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4.3.1 Central Difference method for elastodynamics (an explicit LMS method)

- For the equation (226a) ($M\ddot{U} + C\dot{U} + KU = R$) we use central difference approximations for both $\ddot{U}$ and $\dot{U}$:

Central difference

$${}^{t}\ddot{U} = \frac{1}{\Delta t^2}\left({}^{t+\Delta t}U - 2\,{}^{t}U + {}^{t-\Delta t}U\right)$$

$t+\Delta t$ t_{n+1}

$t \longleftarrow t_n$

$t-\Delta t$ t_{n-1}

$${}^{t}\dot{U} = \frac{1}{2\Delta t}\left({}^{t+\Delta t}U - {}^{t-\Delta t}U\right)$$

plug into $M\ddot{U} + C\dot{U} + KU = R$ for t ($t = t_n$)

- For the equation (226a) ($M\ddot{U} + C\dot{U} + KU = R$) we use central difference approximations for both $\ddot{U}$ and $\dot{U}$:

$$\begin{aligned} {}^{t}\ddot{\mathbf{U}} &= \frac{1}{\Delta t^2}\left({}^{t-\Delta t}\mathbf{U} - 2\,{}^{t}\mathbf{U} + {}^{t+\Delta t}\mathbf{U}\right) \\ {}^{t}\dot{\mathbf{U}} &= \frac{1}{2\Delta t}\left(-{}^{t-\Delta t}\mathbf{U} + {}^{t+\Delta t}\mathbf{U}\right) \end{aligned} \tag{244}$$

- After plugging (244) in (226a) ($M\ddot{U} + C\dot{U} + KU = R$) for time step t_n we obtain,

$$\underbrace{\boxed{\left(\frac{1}{\Delta t^2}\mathbf{M} + \frac{1}{2\Delta t}\mathbf{C}\right)}}_{\text{No } K}{}^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{R} - \left(\mathbf{K} - \frac{2}{\Delta t^2}\mathbf{M}\right){}^{t}\mathbf{U} - \left(\frac{1}{\Delta t^2}\mathbf{M} - \frac{1}{2\Delta t}\mathbf{C}\right){}^{t-\Delta t}\mathbf{U} \tag{245}$$

How about time step 0

$t = \Delta t$ t_1

$t = 0$ t_0

$t = -\Delta t$ t_{-1}

$${}^{0}\ddot{U} = \frac{1}{\Delta t^2}\left({}^{-\Delta t}U - 2\,{}^{0}U + {}^{\Delta t}U\right)$$

$${}^{0}\dot{U} = \frac{1}{2\Delta t}\left(-{}^{-\Delta t}U + {}^{\Delta t}U\right)$$

${}^{-\Delta t}U = ?$

$${}^{-\Delta t}U = \underbrace{{}^{0}U - \Delta t\,{}^{0}\dot{U}}_{\text{IC's}} + \frac{1}{2}(-\Delta t)^2\,{}^{0}\ddot{U}$$

${}^{0}\ddot{U} = ?$

$$M\,{}^{0}\ddot{U} + C\,\underbrace{{}^{0}\dot{U} + K\,{}^{0}U}_{\text{IC's}} = {}^{0}F$$

$\longrightarrow$ End ${}^{0}\ddot{U}$

$$
{}^{-\Delta t}U_i = {}^{0}U_i - \Delta t\ {}^{0}\dot{U}_i + \frac{\Delta t^2}{2}{}^{0}\ddot{U}_i \tag{246}
$$

- Solution strategy from t_{n-1}, t_n to t_{n+1}:
  - From (245) the solution for U at t_{n+1} requires a linear system solution with matrix coefficient:

$$
\boxed{\hat{\mathrm{M}} = \frac{1}{\Delta t^2}\mathrm{M} + \frac{1}{2\Delta t}\mathrm{C},} \quad \text{where} \quad \hat{\mathrm{M}}\mathrm{U}^{n+1} = \mathrm{R}^n \tag{247}
$$

  - If the system is undamped we make the option to have the LHS matrix $\hat{\mathrm{M}} = \frac{1}{\Delta t^2}\mathrm{M}$. If $\mathrm{C} \propto \mathrm{M}$ we still have a similar problem.
  - If besides $\mathrm{C} = 0$ (or it being proportional to M) we have a lumped mass matrix, we do not need a matrix equation and update is followed as,

$$
\underbrace{{}^{t+\Delta t}U_i = {}^{t}\hat{R}_i\left(\frac{\Delta t^2}{m_{ii}}\right)} \quad \text{for} \quad {}^{t}\hat{\mathbf{R}} = {}^{t}\mathbf{R} - \left(\mathbf{K} - \frac{2}{\Delta t^2}\mathbf{M}\right){}^{t}\mathbf{U} - \left(\frac{1}{\Delta t^2}\mathbf{M}\right){}^{t-\Delta t}\mathbf{U} \tag{248}
$$

