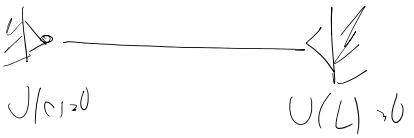
DC2020/03/11

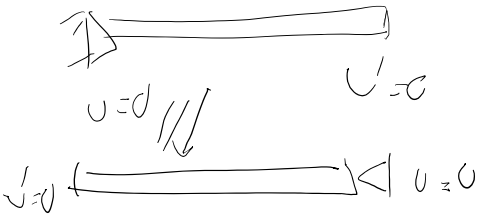
Wednesday, March 11, 2020 1:12 PM

Natural frequency ω_n	Natural mode (Φ_n)	Temporal function $T_n(t)$	$u_n(x,t)$
$\omega_n = n\pi\frac{c}{L}$	$\sin\left(\frac{\omega_n L}{c}\right)$	$\sin(\omega_n t + \phi_{0_n})$	$\sin\left(\frac{\omega_n L}{c}\right)\sin(\omega_n t + \phi_{0_n})$



other cases

more difficult solutions



$U = A_0 f_0(x,\omega) + A_1 - - - -$

Apply BCs

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$M(\omega)$

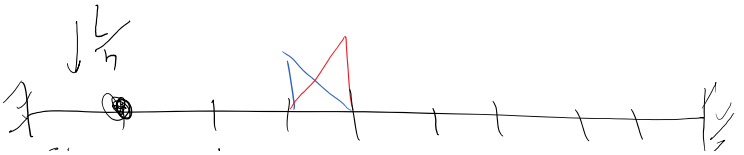
$\det M(\omega) = 0 \longrightarrow$ find natural frequencies &

$A_1 \ldots A_n \longrightarrow$ will determine the form of natural modes

- For other problems a similar analysis can be followed. For example, For the beam problem $\rho A\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - EI\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = 0$ we have a 4^{th} order in space and eigen modes will be build from $\sin\bar{x}, \cos\bar{x}, \sinh\bar{x}, \cosh\bar{x}$ for $\bar{x} = \frac{x}{L}$ and $\bar{L} = \sqrt[4]{\frac{EI}{\rho A\omega^2}}$. That is, $\Phi(x) = A_1\sin\bar{x} + A_2\cos\bar{x} + A_3\sinh\bar{x} + A_4\cosh\bar{x}$. Depending on the problem setup, we have essential or natural BC for

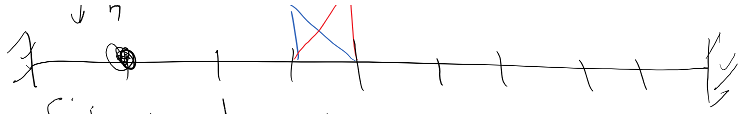
either displacement y or rotation $\theta = \frac{\mathrm{d}y}{\mathrm{d}x}$ at either of the 4 end points. The natural modes will be obtained such that the homogeneous 4×4 system (4 BCs for 4 unknowns A_1 to A_4) has nonzero solutions for $\mathbf{A} = [A_1\ A_2\ A_3\ A_4]^{\mathrm{T}}$; *i.e.*, $|\mathbf{A}| \neq 0$ and $\Phi(x)$ is not trivially zero.

3.1.7 Error analysis for natural frequencies and natural modes



$\omega_n = n\pi \frac{c}{L}$

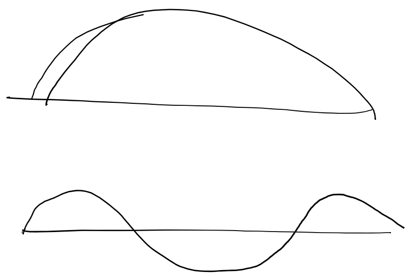
mode n $= \sin\left(\frac{n\pi x}{L}\right)$



finite elements p=1

$K_{n-1 \times n-1}$ $\quad K^e = \frac{AE}{L^e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$, $m^e = \frac{\rho A L^e}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$

$= \frac{AEn}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$ $\quad = \frac{\rho A L}{6n}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$

