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5.2.3 Generalized trapezoidal rule: Consistency of SDOF

$$d_{n+1} = A d_n + L_n \quad \Rightarrow \quad \boxed{d_{n+1} - A d_n - L_n = 0}$$

$$L_n = \Delta t \frac{(1-\alpha) f_n + \alpha f_{n+1}}{1 + \alpha \Delta t \lambda}$$

$$A = \frac{1 - (1-\alpha)\Delta t \lambda}{1 + \alpha \Delta t \lambda}$$

Consistency error in general measures if the solver is "consistent" with the underlying PDE / update equation as more accurate solutions are considered (delta t, h -> 0, p -> infinity)

$\underbrace{d_{n+1}}_{\text{numerical}} = A d_n + L_n$ numerical update

For consistency analysis, we simply insert the exact solution (or a harmonic solution that is exact for the PDE) in the discretization update equation.

$\underbrace{d(t_{n+1})}_{\text{exact}} = A\, d(t_{n+1}) + L_n + \Delta t\, \underbrace{\tau(t_n)}_{\text{truncation error}}$ ★

plug in exact sl'n in numerical update

why? $\underbrace{e_i}_{\text{error}} = \underbrace{d_i}_{\text{numerical}} - \underbrace{d(t_i)}_{\text{exact}} \longrightarrow \left\{ \begin{array}{l} d_n = e_n + d(t_n) \\ d_{n+1} = e_{n+1} + d(t_{n+1}) \end{array} \right.$

$d_{n+1} - A d_n - L_n = 0$ $\longrightarrow$

$d(t_{n+1}) + A d(t_n) - L_n = \underbrace{A e_n - e_{n+1}}_{= \Delta t\, \tau(t_n)}$

$$\boxed{d(t_{n+1}) + A d(t_n) - L_n = \Delta t\, \underbrace{\tau(t_n)}_{\text{truncation error @ time step } n}}$$

$\tau(t_n) = \dfrac{A e_n - e_{n+1}}{\Delta t}$

eg $\tau(t_n) = \dfrac{d(t_{n+1}) - d_{n+1}}{\Delta t}$ if $e_n = 0$

★

1. We want to analyze truncation error for alpha-method
2. Use this knowledge to analyze the convergence of the method

$d(t_{n+1}) = d(t_{n+\alpha\Delta t}) + (1-\alpha)\Delta t\, \dot d(t_{n+\alpha\Delta t}) + \frac{1}{2}[(1-\alpha)\Delta t]^2\, \ddot d(t_{n+\alpha\Delta t}) + \ldots$

f_{n+1} — $d(t_{n+1})$; $(1-\alpha)\Delta t$; $f_{n+\alpha\Delta t}$; $\alpha\Delta t$; f_n — $d(t_n)$

that's where we wrote the update numerical results for $\longleftarrow$ $\boxed{t_{n+\alpha\Delta t}}$

$d(t_n) \longrightarrow d(t_n) = d(t_{n+\alpha\Delta t}) - \alpha\Delta t\, \dot d(t_{n+\alpha\Delta t}) + \frac{1}{2}(-\alpha\Delta t)^2\, \ddot d(t_{n+\alpha\Delta t}) \ldots$

resolvs ...

$$+\frac{1}{2}\left(\frac{-\alpha\Delta t}{2}\right)^2 \boxed{\dddot{d}(t_n+\alpha\Delta t)}$$

$$d(t_{n+1}) + A\,d(t_n) - L_n = \Delta t\, \tau(t_n) \qquad \times[1+\alpha\Delta t\lambda]$$

$$[1+\alpha\Delta t\lambda]\, d(t_{n+1}) + [1-(1-\alpha)\Delta t\lambda]\, d(t_n) - \Delta t[(1-\alpha)f_n + \alpha f_{n+1}]$$

