable to actually start from

$$\ddot{x}_i(t) + 2\xi_i\omega_i\dot{x}_i(t) + \omega_i^2 x_i(t) = r_i(t), \qquad \text{where} \quad r_i(t) = \boldsymbol{\Phi}_i^{\mathrm{T}}\mathbf{R}(t) \tag{211}$$

$$M\ddot{U} + C\dot{U} + KU = R \qquad U = \phi X$$

$$\underbrace{\phi^T M \phi}_{I} \ddot{X} + \phi^T C \phi \dot{X} + \underbrace{(\phi^T K \phi)}_{\mathrm{diag}(\omega_i^2)} U = \phi R \qquad \Lambda^2$$

$$\boxed{(\phi^T C \phi)\dot{X} + \ddot{X} + \Lambda^2 X = r}$$

$r_i(t) = \phi_i^T R(t)$

if $\phi^T C \phi$ is diagonal we're good to go by again solving SDOF problems

$$\phi_i^T C \phi_j = \begin{cases} 0 & i \neq j \\ d_i & i = j \end{cases}$$

:)

$$\ddot{x}_i + \underline{d_i}\dot{x}_i + \omega_i^2 x_i = r_i(t) \qquad \text{no summation}$$

$$d_i = 2\omega_i \xi_i \longrightarrow \xi_i = \frac{d_i}{2\omega_i}$$

If we have Rayleigh damping:

$$C = \alpha M + \beta K$$

$$\phi^T C \phi = \alpha \phi^T M \phi + \beta \phi^T K \phi$$

$$= \alpha + \beta \Lambda^2$$

$$\boxed{\phi_i^T C \phi_i = \alpha + \beta \omega_i^2} = 2 \xi_i \omega_i$$

$$\longrightarrow \boxed{\xi_i = \frac{\alpha}{2\omega_i} + \frac{\beta \omega_i}{2}} \quad C = \alpha M + \beta K$$

$\omega_i \to 0$ $\quad \alpha M$ dominates

$\omega_i \to \infty$ $\quad \beta K$ $\quad \frac{\beta \omega_i}{2}$ dominates

- If for some reason, the explicit form of $\mathbf{C}$ is required, *e.g.*, when (174) ($\mathbf{M\ddot{U} + C\dot{U} + KU = R}$) is numerically integrated in time by explicit or implicit methods, we can form $\mathbf{C}$ by Cauchy series,

$$\mathbf{C} = \mathbf{M} \sum_{k=0}^{r-1} a_k [\mathbf{M}^{-1}\mathbf{K}]^k, \quad \text{where } a_k \text{ are solved from } r \text{ simultaneous equations :} \tag{212a}$$

SDOF ξ_i's are obtained as follows

$$\xi_i = \frac{1}{2}\left(\frac{a_0}{\omega_i} + a_1\omega_i + a_2\omega_i^3 + \cdots + a_{r-1}\omega_i^{2r-3}\right) \quad i = 0, \cdots, (r-1) \tag{212b}$$

and r is the number of damping coefficients given to define $\mathbf{C}$.

$$C = a_0 M + a_1 K + a_2 M (M^{-1} K)^2 + \cdots$$

EXAMPLE 9.9: Assume that for a multiple degree of freedom system $\omega_1 = 2$ and $\omega_2 = 3$, and that in those two modes we require 2 percent and 10 percent critical damping, respectively; i.e., we require $\xi_1 = 0.02$ and $\xi_2 = 0.10$. Establish the constants α and β for Rayleigh damping in order that a direct step-by-step integration can be carried out.

In Rayleigh damping we have

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K} \tag{a}$$

$$\phi^T C \phi = \alpha \phi^T M \phi + \beta \phi^T K \phi$$

$$2\omega_i \xi_i = \phi^T C \phi = \alpha + \beta \omega_i^2$$

$$\boxed{2\omega_i \xi_i = \alpha + \beta \omega_i^2}$$

$$\boxed{2\omega_i \xi_i = \alpha + \beta \omega_i^2}$$

$i=1 \quad \omega_1 = 2, \xi_1 = 0.02$

$i=2 \quad \omega_2 = 3, \xi_2 = .1 \longrightarrow$

$$\alpha + 4\beta = 2 \times 2 \times 0.02$$
$$\alpha + 9\beta = 2 \times 3 \times .1$$

$$\boxed{\begin{aligned}\alpha &= -.336\\ \beta &= .104\end{aligned}}$$

1st SDOF $\ddot{x}_1 + 2 \times (0.02)\dot{x}_1 + 4x_1 = r_1$

$\ddot{x}_2 + 2 \times .1 \dot{x}_2 + 9x_2 = r_2$

$$\boxed{\begin{aligned}\xi_n = ? &= \alpha + \beta\omega_i^2 / 2\omega_i\\ &= (-.336 + .104\,\omega_i^2) / 2\omega_i\end{aligned}}$$