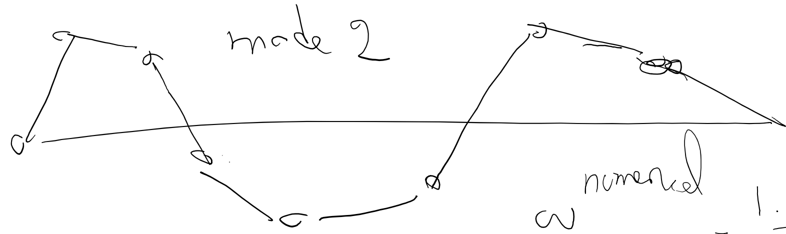


$\omega_3 = \frac{3\pi c}{L}$

$K_{n-1 \times n-1} = \frac{AEn}{L}\begin{bmatrix}2 & -1 & & \\ -1 & \ddots & & \\ & & 2 & -1\\ & & -1 & 2\end{bmatrix}$ $\quad M_{n-1 \times n-1} = \frac{\rho AL}{6n}\begin{bmatrix}4 & 1 & & \\ 1 & 4 & \ddots & \\ & & 4 & 1\\ & & 1 & 4\end{bmatrix}$

$K\phi = \omega^2 M\phi \longrightarrow$ n-1 ω's, n-1 φ's

mode 2



numerical $\omega_2 = \frac{1.8\pi c}{L}$

What is the error between numerical (discrete) modes and frequencies and exact ones?
Preliminary:

3.1.7 Error analysis for natural frequencies and natural modes

- If the differential equation has $2m$ highest spatial derivative, shape functions must be globally $\mathcal{C}^{m-1}$ continuous.
- Below, two cases for bar and beam examples are shown:

-	Bar	Beam
PDE	$\rho A \frac{d^2u}{dt^2} - EA\frac{d^2u}{dx^2} = 0$	$\rho A \frac{d^2y}{dt^2} - EI\frac{d^4y}{dx^4} = 0$
$2m$	2	4
Global continuity $\mathcal{C}^{m-1}$	0	1