$$L_n = \Delta t\frac{(1-\alpha)f_n+\alpha f_{n+1}}{1+\alpha\Delta t\lambda}$$

$$= \Delta t[1+\alpha\Delta t\lambda]\,\tau(t_n)$$

$$A = \frac{1-(1-\alpha)\Delta t\lambda}{1+\alpha\Delta t\lambda}$$

Similarly do Taylor's expansion of f around $t_n + \alpha\Delta t$

$$
\begin{aligned}
&\Delta t(1+\alpha\Delta t\lambda^h)\,\tau(t_n)\\
&= (1+\alpha\Delta t\lambda^h)\,d(t_{n+1}) - (1-(1-\alpha)\Delta t\lambda^h)\,d(t_n) - \Delta t F_{n+\alpha}\\
&= \{(1+\alpha\Delta t\lambda^h) - (1-(1-\alpha)\Delta t\lambda^h)\}\boxed{d(t_{n+\alpha})}\\
&\quad + \{(1+\alpha\Delta t\lambda^h)(1-\alpha)\Delta t - (1-(1-\alpha)\Delta t\lambda^h)(-\alpha\Delta t)\}\boxed{\dot{d}(t_{n+\alpha})}\\
&\quad + \left\{(1+\alpha\Delta t\lambda^h)\frac{((1-\alpha)\Delta t)^2}{2} - (1-(1-\alpha)\Delta t\lambda^h)\frac{(-\alpha\Delta t)^2}{2}\right\}\boxed{\ddot{d}(t_{n+\alpha})}\\
&\quad + \left\{(1+\alpha\Delta t\lambda^h)\frac{((1-\alpha)\Delta t)^3}{3!} - (1-(1-\alpha)\Delta t\lambda^h)\frac{(-\alpha\Delta t)^3}{3!}\right\}\boxed{\dddot{d}(t_{n+\alpha})}\\
&\quad -\Delta t\Big([\alpha+(1-\alpha)]\boxed{F(t_{n+\alpha})} + [\alpha(1-\alpha)\Delta t + (1-\alpha)(-\alpha\Delta t)]\boxed{\dot{F}(t_{n+\alpha})}\\
&\quad + \left\{\alpha\frac{((1-\alpha)\Delta t)^2}{2} + (1-\alpha)\frac{(-\alpha\Delta t)^2}{2}\right\}\boxed{\ddot{F}(t_{n+\alpha})}\\
&\quad + \left\{\alpha\frac{((1-\alpha)\Delta t)}{3!} + (1-\alpha)\frac{(-\alpha\Delta t)^3}{3!}\right\}\boxed{\dddot{F}(t_{n+\alpha})}\Big) + O(\Delta t^4)
\end{aligned}
$$

$$\Delta t\,\lambda\, d(t_{n+\alpha})$$

$$\Delta t\ \dot{d}(t_{n+\alpha})$$

$$\frac{\Delta t^2}{2}\underbrace{((1-\alpha)^2-\alpha^2)} + O(\Delta t^3)$$

$$-\Delta t F(t_{n+\alpha}) + \cdots$$

$$\Delta t(1+\alpha\Delta t\lambda)\,\tau(t_n) = \Delta t\{\dot{d}(t_{n+\alpha}) + \lambda d(t_{n+\alpha}) - F(t_{n+\alpha})\} + \frac{\Delta t^2}{2}\underbrace{(1-2\alpha)} + O(\Delta t^3)$$

HOT

ODE

$\dot{d} + \lambda d = F$

$d(t_i)$ exact s'n

$d(t_{n+\alpha})$ exact s'n @ $t_n + \alpha\Delta t$

$$\boxed{\tau(t_n) = \Delta t\left(\frac{1}{2} - \alpha\right) + O(\Delta t^2)}$$

expansion of truncation error

$$\tau = (1-2\alpha)\mathcal{O}(\Delta t^1) + \mathcal{O}(\Delta t^2) \quad \Rightarrow$$

$$\begin{cases} \alpha = \frac{1}{2} & \tau = O(\Delta t^2) \quad q \\ \alpha \neq \frac{1}{2} & \tau = O(\Delta t^1) \quad q \end{cases}$$