- Another advantage of lumped mass matrix is that:
  - Lumped mass matrix elongates period of moving waves.
  - Explicit method typically shorten the period of moving waves

  so matching explicit integrators and lumped mass matrices to some extend cancels the period error of the numerical method and is preferred from this perspective. On the other hand, if we had used consistent mass matrix that would as well would have shortened the period of moving waves and exaggerate the problem of explicit time integrators.
- Another important implication of not having K appearing in $\hat{\mathrm{M}}$ is that we do not need to actually assemble K.
- We can directly add contributions from stiffness to the global force vector R at the element level:

$$
\mathbf{K}\,{}^{t}\mathbf{U} = \sum_i \mathbf{K}^{(i)}\,{}^{t}\mathbf{U} = \sum_i {}^{t}\mathbf{F}^{(i)} \tag{249}
$$

- The elimination of assembly of K (as its contributions can be directly added to global load vector at the element level) and assembly of a nontrivial M (since it's diagonal only the diagonal values are assembled) substantially reduces computational cost as well as memory as none of these matrices are stored in memory (M is assembled to a vector).

TABLE 9.1 *Step-by-step solution using central difference method (general mass and damping matrices)*

A. *Initial calculations:*

1. Form stiffness matrix $\mathbf{K}$, mass matrix $\mathbf{M}$, and damping matrix $\mathbf{C}$.
2. Initialize ${}^{0}\mathbf{U}$, ${}^{0}\dot{\mathbf{U}}$, and ${}^{0}\ddot{\mathbf{U}}$.
3. Select time step Δt, $\Delta t \le \Delta t_{cr}$, and calculate integration constants:

$$
a_0 = \frac{1}{\Delta t^2}; \qquad a_1 = \frac{1}{2\,\Delta t}; \qquad a_2 = 2a_0; \qquad a_3 = \frac{1}{a_2}
$$

4. Calculate ${}^{-\Delta t}\mathbf{U} = {}^{0}\mathbf{U} - \Delta t\,{}^{0}\dot{\mathbf{U}} + a_3\,{}^{0}\ddot{\mathbf{U}}$.
5. Form effective mass matrix $\hat{\mathbf{M}} = a_0\mathbf{M} + a_1\mathbf{C}$.
6. Triangularize $\hat{\mathbf{M}}$: $\hat{\mathbf{M}} = \mathbf{LDL}^T$.

$\hat{M} = L\,U$

B. *For each time step:*

1. Calculate effective loads at time t:

$$
{}^{t}\hat{\mathbf{R}} = {}^{t}\mathbf{R} - (\mathbf{K} - a_2\mathbf{M})\,{}^{t}\mathbf{U} - (a_0\mathbf{M} - a_1\mathbf{C})\,{}^{t-\Delta t}\mathbf{U}
$$

2. Solve for displacements at time $t + \Delta t$:

$$
\mathbf{LDL}^T\,{}^{t+\Delta t}\mathbf{U} = {}^{t}\hat{\mathbf{R}}
$$

3. If required, evaluate accelerations and velocities at time t:

$$
{}^{t}\ddot{\mathbf{U}} = a_0({}^{t-\Delta t}\mathbf{U} - 2\,{}^{t}\mathbf{U} + {}^{t+\Delta t}\mathbf{U})
$$
$$
{}^{t}\dot{\mathbf{U}} = a_1(-{}^{t-\Delta t}\mathbf{U} + {}^{t+\Delta t}\mathbf{U})
$$

4.3.2 Houbolt method (an implicit LMS method for elastodynamics)

$k = 3$ LMS

$$
{}^{t+\Delta t}\dot{U} = \frac{1}{6\Delta t}\left(11\,{}^{t+\Delta t}U - 18\,{}^{t}U + 9\,{}^{t-\Delta t}U - 2\,{}^{t-2\Delta t}U\right)
$$

$$
{}^{t+\Delta t}\ddot{U} = \frac{1}{\Delta t^2}\left(2\,{}^{t+\Delta t}U - 5\,{}^{t}U + 4\,{}^{t-\Delta t}U - {}^{t-2\Delta t}U\right)
$$

$\boxed{{}^{t+\Delta t}U}$ 1

${}^{t}U$

${}^{t-\Delta t}U$

${}^{t-2\Delta t}U$

K

We write the update equation for $t+\Delta t \longrightarrow$ implicit

$$
\begin{aligned}
{}^{t+\Delta t}\dot{\mathbf{U}} &= \frac{1}{6\Delta t}\left(11\,{}^{t+\Delta t}\mathbf{U} - 18\,{}^{t}\mathbf{U} + 9\,{}^{t-\Delta t}\mathbf{U} - 2\,{}^{t-2\Delta t}\mathbf{U}\right)\\
{}^{t+\Delta t}\ddot{\mathbf{U}} &= \frac{1}{\Delta t^2}\left(2\,{}^{t+\Delta t}\mathbf{U} - 5\,{}^{t}\mathbf{U} + 4\,{}^{t-\Delta t}\mathbf{U} - {}^{t-2\Delta t}\mathbf{U}\right)
\end{aligned}
\tag{251}
$$