$2 = 2m \Rightarrow m = 1$

order $= 4 = 2m \longrightarrow m = 2$

WRS $\int w(\rho A\ddot{u} - EI y^{(4)})$

IBP $\int w'' EI y''$

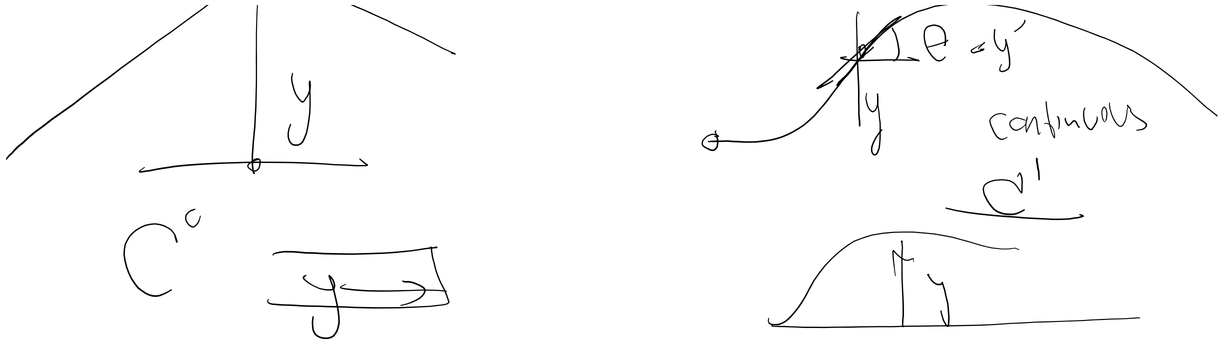
$m = 1$

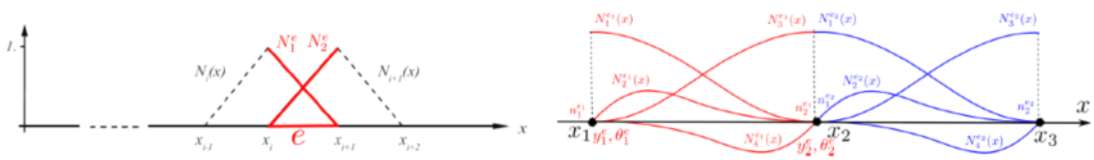
$C^{m-1} = C^0$ enough

H_2: $\int u^2, \int u'^2, \int u''^2 < \infty$

2 derivatives needed (C^2?) is enough C^1

$\theta = y'$



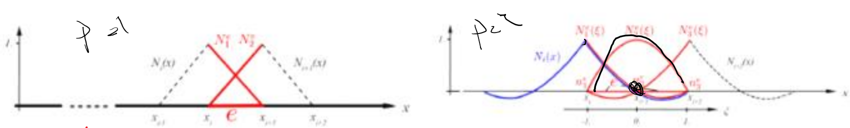


C^0 basis functions for bars

C^1 functions for beams

3.1.7.1 Preliminaries: FEM polynomial order p

-	Bar (1st order)	Bar (2nd)
Sample shape function	$N_1 = 1 - \frac{x^1}{L}$	$N_1 = 1 - \frac{3}{2}x + \frac{1}{2}x^2$ $1-3x+2x^2$
Maximum element order p	1	2



bar element $m=1$ $C^{m-1} = C^0$ continuity needed

- Note that the element maximum polynomial order p is not the same as minimum global required continuity $m-1$.
- For example, in the figure both elements are for the bar element with $m-1=0$ (C^0 global continuity).
- Yet, the element on the left is 0^{th} order $(p=0)$ and on the right 1^{st} order $(p=0)$.

m = half of PDE spatial der. order & elements are C^{m-1}

3.1.7.2 A priori error estimates for natural frequencies and modes

- *A priori* error estimates for natural frequencies and natural modes are in the form,

$$0 \le \omega_i^h - \omega_i \le Ch^{2(p+1-m)}\omega_i^{\frac{2p+2-m}{m}} \quad (222a)$$

$$||\Phi_i^h - \Phi_i||_m \le Ch^{(p+1-m)}\omega_i^{\frac{p+1}{m}} \quad (222b)$$

p order of element

$i \in \{1, 2, ..., N\}$ discrete dof

i is ith natural mode

natural frequency

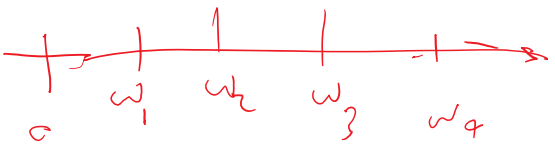
for higher modes expect larger errors

$()^h$ = discrete value

( ) = discrete value

natural frequency converge twice as fast

modes expect larger errors



$$\omega_i^h \geq \omega_i$$

numerical natural frequency is larger ~~smaller~~ than exact one

grid resolution h = the largest element size (size of an element is the radius of its circumscribing circle (2D) / sphere (3D))

1. $0 \leq \omega_i^h - \omega_i$, *i.e.*, having $\omega_i^h \leq \omega_i$ is not preserved once the Galerkin rules are violated [Hughes, 2012] (e.g., when reduced integration or incompatible modes are employed or when lumped mass matrix is used).

2. The rate of convergence (*i.e.*, power of h) of eigenvalues is twice that of eigenfunctions in the H^m (Hilbert m norm) [compare (222a) and (222b)]. That is,

   Natural frequencies converge twice faster than natural modes

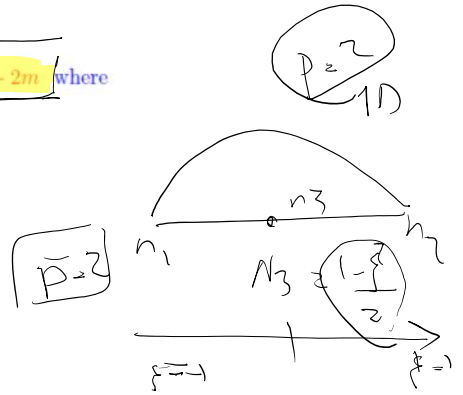
3. The appearance of powers of the natural frequencies on the right-hand sides of (222a) $\omega_i^{\frac{2p+2-m}{m}}$ and (222b) $\omega_i^{\frac{p+1}{m}}$ suggests that the **quality of approximation deteriorates for higher modes**. Recall that $\omega_0 < \omega_1 < \cdots < \omega_n$. This can be explained that higher modes have higher spatial variability (wave number) and for the same resolution of FEM mesh h it is more difficult to capture the exact solution.
4. **K**, **M** (and **C**) are often integrated numerically, *i.e.*, by quadrature.