$\alpha=\frac{1}{2}$ Trapezoidal rule

$\tau=O(\Delta t^{q})$ $q=2$

2nd order

$\alpha\neq\frac{1}{2}$ $\tau=O(\Delta t)$

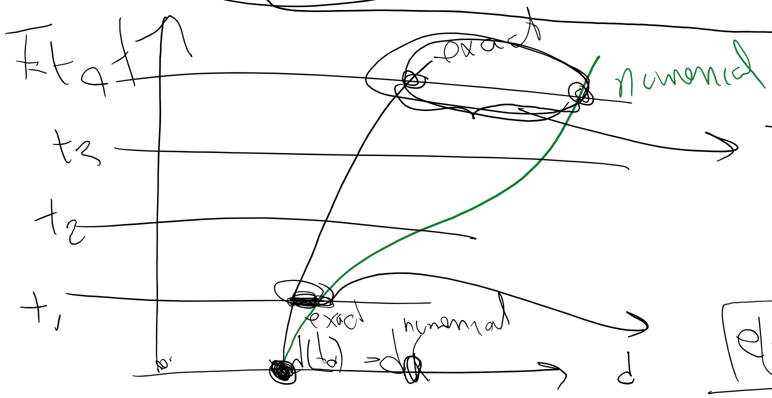
$\alpha=0$ FEuler

$\alpha=1$ B Euler

$\tau=O(\Delta t)$

1st order

$$\boxed{\tau(t_n)=\tau_n=\frac{Ae_n-e_{n+1}}{\Delta t}=O(\Delta t^{q})}$$



$e_{final}=d_{nf}-d(t_{nf})$ $n_f=4$ here

accumulative error at the final time

$\boxed{e_1=d_1-d(t_1)}$

$e_0=0$

$e_1=?$

$$\boxed{e_{n+1}=Ae_n-\Delta t\,\tau_n}$$

$e_1=\cancel{Ae_0}-\Delta t\,\tau_0$

0

$\boxed{e_1=-\Delta t\,\tau_0}$

$e_0=d_0-d(t_0)=0$ we satisfy IC

$e_2=Ae_1-\Delta t\,\tau_1$

$e_2=A(-\Delta t\,\tau_0)-\Delta t\,\tau_1=-A\Delta t\,\tau_0-\Delta t\,\tau_1$

$e_3=Ae_2-\Delta t\,\tau_2=-A^2\Delta t\,\tau_0-A\Delta t\,\tau_1-\Delta t\,\tau_2$

$$\boxed{e_n=-\Delta t\sum_{i=0}^{n-1}A^{i}\,\tau_{n-1-i}}$$

$n\rightarrow nf$

$$\boxed{e_n = -\Delta t \sum_{i=0}^{n-1} A^i \tau_{n-1-i}}$$

$n \to n_f$

$$|e_{n_f}| = \Delta t \left| \sum_{i=0}^{n_f-1} A^i \tau_{n_f-1-i} \right|$$

$|\alpha + \beta \ldots| \leq |\alpha| + |\beta| + \ldots$

triangle inequality

$$|e_{n_f}| \leq \Delta t \sum_{i=0}^{n_f-1} |A|^i \, |\tau_{n_f-1-i}|$$

(1) stability $|A| \leq 1$

$|A|^i \leq 1$

(1) stability $\boxed{d_n = A^n d_0}$ $f=0$

spectral $|A| \leq 1$

$$\leq \Delta t \sum_{i=0}^{n_f-1} |\tau_{n_f-1-i}|$$

need extra Δt in τ_n expression

so this term remains bounded by t_f

$$\leq \underbrace{(\Delta t\, n_f)}_{t_f} \operatorname{Max}_{i=0 \ldots n_f-1} \underbrace{(\tau_{n_f-1-i})}_{C\,\Delta t^q}$$

$\tau(t_i) = O(\Delta t^q)$

(2) consistency $\operatorname{Max}_i(\tau_{n_f-1-i}) = C \Delta t^q$

does not depend on Δt

$$\boxed{|e(t_f)| \leq (t_f C)\, \Delta t^q}$$

$n_f = 4$ $t_f = 2$

$\Delta t\, n_f =$

$.5 \times 4 = t_f$

$\Delta t = .5$

$\Delta t = .5$

consistency $\Longleftrightarrow$ stability $\Longleftrightarrow$ convergence

consistency $\to$ Algebraic Taylor's expansion

consistency + stability $\Longrightarrow$ $\boxed{\text{Convergence}}$ difficult to prove on its own

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we proved Lax theorem for Gen. trap.

$$|e(t_f)| = |\underset{\text{numerical}}{d_{t_f}} - \underset{\text{exact}}{d(t_f)}| \leq \underbrace{t_f C}_{C(t_f)} \Delta t^q$$

$\Delta t \to 0$ $\quad |e(t_f)| \to 0$ $\quad t_f = C_1$ $\quad C t_f = DC$

$\Delta t \to 0$ $\quad\quad$ $|e(t_f)| \to 0$

$t_f = 10$	$C_f = 10C$
$t_f = 5$	$C_f = 5C$
$1 = t_f$	$C_f = C$

5.2.4 Generalized trapezoidal rule: Convergence of SDOF

- Remembering Lax-Richtmyer equivalence theorem for FD methods, we asserted in (288) that for a consistent method stability and convergence are equivalent.
- As shown in (289), in practice we prove the convergence of a method by establishing that it is both consistent and stable:

$$\text{Consistency and Stability} \quad \Rightarrow \quad \text{Convergence}$$

- Below we prove this for a SDOF problem with $\lambda^h \geq 0$ for a general 1-step time integration scheme in the form of (316a).
- Let $t_n = n\Delta t$ be fixed but Δt be allowed to vary. Assume the time integration is,
  1. stable, *i.e.*, $|A| \leq 1$.
  2. consistent, *i.e.*, there exists a $k > 0, c \geq 0$ such that $|\tau(t)| \leq c\Delta t^k$ for all $t \in [0, T]$; *cf.* (318a).