- After plugging these values in (226a) $(M\ddot{U} + C\dot{U} + KU = R)$ for t_{n+1} we obtain,

$$
\begin{aligned}
\left(\frac{2}{\Delta t^2}\mathbf{M} + \frac{11}{6\Delta t}\mathbf{C} + \mathbf{K}\right){}^{t+\Delta t}\mathbf{U} &= {}^{t+\Delta t}\mathbf{R} + \left(\frac{5}{\Delta t^2}\mathbf{M} + \frac{3}{\Delta t}\mathbf{C}\right){}^{t}\mathbf{U}\\
&\quad - \left(\frac{4}{\Delta t^2}\mathbf{M} + \frac{3}{2\Delta t}\mathbf{C}\right){}^{t-\Delta t}\mathbf{U} + \left(\frac{1}{\Delta t^2}\mathbf{M} + \frac{1}{3\Delta t}\mathbf{C}\right){}^{t-2\Delta t}\mathbf{U}
\end{aligned}
\tag{252}
$$

because it's implicit

- We observe that that $\mathbf{K}$ appears at the LHS and must be assembled.

- In addition if the problem were nonlinear, this methods update equation would have been nonlinear.
- This implicit method is unconditionally stable. Table from [Bathe, 2006] summarized the system update for time step t_{n+1}:

TABLE 9.2 *Step-by-step solution using Houbolt integration method*

A. *Initial calculations:*
1. Form stiffness matrix $\mathbf{K}$, mass matrix $\mathbf{M}$, and damping matrix $\mathbf{C}$.
2. Initialize ${}^0\mathbf{U}$, ${}^0\dot{\mathbf{U}}$, and ${}^0\ddot{\mathbf{U}}$.
3. Select time step Δt and calculate integration constants:

$$a_0 = \frac{2}{\Delta t^2};\quad a_1 = \frac{11}{6\Delta t};\quad a_2 = \frac{5}{\Delta t^2};\quad a_3 = \frac{3}{\Delta t};\quad a_4 = -2a_0;$$
$$a_5 = \frac{-a_3}{2};\quad a_6 = \frac{a_0}{2};\quad a_7 = \frac{a_3}{9}$$

4. Use special starting procedure to calculate ${}^{\Delta t}\mathbf{U}$ and ${}^{2\Delta t}\mathbf{U}$.
5. Calculate effective stiffness matrix $\hat{\mathbf{K}}$: $\hat{\mathbf{K}} = \mathbf{K} + a_0\mathbf{M} + a_1\mathbf{C}$.
6. Triangularize $\hat{\mathbf{K}}$: $\hat{\mathbf{K}} = \mathbf{LDL}^T$.

For these LMS methods with high value of k, many times we use a simple time marching scheme for the first k time steps to build enough time step values (for example central difference) and after that we can switch to our high order LMS method.

Disadvantages of LMS methods:
1. For early steps we need to use a lower order method to build enough prior steps.
2. The time step is fixed because the difference formulas for multiple steps are obtained by assuming Delta t being fixed.

We want to discuss methods that remove the fixed time step constraint

4.4 Multivariate single-step methods

a) For time step n + 1, they only require the solutions from time step n. Unlike LMS methods it does not go further back.
b) Primary unknowns are values, time derivative, second time derivatives, etc. of the unknown.

$M\ddot{U} + C\dot{U} + KU = F$

time step n: unknowns are $\underbrace{\boxed{{}^n\ddot{U},\ {}^n\dot{U}},\ {}^nU}_{\text{new unknowns}}$

c) Since only one step back is needed, we can easily change the time step (for example when more accuracy is needed, refine the time step).

- In contrast to the explicit central different and implicit Houbolt methods that require values for t_{n-1} and earlier for the solution of t_n, we are looking for solution schemes that only use values for t_n.
- To make this approach to work, we need to add $\dot{U}$ and $\ddot{U}$ (velocity and acceleration) to U as other variables of the problem that should be updated from step t_n to t_{n+1}.
- The values for $\dot{U}$ and $\ddot{U}$ may be kept in the formulation (as they are anyhow generally needed) or eliminated in the final form of update from t_n to t_{n+1}.
- We discuss two very important examples from these approaches: θ-Wilson and Newmark methods.

The θ-Wilson method

4.4.1 The θ-Wilson method

- In θ-Wilson method acceleration is linearly interpolated between time step t_n (t) to $\theta\Delta t$ after that

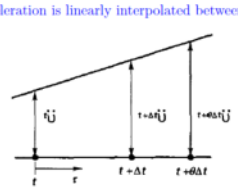
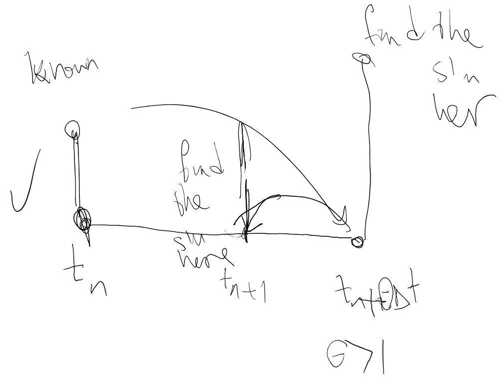
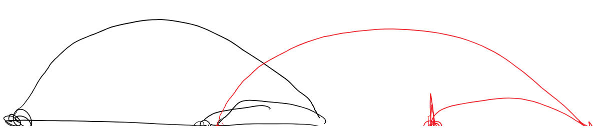
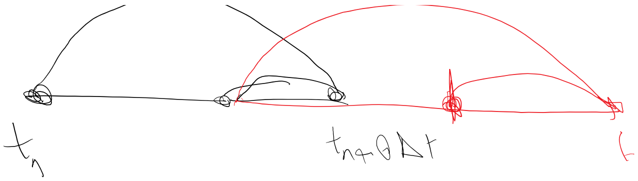