4. **K**, **M** (and **C**) are often integrated numerically, *i.e.*, by quadrature.

   (a) For the convergence rates in h in (222) to hold:

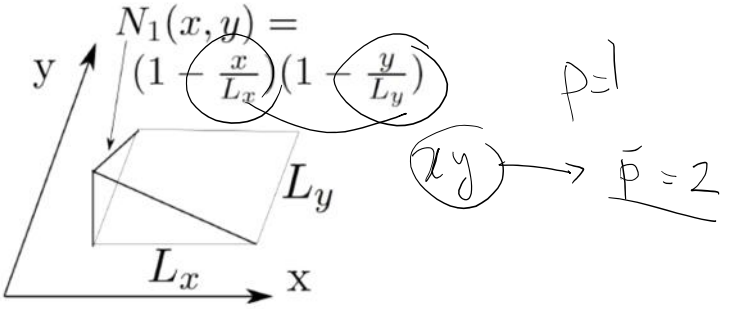
   The quadrature rule must be accurate enough to exactly integrate all monomials through order $\bar{p} + p - 2m$ where

   $\bar{p}$ = Order of the highest-order monomial appearing in the element shape functions,
   p = Order of the element
   $m - 1$ = Level of global continuity of FEM shape functions



Example for evaluating p and $\bar{p}$ from 2D bilinear finite element:

- For this bilinear element we observe the highest monomial order in shape functions is two. For example, N_1 has a term $\frac{xy}{L_x L_y}$ $\bar{p} = 2$.
- At the same time, this element is considered linear $p = 1$ because the highest complete polynomial space covered (for this given Cartesian geometry) is 1st. For example for it to be second order it should have terms like x^2y^2 which the shape functions of this element do not have such monomials.



Focus on 1D bar elements

$n=1$ $\quad$ λ $m=1$

$\bar{p} = p$ & $m = 1$

need to integrate things as accurate as

$$M_{ij} = \int_e N_i(x) N_j(x)\, dx$$

order p, order p

$2p$ order

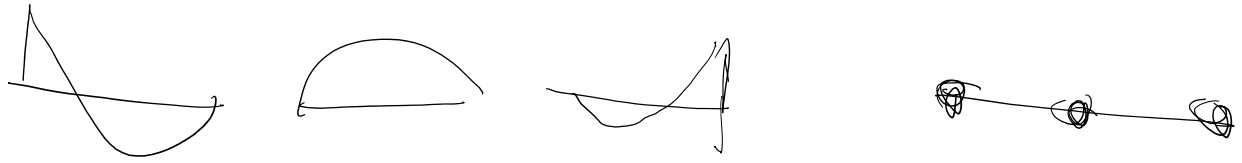
$2p-2$ is enough

Equal

$$\tilde{k}_{ij} = \int \frac{dN_i(x)}{dx} EA \frac{dN_j(x)}{dx}$$

k $2p-2$

$p=2$



$M(\text{order}) = 4$

$K(\text{order}) = 2$

Simpson's rule → integrate 2nd order polynomial exactly

$2p-2=2$ is needed for eigenmode estimates

- The sum of components of $\mathbf{M}^e$ is again equal to m^e.
- The mass matrix IS diagonal.
- As we will see this will results in DIAGONAL global level mass matrix.
- For the second order bar element we use the Simpson rule,

$$\text{Quadrature}\left(\int_0^L f(x)\,dx\right) = \frac{L}{6} f(0) + \frac{4L}{6} f(L/2) + \frac{L}{6} f(L) \tag{168}$$

which results in

$$\mathbf{M}^e = \frac{m^e}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ for second order 1D element} \tag{169}$$

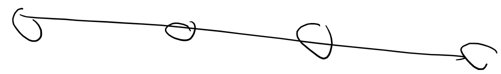
lumped mass matrices

note that,

- The lumped mass matrix is again diagonal.
- The diagonal values may NOT necessarily be equal.