$\ddot{x}_n + 2\omega_n \xi_n \dot{x}_n + \omega_n^2 x_n = r_n$

If C is nasty enough that we cannot use the Rayleigh damping or the expansion 212) with decent accuracy and number of terms [example: very different dampings in different parts of the problem, e.g. structure and foundation) then we can use complex analysis

$$\boxed{M\ddot{U} + C\dot{U} + KU = 0} \qquad (R)$$

$U = \tilde{U}e^{i\omega t}$ → complex

$\dot{U} - V = 0$

$\dot{U} = V \qquad M\dot{V} + CV + KU = 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & M \end{bmatrix}\dot{A} + \begin{bmatrix} 0 & -1 \\ K & C \end{bmatrix} A = \begin{bmatrix} 0 \\ R \end{bmatrix}$$

$$\boxed{\tilde{M}\dot{A} + \tilde{K}A = \tilde{R}}$$ Do eigen decomposition

or $\tilde{M}, \tilde{K}$

$A = \Phi e^{i\omega t}$

$$\boxed{(\tilde{M}i\omega + \tilde{K})\,\Phi = 0}$$

Complex eigen analysis

- For further discussion on this topic refer to [Bathe, 2006] section 9.3.3.
- Examples 9.9 and 9.10 from [Bathe, 2006] provide an example of this process.

3.1.6 Continuum (exact) natural frequencies and modes

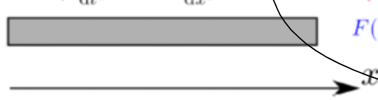
- We discussed how to obtain natural frequencies ω_i and modes $\boldsymbol{\Phi}_i$ for the discretized system $\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = 0$ (if $\mathbf{C} = 0$ or natural modes are $\mathbf{C}$-orthogonal).
- That is, if we discretize the system with an n dof system there will be n natural frequency, natural mode pairs $(\omega_i, \boldsymbol{\Phi}_i)$. $\Rightarrow$
- As we have a system with more dofs we obtain more natural frequency / mode pairs.
- These in fact approximate the continuum level natural frequencies / modes
- Below contrasts continuum and discrete modal analysis:

System Type	Basis for natural mode analysis	Number of natural modes/frequencies
Continuum	PDE, *e.g.*, $\rho A\frac{\mathrm{d}^2u}{\mathrm{d}t^2} - EA\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = 0$ for 1D elastic bar	∞
Discrete	ODE, $\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = 0$	finite n

$u(x = 0) = 0$ or

$F(x = 0) = EA\frac{\mathrm{d}u}{\mathrm{d}x}(x = 0) = 0$

$\rho\frac{\mathrm{d}^2u}{\mathrm{d}t^2} - EA\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = 0$



$u(x = L) = 0$ or

$F(x = L) = EA\frac{\mathrm{d}u}{\mathrm{d}x}(x = L) = 0$

- Consider the 1D bar example,

$$\rho A\frac{\mathrm{d}^2u}{\mathrm{d}t^2} - EA\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = 0 \qquad (216)$$

$u(x,t) = \phi(x)\,T(t)$

$\rho A\phi T'' - EA\phi'' T = 0$

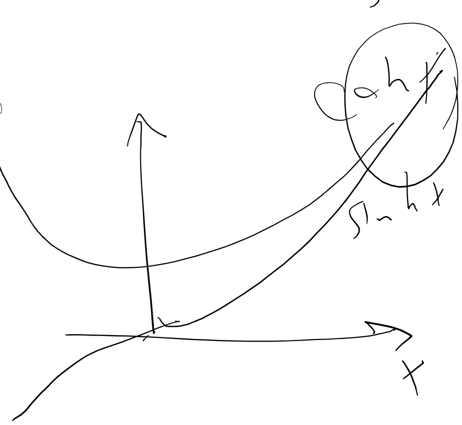
$c^2\frac{\phi''(x)}{\phi(x)} = \frac{T''(t)}{T(t)} = \alpha \rightarrow$ constant!

$c = \sqrt{\frac{E}{\rho}}$

function of

$\alpha > 0$

$\phi(x) = A\cosh\omega x + \sinh\omega x$



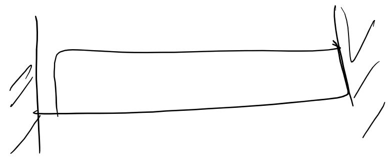
$\alpha = -\omega^2$

$$\frac{T''(t)}{T(t)} = -\omega^2 \longrightarrow T(t) = \sin(\omega t + \phi_0)$$

$$c^2 \frac{\phi''(x)}{\phi(x)} = -\omega^2 \longrightarrow \boxed{\phi(x) = A\sin\frac{\omega x}{c} + B\cos\frac{\omega x}{c}}$$

Consider $\phi(0)=0$ $\phi(L)=0$ case

Dirichlet



$$A\times 0 + B\times 1 = 0 \qquad B=0$$

$$A\sin\frac{\omega L}{c} + \cancel{B\cos\frac{\omega L}{c}} = 0 \qquad \boxed{A\sin\frac{\omega L}{c} = 0}$$

$$\boxed{A \neq 0} \qquad \boxed{\sin\frac{\omega L}{c} = 0}$$

Natural frequency ω_n	Natural mode (Φ_n)	Temporal function $T_n(t)$	$u_n(x,t)$
$\omega_n = n\pi\frac{c}{L}$	$\sin\left(\frac{\omega_n L}{c}\right)$	$\sin(\omega_n t + \phi_{0_n})$	$\sin\left(\frac{\omega_n L}{c}\right)\sin(\omega_n t + \phi_{0_n})$