Figure 9.1 Linear acceleration assumption of Wilson θ method







$\theta > 1$

The assumption is that acceleration is linear between tn and tn + $\theta\Delta t$

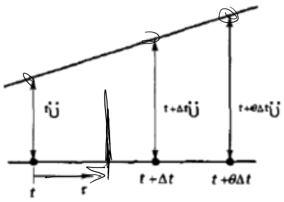


Figure 9.1 Linear acceleration assumption of Wilson θ method

$$^{t+\tau}\ddot{U} = {}^t\ddot{U} + \frac{\tau}{\theta\Delta t}\left({}^{t+\theta\Delta t}\ddot{U} - {}^t\ddot{U}\right) \qquad \left(\tau = \theta\Delta t \quad {}^{t+\theta\Delta t}\ddot{U} = 1\right)$$

$$^{t+\tau}\dot{U} = {}^t\dot{U} + \tau\,{}^t\ddot{U} + \frac{\tau^2}{2\theta\Delta t}\left({}^{t+\theta\Delta t}\ddot{U} - {}^t\ddot{U}\right)$$

integrate 1 more time

$$^{t+\tau}U = {}^tU + \tau\,{}^t\dot{U} + \frac{\tau^2}{2}\,{}^t\ddot{U} + \frac{\tau^3}{6\theta\Delta t}\left({}^{t+\theta\Delta t}\ddot{U} - {}^t\ddot{U}\right)$$

unknown

use $\tau = \theta\Delta T$ consider this point

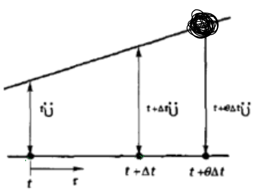


Figure 9.1 Linear acceleration assumption of Wilson θ method

expressed in terms of $^{t+\theta\Delta t}U$

$$\boxed{^{t+\theta\Delta t}\dot{\mathbf{U}}} = {}^t\dot{\mathbf{U}} + \frac{\theta\,\Delta t}{2}\left({}^{t+\theta\Delta t}\ddot{\mathbf{U}} + {}^t\ddot{\mathbf{U}}\right) \qquad \text{(a)}$$

$$^{t+\theta\Delta t}\mathbf{U} = {}^t\mathbf{U} + \theta\,\Delta t\,{}^t\dot{\mathbf{U}} + \frac{\theta^2\,\Delta t^2}{6}\left({}^{t+\theta\Delta t}\ddot{\mathbf{U}} + 2\,{}^t\ddot{\mathbf{U}}\right) \qquad \text{(b)} \tag{255}$$

$^{t+\theta\Delta t}\ddot{U} = \dots {}^{t+\theta\Delta t}U$

$^{t+\theta\Delta t}U$, $^{t+\theta\Delta t}\dot{U}$, $^{t+\theta\Delta t}\ddot{U}$ are unknowns

eventually $^{t+\theta\Delta t}\dot{U}$ & $^{t+\theta\Delta t}\ddot{U}$ are expressed in terms of $^{t+\theta\Delta t}U$

- To obtain $^{t+\theta\Delta t}\dot{\mathbf{U}}$ and also have values for the next time step t_{n+1}, we plug in $t = t + \Delta t$ (t refers to t_n) in (254) to obtain,

$$\boxed{^{t+\theta\Delta t}\dot{\mathbf{U}}} = {}^t\dot{\mathbf{U}} + \frac{\theta\,\Delta t}{2}\left({}^{t+\theta\Delta t}\ddot{\mathbf{U}} + {}^t\ddot{\mathbf{U}}\right) \qquad \text{(a)}$$

$$^{t+\theta\Delta t}\mathbf{U} = {}^t\mathbf{U} + \theta\,\Delta t\,{}^t\dot{\mathbf{U}} + \frac{\theta^2\,\Delta t^2}{6}\left({}^{t+\theta\Delta t}\ddot{\mathbf{U}} + 2\,{}^t\ddot{\mathbf{U}}\right) \qquad \text{(b)} \tag{255}$$

- To obtain $^{t+\theta\Delta t}\ddot{\mathbf{U}}$ and $^{t+\theta\Delta t}\dot{\mathbf{U}}$ we do:

– First find ${}^{t+\theta\Delta t}\ddot{\mathrm{U}}$ from (255)(b).
– Plug ${}^{t+\theta\Delta t}\ddot{\mathrm{U}}$ in (255)(a) to obtain ${}^{t+\theta\Delta t}\dot{\mathrm{U}}$.