$p=3$ element

p = 3 element



Mass = 6 th order

K 4 th order

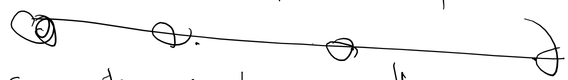
$p + p - 2m$ = 4 th order integration is needed

$2p - 2m$

NC

NC 4 pts

$p = p+1$



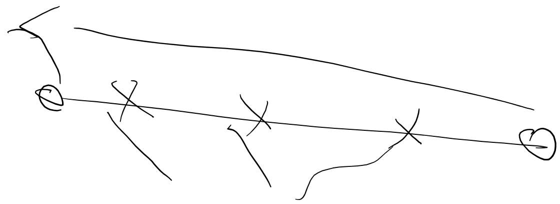
$1, x, x^2, x^3$ can be integrated correctly

Mass lumping order = 3

We need order 4

Gauss Lobatto

end points



we need

$2p-2$ order

$2p-3$

Counted twice

1D $p > 3, 4$ Gauss Lobatto just lacks one order

& still gives Mass Lumped matrix

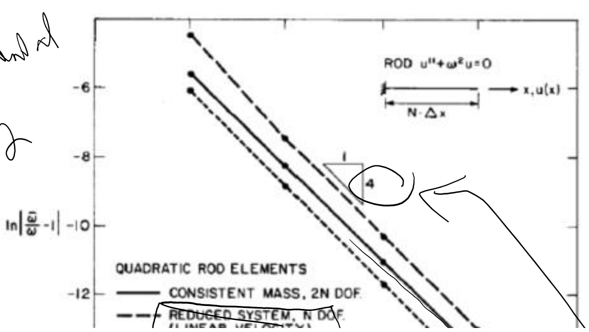
If reduced order quadrature is used, we need to enforce condition (a) above to ensure preserving the maximum convergence rates for modal quantities and enforce condition (b) to ensure convergence of modal quantities to exact values as $h \to 0$ (the minimum requirement)

This becomes critically important in the formation of lumped mass matrices where at times special quadrature rules, *e.g.*, Lobatto quadrature (it is similar to Gauss quadrature but maintains the end point values of the interval as quadrature points) need to be used to preserve maximum modal quantity convergence rates.

$\ell = 1$

fundamental frequency

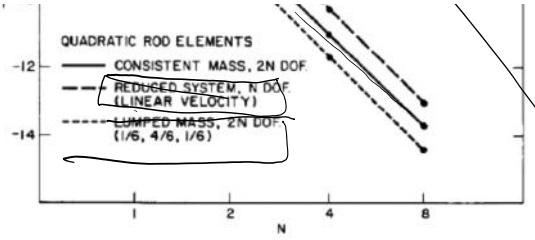
ω_1



p = 2 element bar, m = 1

$\omega_i^h - \omega_i \le C h^{2(p+1-m)}$ $\frac{2p+2-m}{\omega_i^m}$

$\omega_1^h - \omega_1 \le C h^{2(2+1-1)}$ ω_1



Convergence of fundamental (first) natural frequency for a quadratic 1D bar problem