  Then the method is convergent ($e(t_n) \to 0$ as $\Delta t \to 0$) with the rate of convergence k.

Proof:

- First we want to form an update equation for the error from time step t_n to t_{n+1}:

$$\left.\begin{array}{lr} d_{n+1} - Ad_n - L_n = 0 & cf.\ (316a) \\ d(t_{n+1}) - Ad(t_n) - L_n = \Delta t\tau(t_n) & cf.\ (317) \\ e(t_{n+1}) = d_{n+1} - d(t_{n+1}), \quad e(t_n) = d_n - d(t_n) & \text{Definition of error; } cf.\ (296c) \end{array}\right\} \Rightarrow \boxed{e(t_{n+1}) = Ae(t_n) - \Delta t.\tau(t_n)} \tag{322}$$

- By using $n-1$ instead of n in equation (322) (*i.e.*, previous time step) we obtain,

$$e(t_n) = Ae(t_{n-1}) - \Delta t.\tau(t_{n-1}) \quad \text{and knowing} \quad e(t_{n+1}) = Ae(t_n) - \Delta t.\tau(t_n) \quad \text{from (322)} \quad \Rightarrow$$
$$e(t_{n+1}) = A^2 e(t_{n-1}) - \Delta t A\tau(t_{n-1}) - \Delta t\tau(t_n)$$

- By repeating this equation to eliminate $e(t_{n-1})$ from the RHS (by writing (322) for $n \to n-2$) we obtain,

$$e(t_{n+1}) = A^3 e(t_{n-2}) - \Delta t A^2 \tau(t_{n-2}) - \Delta t A\tau(t_{n-1}) - \Delta t\tau(t_n)$$

  and so on,

- So we would have,

$$\boxed{e(t_{n+1}) = A^{n+1}e(t_0) - \Delta t \sum_{i=0}^{n} A^i \tau(t_{n-i})} \tag{323}$$

- But $e(t_0) = 0$ because we initialize the time marching scheme at the first step with the exact solution, *i.e.*, IC.

- Expressing (323) for time step t_n instead of t_{n+1} and taking its absolute value we obtain,

$$
\begin{aligned}
|e(t_n)| &= \Delta t \left| \sum_{i=0}^{n-1} A^i \tau(t_{n-1-i}) \right| && \text{(a)} \\
&\le \Delta t \sum_{i=0}^{n-1} |A|^i \, |\tau(t_{n-1-i})| && \text{(b)} \\
&\le \Delta t \sum_{i=0}^{n-1} |\tau(t_{n-1-i})| \quad \text{(stability)} && \text{(c)} \\
&\le \boxed{t_n} \max |\tau(t)| \quad t \in [0, T] && \text{(d)} \\
&\le \boxed{t_n c} \Delta t^k \quad \text{(consistency)} && \text{(e)}
\end{aligned}
\tag{324}
$$

- We observe,
  1. A stable SDOF one step time integration scheme is convergent iff it is stable and consistent (the converse, *i.e.*, convergence $\Rightarrow$ stability and consistency is trivial; we only showed that stability and consistency proved convergence). Compare this with slightly different versions (288) and (289).
  2. We observe, rate of convergence is the same as k in the definition of consistency in (317) ($d(t_{n+1}) - Ad(t_n) - L_n = \Delta t\, \tau(t_n)$).

3. The extra Δt that we introduced in the definition of consistency condition on the RHS in (317) ($d(t_{n+1}) - Ad(t_n) - L_n = \Delta t \tau(t_n)$) is needed. Otherwise in (324)(d) we would have got $e(t_n) \le n \max|\tau(t)|$, which clearly makes the RHS unbounded as we can have a very small time step Δt so that in $t_n = n\Delta t$, $n \to \infty$.
4. The bound on the error term in (324)(e) can be written as,

$$|e(t_n)| \le C_{t_n} \Delta t^k, \qquad \text{for a fixed } t_n = n\Delta t, \text{where} \quad C_{t_n} = ct_n \tag{325}$$

we observe,

(a) We observe C_{t_n} in general **depends** on the time value t_n and can grow with the the observation time t_n.
(b) But for a **fixed** time t_n the error is bounded no matter what time step value (assuming stability is satisfied) is used!
(c) So, the error constant in general depends on time in convergence analysis, **but must NOT depend on the time step size Δt.**

SDOF to MDOF convergence rate:

- From (298) we have,

$$\mathbf{e}(t_n)^{\mathrm{T}} \mathbf{M} \mathbf{e}(t_n) = \sum_{i=1}^{n} \left(e_{\{i\}}(t_n)\right)^2 \tag{326}$$

  where from (325) we know that all SDOF problems $i = 1$ to n have convergence rate of k for their error $e_{\{i\}}(t_n)$ if their local truncation convergence order is k and stable time step is used for all of them.
- As mentioned before, if time integration is conditionally stable, by using the most stringent time-step (from the highest λ_i^h we ensure that all SDOFs are stable.
- In addition, if we directly integrate the underlying MDOF with time step Δt it is equivalent to integrating all SDOFs with time step Δt.
- Given that all SDOFs have the same convergence rates but potentially different constants $(C_{t_n})_i$ we bound the RHS of (326) from (325) in the form,

$e_i \sim C_i \Delta t^k$

$$\mathbf{e}(t_n)^{\mathrm{T}} \mathbf{M} \mathbf{e}(t_n) = \sum_{i=1}^{n} \left(e_{\{i\}}(t_n)\right)^2 \le \sum_{i=1}^{n} (C_{t_n})_i^2 \Delta t^{2k} = C_{t_n}^2 \Delta t^{2k}, \quad \text{where} \quad \boxed{C_{t_n}^2 = \sum_{i=1}^{n} (C_{t_n})_i^2} \Rightarrow$$
$$\boxed{\sqrt{\mathbf{e}(t_n)^{\mathrm{T}} \mathbf{M} \mathbf{e}(t_n)} \le C_{t_n} \Delta t^k} \tag{327}$$

- but we know,

$$L_2(\mathbf{e}(t_n)) = \sqrt{\mathbf{e}(t_n).\mathbf{e}(t_n)} \le \frac{1}{m_{\min}} \sqrt{\mathbf{e}(t_n)^{\mathrm{T}} \mathbf{M} \mathbf{e}(t_n)} \le \frac{C_{t_n}}{m_{\min}} \Delta t^k \tag{328}$$

If individual SDOFs converge with order k, the global system converges with order k.

where $m_{\min} > 0$ is the minimum eigenvalue of $\mathbf{M}$. How do we get the first inequality in (328) and why $m_{\min} > 0$?

- Basically L_2 norm and norm with $\mathbf{M}$ are equivalent $\Rightarrow$
- from (328) we observe that MDOF $\mathbf{M}\ddot{\mathbf{d}} + \mathbf{K}\mathbf{d} = \mathbf{F}$ error vector $\mathbf{e}(t_n)$ converges with order k in both L_2 norm and norm with $\mathbf{M}$ kernel provided that the integration scheme is stable and consistent with the rate k.

5.3 Stability analysis of SDOF problems involving matrix update equation

- In general we can have update equations from time step t_n to time step t_{n+1} which has an update equation of the form,
- As mentioned before, stability analysis of a MDOF problem, reduces to the stability analysis of its modal SDOFs:

$$\begin{aligned}
&\text{MDOF} && \Rightarrow && \text{SDOF} && (329a)\\
&\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R} && && \ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = f(t) && (329b)\\
&\mathbf{M}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R} && && \dot{x} + \lambda x = f(t) && (329c)
\end{aligned}$$

- The stability, consistency, and convergence of (329c) was discussed in §5.2 in the context of generalized trapezoidal method.
- Herein, we analyze the stability of time marching methods from §4.1. by analyzing their corresponding SDOF from (329b).
- In the analysis of higher order ODEs, multi-step method methods with more than 2 steps, and many other instances we deal with update equations of the form,

$$\boxed{{}^{t+\Delta t}\hat{\mathbf{X}} = \mathbf{A}\,{}^{t}\hat{\mathbf{X}} + \mathbf{L}({}^{t+\nu}r)} \tag{330}$$

where ${}^{t+\Delta t}\hat{X}$ and ${}^{t}\hat{X}$ correspond to generalized vector update values for time steps t_n and t_{n+1} and

- $\mathbf{A}$ is the matrix amplification factor.
- Examples of ${}^{t}\hat{X}$ are
  1. Value and subsequent time derivatives of x in (329b): ${}^{t}\hat{X} = [{}^{t}x\ {}^{t}\dot{x}\ {}^{t}\ddot{x}]$. Examples be from Newmark and θ-Wilson methods.
  2. Value and previous step values of x in (329b): ${}^{t}\hat{X} = [{}^{t+\Delta t}x\ {}^{t}x\ {}^{t-\Delta t}x\ \cdots]$. This will be the form of ${}^{t}\hat{X}$ for LMS methods.
- In either case, since ${}^{t}\hat{X}$ is a vector, unlike (316a) ($d_{n+1} = Ad_n + L_n$) where the update equation was for a scalar variable d_{n+1} and involved a scalar amplification factor A, in (330) $\mathbf{A}$ is a matrix.
- Applying (330) multiple times we obtain,

$$\boxed{\begin{aligned}{}^{t+n\Delta t}\hat{\mathbf{X}} &= \mathbf{A}^n\,{}^{t}\hat{\mathbf{X}} + \mathbf{A}^{n-1}\mathbf{L}({}^{t+\nu}r) + \mathbf{A}^{n-2}\mathbf{L}({}^{t+\Delta t+\nu}r) + \cdots\\ &+ \mathbf{A}\,\mathbf{L}({}^{t+(n-2)\Delta t+\nu}r) + \mathbf{L}({}^{t+(n-1)\Delta t+\nu}r)\end{aligned}} \tag{331}$$