- This provides values for the unknowns:

$$ {}^{t+\theta\Delta t}\ddot{\mathbf{U}} = \frac{6}{\theta^2\,\Delta t^2}({}^{t+\theta\Delta t}\mathbf{U} - {}^t\mathbf{U}) - \frac{6}{\theta\,\Delta t}{}^t\dot{\mathbf{U}} - 2\,{}^t\ddot{\mathbf{U}} $$
$$ {}^{t+\theta\Delta t}\dot{\mathbf{U}} = \frac{3}{\theta\,\Delta t}({}^{t+\theta\Delta t}\mathbf{U} - {}^t\mathbf{U}) - 2\,{}^t\dot{\mathbf{U}} - \frac{\theta\,\Delta t}{2}{}^t\ddot{\mathbf{U}} $$

the only unknown is ${}^{t+\theta\Delta t}U$

How to solve it?

$M\ddot{U} + C\dot{U} + KU = F$ solve this for $t+\theta\Delta t$

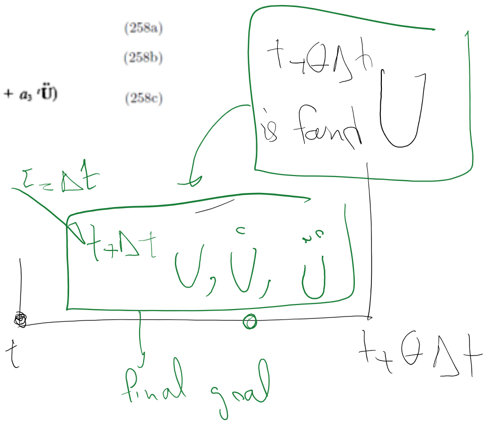
- Thus from (256) the unknowns ${}^{t+\theta\Delta t}\ddot{\mathrm{U}}$ and ${}^{t+\theta\Delta t}\dot{\mathrm{U}}$ are written in terms of one unknown vector ${}^{t+\theta\Delta t}\mathrm{U}$. Subsequently, we plug these values in (226a) ($\mathrm{M}\ddot{\mathrm{U}} + \mathrm{C}\dot{\mathrm{U}} + \mathrm{KU} = \mathrm{R}$) for $t + \theta\Delta t$,

$$ \mathbf{M}\,{}^{t+\theta\Delta t}\ddot{\mathbf{U}} + \mathbf{C}\,{}^{t+\theta\Delta t}\dot{\mathbf{U}} + \mathbf{K}\,{}^{t+\theta\Delta t}\mathbf{U} = {}^{t+\theta\Delta t}\overline{\mathbf{R}} $$
$$ {}^{t+\theta\Delta t}\overline{\mathbf{R}} = {}^t\mathbf{R} + \theta({}^{t+\Delta t}\mathbf{R} - {}^t\mathbf{R}) \qquad (257) $$

- to obtain ${}^{t+\theta\Delta t}\mathrm{U}$ from the system below,

$$ \hat{\mathrm{K}}\,{}^{t+\theta\Delta t}\mathrm{U} = {}^{t+\theta\Delta t}\hat{\mathrm{R}} \qquad (258a) $$
$$ \boxed{\hat{\mathrm{K}} = \mathrm{K} + \frac{6}{(\theta\Delta t)^2}\mathrm{M} + \frac{3}{\theta\Delta t}\mathrm{C}} \qquad (258b) $$
$$ {}^{t+\theta\Delta t}\hat{\mathbf{R}} = {}^t\mathbf{R} + \theta({}^{t+\Delta t}\mathbf{R} - {}^t\mathbf{R}) + \mathbf{M}(a_0\,{}^t\mathbf{U} + a_2\,{}^t\dot{\mathbf{U}} + 2\,{}^t\ddot{\mathbf{U}}) + \mathbf{C}(a_1\,{}^t\mathbf{U} + 2\,{}^t\dot{\mathbf{U}} + a_3\,{}^t\ddot{\mathbf{U}}) \qquad (258c) $$

obtain ${}^{t+\theta\Delta t}U$



- Once we obtain ${}^{t+\theta\Delta t}\mathrm{U}$ we obtain ${}^{t+\theta\Delta t}\dot{\mathrm{U}}$ and ${}^{t+\theta\Delta t}\ddot{\mathrm{U}}$ from (256).