$\omega_1^h - \omega_1 \leq C h^{2(2p-q)}$

$\left| \frac{\omega_1^h}{\omega_1} - 1 \right| \leq C h^4$

ω_1

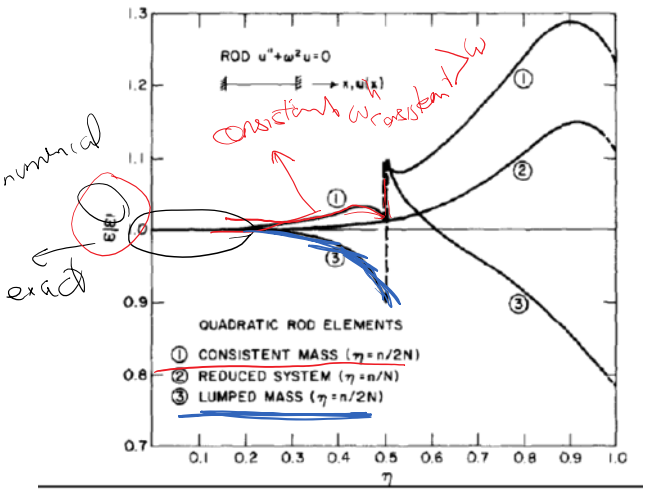
ω_1^4

spatial elements $1/h$

Check another condition $\omega_i^h \leq \omega_i$

require consistent mass

+ enough integration order $p+\bar{p}-2m$



N nodes

$L = 1$

$h = \frac{L}{N}$

$h = \frac{L}{N}$

natural mode #n

$= \sin \frac{n\pi x}{L}$

$n = 2$

$n = 3$

Δ

$\frac{n\pi\Delta}{L} = \pi$

$\Delta = \frac{L}{n}$

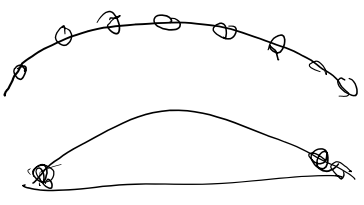
$h_{FEM} = \frac{L}{N}$

$h_{mode} = \Delta = \frac{L}{n}$

$\eta = \frac{h_{FEM}}{h_{mode}} = \frac{n}{N}$

$\eta \approx 0$ $h_{FEM} \ll h_{mode}$ ☺

$\eta \to 0$



- The plot demonstrate how the accuracy is modal frequencies decreases as higher mode frequencies are computed and compared with analytical ones.
- n is the mode number in the figure.
- N is the number of elements in the figure.
- As $\eta \propto \frac{n}{N}$ increases, *i.e.*, higher modes RELATIVE to the element size $h \propto 1/N$ are considered the error starts to increase.
- That is, the error is not merely a function of which natural mode is considered, but more on how accurately an element can approximate spatial variability (wave number) of a given natural mode.
- Again the violation of $\omega_i^h \leq \omega_i$ from (222b) for lumped mass matrix option is due to its violation of Galerkin methods required for of $\omega_i^h \leq \omega_i$ in (222b).
- Interestingly, we observe that the reduced integration order (of the mass matrix) performs better than full integration order with consistent mass option.
- The sharp jump of the error at $\eta = 0.5$ signals quick deterioration of calculating high natural frequencies (relative to the number of elements). This is one point of concern, and one of the advantages of isogeometric FEMs (a relatively new FE method) is having a much better performance in solving high natural frequencies (relative to the number of elements).

- To conclude figure below compares axial and flexural frequencies using different mass matrix options:
  - While relative error in the 1D bar problem is directly related to how many wave numbers an element can model, the absolute error quickly increases as mode number increases.
  - Consistent and lumped mass matrices option again provides higher and lower frequencies than exact ones while an average between two can provide both higher and lower values.
  - The errors in flexural natural frequencies are higher than axial ones for the same number of elements.
  - The unavailable natural frequency for beam problem with $n = 10$ and lumped mass matrix is that for this particular computational set-up natural modes beyond 4 are infinite, implying care that must be taken in using lumped mass matrix in practice.

TABLE 11.3-1. PERCENTAGE ERRORS OF COMPUTED NATURAL FREQUENCIES, USING [Cook, 2002] DIFFERENT MASS MATRICES [11.7]. FOR BEAM ELEMENTS WITH PARTICLE-MASS LUMPING, $\alpha = 0$ IN EQ. 11.3-3. STRUCTURES ARE UNIFORM AND MODELED BY ELEMENTS OF EQUAL LENGTH.