- For the moment, by assuming the force operator $\mathbf{L} = 0$ we obtain,

$$\boxed{{}^{t+n\Delta t}\hat{\mathbf{X}} = \mathbf{A}^n\,{}^{t}\hat{\mathbf{X}}} \tag{332}$$

${}^{t}X = \begin{Bmatrix} {}^{t}u \\ {}^{t}\dot{u} \\ {}^{t}\ddot{u} \end{Bmatrix}$ for multi-variate methods (θ-Wilson, Newmark)

- The stability of the time marching scheme requires $\mathbf{A}^n$ does not blow up.
- For the moment assume $\mathbf{A}$ is diagonalizable: $\mathbf{A} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1}$ ($\mathbf{P} = [\mathbf{p}_1\ \mathbf{p}_2 \cdots \mathbf{p}_m]$ is the matrix of right eigenvectors $\mathbf{p}_i$ and $\mathbf{J} = \text{diag}(a_1, \cdots, a_m)$ and a_i are the corresponding eigenvalues. The matrix $\mathbf{A}$ is $m \times m$.
- In this case, we have

$$\mathbf{A} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1} \quad \Rightarrow \mathbf{A}^n = \mathbf{P}\mathbf{J}^n\mathbf{P}^{-1} = \mathbf{P}\begin{bmatrix} a_1^n & & & \\ & a_2^n & & \\ & & \ddots & \\ & & & a_m^n \end{bmatrix}\mathbf{P}^{-1} \quad \text{for diagonal } \mathbf{J} \tag{333}$$

- Recalling the definition of spectral radius (45)

$$\rho(\mathbf{A}) = \max\{|a_i|\ i \in \{1, \cdots, m\} | a_i \text{ are eigenvalues of } A\} \tag{334}$$

for a diagonalizable matrix $\rho(A) \leq 1$

- Clearly, from the definition (334) the stability condition for a **diagonalizable** update matrix A is,

$$\text{Update by diagonalizable matrix amplification factor } \mathbf{A} \text{ is stable iff} \qquad \rho(\mathbf{A}) \leq 1 \tag{335}$$

- Now, what happens if **A** is not diagonalizable and **when A is not** diagonalizable: **A** is **not** diagonalizable iff
  1. Eigenvalues a_k with eigenvalues with $n_k^A > 1$ **algebraic** multiplicity.
  2. Smaller **geometric** multiplicity $n_k^G < n_k^A$
- Basically, the matrix **A** is diagonalizable if it has repeated eigenvalues whose geometric multiplicity is smaller than algebraic one. Clearly, if **A** has distinct eigenvalues all algebraic and geometric multiplicities are are one and it's diagonalizable. Also, if it has repeated eigenvalues but with the same geometric and algebraic multiplicity that is not a problem either. A good example is α multiple of identity matrix $\alpha\mathbf{I}$ which clearly is already diagonal.
- So, what can we do if **A** is not diagonalizable?
- Can we still write $\mathbf{A} = \mathbf{PJP}^{-1} \Rightarrow \mathbf{A} = \mathbf{PJ}^n\mathbf{P}^{-1}$ for $\mathbf{J}$ being a more manageable matrix than **A**?
- The answer is yes. The transformation is done by Jordan normal form.

- Any complex matrix **A** can be decomposed in the form,

$$\text{for arbitrary } \mathbf{A}: \quad \underline{\mathbf{A} = \mathbf{PJP}^{-1}}, \quad \text{where } \mathbf{J} \text{ is a } \textbf{Jordan normal form} \text{ that is similar to } \mathbf{A} \text{ by } \mathbf{P} \tag{336}$$

- where **Jordan normal form** J is a block diagonal matrix of the form,

block matrices

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{bmatrix} \quad \text{where each block } J_i \text{ is a square matrix of the form} \quad J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \tag{337}$$

m×m

eigenvalue m is repeated m times & has geometric multiplicity $\underline{1}$

- an example can be seen below,

λ=5 repeated 10 times
geometric multiplicity is 4

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \; 3\times3$$

$$\underbrace{\begin{bmatrix} 5 & 1 & & & & & \\ & 5 & 1 & & & & \\ & & 5 & 1 & & & \\ & & & 5 & 1 & & \\ & & & & 5 & 1 & \\ & & & & & 5 & 1 \\ & & & & & & 5 \end{bmatrix}}_{} \; 7\times7$$