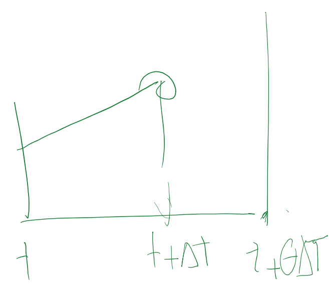
$$ {}^{t+\theta\Delta t}\ddot{\mathbf{U}} = \frac{6}{\theta^2\,\Delta t^2}({}^{t+\theta\Delta t}\mathbf{U} - {}^t\mathbf{U}) - \frac{6}{\theta\,\Delta t}{}^t\dot{\mathbf{U}} - 2\,{}^t\ddot{\mathbf{U}} $$
$$ {}^{t+\theta\Delta t}\dot{\mathbf{U}} = \frac{3}{\theta\,\Delta t}({}^{t+\theta\Delta t}\mathbf{U} - {}^t\mathbf{U}) - 2\,{}^t\dot{\mathbf{U}} - \frac{\theta\,\Delta t}{2}{}^t\ddot{\mathbf{U}} \qquad (256) $$

Can be calculated

- Finally, we plug $\tau = \Delta t$ in (254) to obtain ${}^{t+\Delta t}\mathrm{U}$ and ${}^{t+\Delta t}\dot{\mathrm{U}}$ and and in (253) to obtain ${}^{t+\Delta t}\ddot{\mathrm{U}}$ and be ready for the next time step.

$$ {}^{t+\tau}\dot{\mathbf{U}} = {}^t\dot{\mathbf{U}} + {}^t\ddot{\mathbf{U}}\tau + \frac{\tau^2}{2\theta\,\Delta t}({}^{t+\theta\Delta t}\ddot{\mathbf{U}} - {}^t\ddot{\mathbf{U}}) $$
$$ {}^{t+\tau}\mathbf{U} = {}^t\mathbf{U} + {}^t\dot{\mathbf{U}}\tau + \frac{1}{2}{}^t\ddot{\mathbf{U}}\tau^2 + \frac{1}{6\theta\,\Delta t}\tau^3({}^{t+\theta\Delta t}\ddot{\mathbf{U}} - {}^t\ddot{\mathbf{U}}) $$

$\tau = \Delta t$



- The θ-Wilson method is unconditionally stable for $\theta \geq 1.37$ and usually we use $\theta = 1.40$.

TABLE 9.3 *Step-by-step solution using Wilson θ integration method*

A. *Initial calculations:*

1. Form stiffness matrix $\mathbf{K}$, mass matrix $\mathbf{M}$, and damping matrix $\mathbf{C}$.
2. Initialize ${}^0\mathbf{U}$, ${}^0\dot{\mathbf{U}}$, and ${}^0\ddot{\mathbf{U}}$.
3. Select time step Δt and calculate integration constants, $\theta = 1.4$ (usually):

$$ a_0 = \frac{6}{(\theta\,\Delta t)^2}; \quad a_1 = \frac{3}{\theta\,\Delta t}; \quad a_2 = 2a_1; \quad a_3 = \frac{\theta\,\Delta t}{2}; \quad a_4 = \frac{a_0}{\theta}; $$
$$ a_5 = \frac{-a_2}{\theta}; \quad a_6 = 1 - \frac{3}{\theta}; \quad a_7 = \frac{\Delta t}{2}; \quad a_8 = \frac{\Delta t^2}{6} $$

4. Form effective stiffness matrix $\hat{\mathbf{K}}$: $\hat{\mathbf{K}} = \mathbf{K} + a_0\mathbf{M} + a_1\mathbf{C}$.
5. Triangularize $\hat{\mathbf{K}}$: $\hat{\mathbf{K}} = \mathbf{LDL}^T$.

B. *For each time step:*

1. Calculate effective loads at time $t + \theta\,\Delta t$:

$$ {}^{t+\theta\Delta t}\hat{\mathbf{R}} = {}^{t}\mathbf{R} + \theta({}^{t+\Delta t}\mathbf{R} - {}^{t}\mathbf{R}) + \mathbf{M}(a_0\,{}^{t}\mathbf{U} + a_2\,{}^{t}\dot{\mathbf{U}} + 2\,{}^{t}\ddot{\mathbf{U}}) + \mathbf{C}(a_1\,{}^{t}\mathbf{U} + 2\,{}^{t}\dot{\mathbf{U}} + a_3\,{}^{t}\ddot{\mathbf{U}}) $$

2. Solve for displacements at time $t + \theta\,\Delta t$:

$$ \mathbf{LDL}^T\,{}^{t+\theta\Delta t}\mathbf{U} = {}^{t+\theta\Delta t}\hat{\mathbf{R}} $$

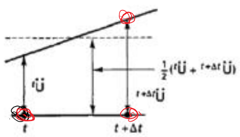
3. Calculate displacements, velocities, and accelerations at time $t + \Delta t$:

$$ {}^{t+\Delta t}\ddot{\mathbf{U}} = a_4({}^{t+\theta\Delta t}\mathbf{U} - {}^{t}\mathbf{U}) + a_5\,{}^{t}\dot{\mathbf{U}} + a_6\,{}^{t}\ddot{\mathbf{U}} $$

$$ {}^{t+\Delta t}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + a_7({}^{t+\Delta t}\ddot{\mathbf{U}} + {}^{t}\ddot{\mathbf{U}}) $$

$$ {}^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + \Delta t\,{}^{t}\dot{\mathbf{U}} + a_8({}^{t+\Delta t}\ddot{\mathbf{U}} + 2\,{}^{t}\ddot{\mathbf{U}}) $$