Mode number	Type of mass matrix used		
	Particle-mass lumps (%)	Average [m] (%)	Consistent [m] (%)
	Axial vibration of an eight-element bar, one end fixed, the other free		
1	−0.16	0.00	+0.16
2	−1.44	−0.03	+1.45
3	−3.97	−0.20	+4.05
4	−7.69	−0.79	+7.92
8	−32.42	−17.43	+15.94
	Flexural vibration of a five-element cantilever beam		
1	−1.80	−0.91	0.00
2	−5.90	−3.07	+0.05
3	−9.31	−5.03	+0.36
4	−13.62	−7.69	+1.17
10	Unavailable	+91.77	+67.83

4.1 Introduction to time marching schemes

- Our interest in this section is solving ODEs in time.
- Consider the following system of equations obtained by FEM discretization ((174))

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R} \tag{225}$$

- This equation is an ODE in time.
- Spatial derivative terms are already eliminated by using FE discretization in space.
- In this section we discuss methods by which we can solve first and second order temporal ODEs, *i.e.*, ODEs with initial conditions.
- There are different aspects for which we want to classify the solution of (225):

1. Hyperbolic vs. Parabolic: FE discretization of a hyperbolic and parabolic problems are often reduced to the following ODEs,

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R} \quad \text{Hyperbolic} \quad \text{Example: Elastodynamics } \mathbf{M} = \text{ mass, } \mathbf{C} = \text{ damping, } \mathbf{K} = \text{ stiffness matrices} \tag{226a}$$

$$\mathbf{M}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R} \quad \text{Parabolic} \quad \text{Example: Heat equation } \mathbf{M} = \text{ capacity, } \mathbf{K} = \text{ conductivity matrices} \tag{226b}$$

Note that any second order (or n^{th} order) ODE can be written in the form (226b) and the form itself does not directly imply if the underlying PDE is hyperbolic or parabolic. For example, we can express (226a) as,

$$\dot{\mathbf{U}} - \mathbf{V} = 0 \tag{227a}$$

$$\mathbf{M}\dot{\mathbf{V}} + (\mathbf{K}\mathbf{U} + \mathbf{C}\mathbf{V}) = \mathbf{R} \tag{227b}$$

where $\mathbf{V}$ represents the temporal derivative of $\mathbf{U}$, *i.e.*, velocity when $\mathbf{U}$ is displacement. We can express this in the form (226b):

$$\tilde{\mathbf{M}}\dot{\tilde{\mathbf{U}}} + \tilde{\mathbf{K}}\tilde{\mathbf{U}} = \tilde{\mathbf{R}}, \text{where} \tag{228a}$$

$$\tilde{\mathbf{U}} = \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix}, \quad \tilde{\mathbf{M}} = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{M} \end{bmatrix} \quad \tilde{\mathbf{K}} = \begin{bmatrix} 0 & -1 \\ \mathbf{K} & \mathbf{C} \end{bmatrix} \quad \tilde{\mathbf{R}} = \begin{bmatrix} 0 \\ \mathbf{R} \end{bmatrix} \tag{228b}$$

thus

- Having only one or two temporal derivatives on itself does not imply whether the underlying PDE is hyperbolic or PDE.
- Whether the underlying equation is hyperbolic or parabolic results in how the (smallest) element frequency scales versus its size.
  - Hyperbolic PDEs: $\omega^h \propto h \quad \Rightarrow \quad \Delta t \propto \frac{1}{h_{stm}}$
  - Parabolic PDEs: $\omega^h \propto h^2 \quad \Rightarrow \quad \Delta t \propto \frac{1}{h^2_{stm}}$

$_{stm}$

2. Single-degree-of-freedom vs. Multi-degree-of-freedom (SDOF vs. MDOF)

- For the temporal solution of PDEs we use a variety of different time marching schemes.
- As we observed from (226) they are often expressed in the form $\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$ or $\mathbf{M}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$.
- That is they are ODEs after FEM discretization.
- Similar to modal superposition from §3.1 for hyperbolic case ($\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$), we can reduce the general equation to n SDOF equations of the form:

$$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = f(t) \quad \text{corresponding to} \quad \mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R} \tag{229a}$$

$$\dot{x} + \lambda x = f(t) \quad \text{corresponding to} \quad \mathbf{M}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R} \tag{229b}$$