- As an example consider,

$$\mathbf{A} = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix} \Rightarrow$$

$a_1 = 1, a_2 = 2, a_3 = 4$(with **algebraic** multiplicity 2 $[n_3^A = 2]$ but **geometric** multiplicity 1 $[n_3^G = 1]$)

there is a $\mathbf{P}$ matrix: $\mathbf{A} = \mathbf{PJP}^{-1}$ such that $\mathbf{J} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

4 is repeated twice but geom m. = 1

- So, the question of stability of update in (332) ${}^{t+n\Delta t}\hat{\mathbf{X}} = \mathbf{A}^{n\,t}\hat{\mathbf{X}}$.
- We have

$$\mathbf{A} = \mathbf{PJP}^{-1} \qquad \Rightarrow \qquad \mathbf{A}^n = \mathbf{P}\mathbf{J}^n\mathbf{P}^{-1}$$

- So the stability of update (332) reduces to the behavior (boundedness) of powers of Jordan block diagonal matrix $\mathbf{J}6n$.
- We can have the following statement for stability of the update:

Spectral stability:

$$J^{10} = \begin{bmatrix} 1^{10} & & \\ & 2^{10} & \\ & & \begin{bmatrix} 4 & 1 \\ & 4 \end{bmatrix}^{10} \end{bmatrix} \Rightarrow \begin{bmatrix} 4^{10} & 10\times4^{9} \\ & 4^{10} \end{bmatrix}$$

$$\boxed{{}^{t+n\Delta t}\hat{\mathbf{X}} = \mathbf{A}^{n\,t}\hat{\mathbf{X}} \qquad \text{is stable iff } \rho(\mathbf{A}) \leq 1 \text{ and if } \mathbf{A} \text{ is not diagonalizable eigenvalues } a_i \text{ with } n_i^A > n_i^G \text{ satisfy } |a_i| < 1} \quad (338)$$

- The condition that for eigenvalues a_i with $n_i^A > n_i^G$ we require $|a_i| < 1$ becomes apparent from the example below,

$$\mathbf{J} = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \qquad \boxed{\Rightarrow \mathbf{J}^n = \begin{bmatrix} a^n & \boxed{na^{n-1}} \\ 0 & a^n \end{bmatrix}} \qquad (339)$$

$a = 1 \to n \to$ still ∞

which correspond to a 2×2 $\mathbf{A}$ with $n_1^A = 2, n_1^G = 1$. Same argument can be applied to $n_i^A > 2$ and $m > 2$ as $\mathbf{J}^n$ only involves the powers of the Jordan block diagonal matrices similar to the one in (339).

- There are three cases:
  1. If $a > 1$ both diagonal and off-diagonal (J_{12}) values blow up.
     - This instability is called exponential of "explosive" and very fast shows up in the numerical results.
     - The growth of this instability is of the form $\mathcal{O}(a^n)$.
  2. If $a < 1$ both diagonal and off-diagonal (J_{12}) are bounded (and in fact approach zero as $n \to \infty$).

$[a]$ a^n

$\underbrace{|a| > 1}_{|a| = 1}$

3. If $a = 1$, diagonal values are 1 but off-diagonal value $J_{12} = n$ weakly blows up:
   - This instability is called weak / algebraic and unlike exponential instability may not be easily detected in numerical results.
   - The growth of this instability is of the form $\mathcal{O}(n^s)$, $s = \max_i(n_i^A - n_i^G)$. Compare this with much more severe exponential instability in case 1: $\mathcal{O}(a^n)$.

- In the discussion of the stability of various methods that have a matrix amplification factor $\mathbf{A}$ we refer to (338).