4.4.2 The Newmark method



${}^{t}\ddot{U}$ is known

- In Newmark method, U, $\dot{U}$ are expressed in terms of $U, \dot{U}, \ddot{U}$ at t_n and ${}^{t+\Delta t}\ddot{U}$:

$$ \boxed{{}^{t+\Delta t}\dot{\mathbf{U}}} = {}^{t}\dot{\mathbf{U}} + [(1-\delta)\,{}^{t}\ddot{\mathbf{U}} + \delta\,\boxed{{}^{t+\Delta t}\ddot{\mathbf{U}}}]\,\Delta t $$

$$ {}^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + {}^{t}\dot{\mathbf{U}}\,\Delta t + [(\tfrac{1}{2}-\alpha)\,{}^{t}\ddot{\mathbf{U}} + \alpha\,\boxed{{}^{t+\Delta t}\ddot{\mathbf{U}}}]\,\Delta t^2 $$

$$ {}^{\Delta t + t}\dot{U} = {}^{t}\dot{U} + \Delta t\,{}^{t}\ddot{U} $$

$$ {}^{t+\Delta t}U = {}^{t}U + \Delta t\,{}^{t}\dot{U} + \frac{\Delta t^2}{2}\,\boxed{{}^{t}\ddot{U}} $$

the only unknown is ${}^{t+\Delta t}\ddot{U}$

$$ \boxed{{}^{t+\Delta t}\dot{\mathbf{U}}} = {}^{t}\dot{\mathbf{U}} + [(1-\delta)\,{}^{t}\ddot{\mathbf{U}} + \delta\,\boxed{{}^{t+\Delta t}\ddot{\mathbf{U}}}]\,\Delta t $$

$$ {}^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + {}^{t}\dot{\mathbf{U}}\,\Delta t + [(\tfrac{1}{2}-\alpha)\,{}^{t}\ddot{\mathbf{U}} + \alpha\,\boxed{{}^{t+\Delta t}\ddot{\mathbf{U}}}]\,\Delta t^2 $$

$\longrightarrow$ express ${}^{t+\Delta t}\ddot{U}$ in terms of ${}^{t+\Delta t}U$

${}^{t+\Delta t}\ddot{U}$ & ${}^{t+\Delta t}\dot{U}$ expressed in terms of ${}^{t+\Delta t}U$

How to find ${}^{t+\Delta t}U$?

$$ M\,{}^{t+\Delta t}\ddot{U} + C\,{}^{t+\Delta t}\dot{U} + K\,{}^{t+\Delta t}U = {}^{t+\Delta t}F $$

expressed in terms of ${}^{t+\Delta t}U$

TABLE 9.4 *Step-by-step solution using Newmark integration method*

A. *Initial calculations:*

1. Form stiffness matrix $\mathbf{K}$, mass matrix $\mathbf{M}$, and damping matrix $\mathbf{C}$.
2. Initialize ${}^0\mathbf{U}$, ${}^0\dot{\mathbf{U}}$, and ${}^0\ddot{\mathbf{U}}$.
3. Select time step Δt and parameters α and δ and calculate integration constants:

$$ \delta \geq 0.50; \qquad \alpha \geq 0.25(0.5 + \delta)^2 $$

$$ a_0 = \frac{1}{\alpha\,\Delta t^2}; \quad a_1 = \frac{\delta}{\alpha\,\Delta t}; \quad a_2 = \frac{1}{\alpha\,\Delta t}; \quad a_3 = \frac{1}{2\alpha} - 1; $$

$$ a_4 = \frac{\delta}{\alpha} - 1; \quad a_5 = \frac{\Delta t}{2}\left(\frac{\delta}{\alpha} - 2\right); \quad a_6 = \Delta t(1-\delta); \quad a_7 = \delta\,\Delta t $$

4. Form effective stiffness matrix $\hat{\mathbf{K}}$: $\hat{\mathbf{K}} = \boxed{\mathbf{K} + a_0\mathbf{M} + a_1\mathbf{C}}$.
5. Triangularize $\hat{\mathbf{K}}$: $\hat{\mathbf{K}} = \mathbf{LDL}^T$.

Find ${}^{t+\Delta t}U$ is found $\longrightarrow$

$^{t+\Delta t}\dot{U}$ & $^{t+\Delta t}\ddot{U}$

$$\boxed{^{t+\Delta t}\dot{\mathbf{U}}} = {}^{t}\dot{\mathbf{U}} + [(1-\delta)\,{}^{t}\ddot{\mathbf{U}} + \delta\,\boxed{^{t+\Delta t}\ddot{\mathbf{U}}}]\,\Delta t$$

$$^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + {}^{t}\dot{\mathbf{U}}\,\Delta t + [(\tfrac{1}{2}-\alpha)\,{}^{t}\ddot{\mathbf{U}} + \alpha\,\boxed{^{t+\Delta t}\ddot{\mathbf{U}}}]\,\Delta t^2$$

$\left.\begin{array}{l} {}^{t}U \\ {}^{t}\dot{U} \\ {}^{t}\ddot{U} \end{array}\right\} \longrightarrow \begin{array}{l} {}^{t+\Delta t}U \\ {}^{t+\Delta t}\dot{U} \\ {}^{t+\Delta t}\ddot{U} \end{array}$

For Newmark & (θ_Wilson) matrix solve only involves the solution of $^{t+\Delta t}U$ ($^{t+\theta\Delta t}U$)

Method	Type	α	δ	Stability condition(2)	Order of accuracy(3)
Average acceleration (trapezoidal rule)	Implicit	$\frac{1}{4}$	$\frac{1}{2}$	Unconditional	2
Linear acceleration	Implicit	$\frac{1}{6}$	$\frac{1}{2}$	$\Omega_{crit} = 2\sqrt{3} \cong 3.464$	2
Fox-Goodwin (royal road)	Implicit	$\frac{1}{12}$	$\frac{1}{2}$	$\Omega_{crit} = \sqrt{6} \cong 2.449$	2
Central difference	Explicit(1)	0	$\frac{1}{2}$	$\Omega_{crit} = 2$	2

$$\boxed{^{t+\Delta t}\dot{\mathbf{U}}} = {}^{t}\dot{\mathbf{U}} + [(1-\underset{1/2}{\delta})\,{}^{t}\ddot{\mathbf{U}} + \underset{1/2}{\delta}\,\boxed{^{t+\Delta t}\ddot{\mathbf{U}}}]\,\Delta t$$

$$^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + {}^{t}\dot{\mathbf{U}}\,\Delta t + [(\tfrac{1}{2}-\underset{1/4}{\alpha})\,{}^{t}\ddot{\mathbf{U}} + \underset{1/4}{\alpha}\,\boxed{^{t+\Delta t}\ddot{\mathbf{U}}}]\,\Delta t^2$$

a.a.

$^{t+\Delta t}\dot{U} = {}^{t}\dot{U} + \frac{1}{2}\left({}^{t}\ddot{U} + {}^{t+\Delta t}\ddot{U}\right)$

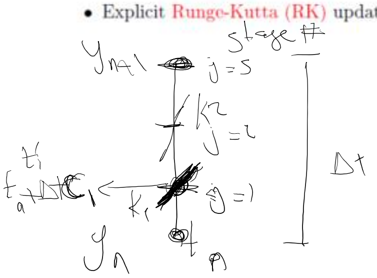
where Ω_{crit} is compared with $\overline{\Delta t} = \omega_{max}\Delta t$ where ω_{max} is the maximum frequency from modal analysis which can conservatively replaced by the highest element of the smallest element size $\omega_{h_{min}}$ (if different elements are used the maximum natural frequency of the individual elements).

4.5 Runge-Kutta (RK) methods

$\frac{dy}{dt} = f(t, y)$ 1st order ODE

$y(0) = y_0$

- Explicit Runge-Kutta (RK) update the solution from time step t_n to t_{n+1} through $s \geq 1$ stages:



$$y_{n+1} = y_n + \Delta t \sum_{i=1}^{s} b_i k_i \quad \text{where} \tag{262a}$$

$$k_i = f\Big(\underbrace{t_n + \Delta t c_i}_{t_i},\; y_n + \Delta t \sum_{j=1}^{i-1} a_{ij} k_j\Big), \qquad 1 \leq i \leq s \tag{262b}$$

$y' = f(t, y)$

b_i a_{ij}, c_i are algorithm numbers

K_i's are the slopes of solution at intermediate stages

$$y_{n+1} = y_n + \Delta t \underbrace{\left(\sum_{i=1}^{s} b_i k_i \right)}_{\text{average derivative}}$$

$$y_{n+1} = y_n + \Delta t \, (y')$$

- The intermediate values k_i represent intermediate slopes ($\frac{dy}{dt}$) at intermediate independent coordinate $t_n + \Delta t c_i$ (which fall between t_n and t_{n+1}) and dependent variable $\Delta t \sum_{j=1}^{i-1} a_{ij} k_j$.

- The fact that the upper limit of summation is $i-1$ is that each k_i **only** depends on prior k_j $(j < i)$ values.
- This enables a step-by-step solution strategy where k_i are solved for $i = 1$ to $i = s$ an finally the new update is computed from (5a) ($y_{n+1} = y_h + \Delta t \sum_{i=1}^{s} b_i k_i$ where).
- RK parameters are,
  1. Size $s \times s$ matrix a_{ij}
  2. Size s vector b_i
  3. Size s vector c_i
- For an explicit RK method $a_{ij} = 0$ for diagonal and upper diagonal members $(i \leq j)$.
- This is what enables the method to become explicit and require a simple and linear update equation for each k_i **(even if f is nonlinear in y)**.

- **Butcher tableau**: The parameters of a RK method are shown in a butcher tableau:

$$\begin{array}{c||cccc}
c_1 & 0 & \cdots & 0 & 0 \\
c_2 & a_{21} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & 0 & 0 \\
c_s & a_{s1} & \cdots & a_{s,s-1} & 0 \\
\hline\hline
 & b_1 & \cdots & b_{s-1} & b_s
\end{array}$$

K_1 = depends on y_n
k_2 " " y_n & k_1
k_3 " " y_n, k_1, k_2