Spectral stability

$\rho(A) < 1$

or if $\rho(A) = 1$ A must be diagonalizable

5.3.1 Stability analysis of LMS methods

- General first and second order linear ODEs applied to a vector $\mathbf{y}$ can be represented as follows, (*cf.* (240) for general nonlinear first order expression of an ODE),

$$\dot{\mathbf{y}} = f(\mathbf{y}, t) = \mathbf{G}_0\mathbf{y} + \mathbf{H}(t) \qquad \text{Linear first order ODE} \qquad (340a)$$

$$\ddot{\mathbf{y}} = f(\mathbf{y}, \dot{\mathbf{y}}, t) = \mathbf{G}_1\dot{\mathbf{y}} + \mathbf{G}_0\mathbf{y} + \mathbf{H}(t) \qquad \text{Linear second order ODE} \qquad (340b)$$

- When a k-step LMS method is applied to an ODE $\mathbf{y}_{n+1}$ in terms of $\mathbf{y}_n, \mathbf{y}_{n-1}, \cdots, \mathbf{y}_{n-k+1}$.
- Formally, the expressions of an k-step LMS method applied to linear first and second order ODEs in (340) are,

$y' = f(y,t)$

$$\sum_{i=0}^{k} \{\alpha_i \mathbf{y}_{n+1-i} + \Delta t\beta_i[\mathbf{G}_0\mathbf{y}_{n+1-i} + \mathbf{H}(t_{n+1-i})]\} = 0 \qquad \text{LMS applied Linear first order ODE} \qquad (341a)$$

$$\sum_{i=0}^{k} \{\alpha_i \mathbf{y}_{n+1-i} + \Delta t\beta_i\mathbf{G}_1\mathbf{y}_{n+1-i} + \Delta t^2\gamma_i[\mathbf{G}_0\mathbf{y}_{n+1-i} + \mathbf{H}(t_{n+1-i})]\} = 0 \qquad \text{LMS applied Linear second order ODE} \qquad (341b)$$

- For the use of LMS methods, we focus on either second order $\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$ or first order $\mathbf{M}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$ MDOF ODEs.
- As mentioned several times, for the analysis of these MDOF problems, we analyze their stability and convergence properties by reducing them to SDOFs,

MDOF		SDOF	Parameters in (340)	
$\mathbf{M}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$	$\Rightarrow$	$\dot{x} + \lambda x = f(t)$	$\mathbf{G}_0 = -\lambda, \mathbf{H}(t) = f(t)$	(342a)
$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$	$\Rightarrow$	$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = f(t)$	$\mathbf{G}_1 = -2\xi\omega, \mathbf{G}_0 = -\omega^2, \mathbf{H}(t) = f(t)$	(342b)

- Thus, plugging $\mathbf{H}(t), \mathbf{G}_0$ (and $\mathbf{G}_0$) from (342) in (341) we obtain the following k-step LMS method applied to SDOFs in (342),

$$\sum_{i=0}^{k} \{\alpha_i x_{n+1-i} + \Delta t\beta_i[-\lambda x_{n+1-i} + f(t_{n+1-i})]\} = 0 \qquad \text{LMS for } \dot{x} + \lambda x = f(t) \qquad (343a)$$

$$\sum_{i=0}^{k} \{\alpha_i x_{n+1-i} - 2\Delta t\beta_i\xi\omega x_{n+1-i} + \Delta t^2\gamma_i[-\omega^2 x_{n+1-i} + f(t_{n+1-i})]\} = 0 \qquad \text{LMS for } \ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = f(t) \qquad (343b)$$

- For stability analysis, for simplicity of analysis herein, we assume $f(t) = 0$.
- Equation (343) for $f(t) = 0$ can be written in the short form,

$$c_0x_{n+1} + c_1x_n + c_1x_{n-1} + \cdots + c_kx_{n+1-k} = 0 \qquad \text{where } c_i = \begin{cases} \alpha_i - \Delta t\beta_i\lambda & \text{LMS for } \dot{x} + \lambda x = 0 \end{cases}$$

- Equation (343) for $f(t) = 0$ can be written in the short form,

$$\boxed{c_0 x_{n+1} + c_1 x_n + c_1 x_{n-1} + \cdots + c_k x_{n-k+1} = 0} \quad \text{where } c_i = \begin{cases} \alpha_i - \Delta t \beta_i \lambda & \text{LMS for } \dot{x} + \lambda x = 0 \\ \alpha_i - 2\Delta t \beta_i \xi\omega - \Delta t^2 \gamma_i \omega^2 & \text{LMS for } \ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = 0 \end{cases} \tag{344}$$

- We can express (353) as the following update equation,

assume

$$\boxed{X_n = A^n X_0}$$

$$\boxed{x_{n+1} = \bar{c}_1 x_n + \bar{c}_1 x_{n-1} + \cdots + \bar{c}_k x_{n-k+1}, \quad \bar{c}_i = -\frac{c_i}{c_0}}$$

$$A^{n+1} = \bar{C}_1 A^n + \bar{C}_1 A^{n-1} \cdots + \bar{C}_k A^{n-k+1}$$

order K eqn

$$\boxed{A^k = \bar{C}_1 A^{k-1} + \cdots + \bar{C}_k}$$

K roots

$$|A_{(1)}| \cdots |A_{(k)}| \leq 1$$