DC2020/04/08

Monday, April 6, 2020 1:11 PM

From last time we had:

- For stability analysis, for simplicity of analysis herein, we assume $f(t) = 0$.
- Equation (343) for $f(t) = 0$ can be written in the short form,

$$c_0 x_{n+1} + c_1 x_n + c_1 x_{n-1} + \cdots + c_k x_{n-k+1} = 0 \quad \text{where } c_i = \begin{cases} \alpha_i - \Delta t \beta_i \lambda & \text{LMS for } \dot{x} + \lambda x = 0 \\ \alpha_i - 2\Delta t \beta_i \xi \omega - \Delta t^2 \gamma_i \omega^2 & \text{LMS for } \ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = 0 \end{cases} \tag{344}$$

- We can express (353) as the following update equation,

$$x_{n+1} = \bar{c}_1 x_n + \bar{c}_2 x_{n-1} + \cdots + \bar{c}_k x_{n-k+1}, \quad \bar{c}_i = -\frac{c_i}{c_0}$$

$$\underbrace{\begin{pmatrix} x_{n+1} \\ x_n \\ x_{n-1} \\ \vdots \\ \boxed{x_{n-k+1}} \end{pmatrix}}_{X_{n+1}} = \begin{pmatrix} \bar{c}_1, \bar{c}_2, & \cdots, & \bar{c}_k \\ 1 & 0 & \\ 0 & 1 & \\ & & \\ & 1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ \vdots \\ \boxed{1} \\ x_{n-k} \end{pmatrix}}_{X_n}$$

- For a k-step LMS method we define the vector of variables $\hat{\mathbf{X}}_{n+1}$ as,

$$\hat{\mathbf{X}}_n = \begin{bmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_{n-k+1} \end{bmatrix}$$

- In this case from (353) we can write the update for $\hat{\mathbf{X}}_{n+1}$:

$$\boxed{\hat{\mathbf{X}}_{n+1} = \mathbf{A}\hat{\mathbf{X}}_n,} \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} \bar{c}_1 & \bar{c}_2 & \cdots & \bar{c}_{k-1} & \bar{c}_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \end{bmatrix}$$

$\rho(A) < 1$

or if $\rho(A) = 1$ $\quad n_G(\lambda=1) = n_A(\lambda=1)$

based of the form of matrix $\quad \underbrace{n_G}_{\text{\# eigenvectors for } \lambda}(\lambda) = 1$

$\rho(A) < 1$

eigenvectors for λ

$\rho(A) < 1$

or $\rho(A) = 1$ & $\lambda = 1$ is a simple eigenvalue $(n_A(\lambda=1) = 1)$ (*)

again c_i depend on Δt and parameters of SDOF parameters and time integration model and it parameters.

- It can be shown (will be a HW problem) that the eigenvalues of (347) satisfy,

$$a_i \text{ is an eigenvalue of } \mathbf{A} \quad \Leftrightarrow \tag{348a}$$

$$\exists \mathbf{v}_i \ \mathbf{A}\mathbf{v}_i = a_i \mathbf{v}_i \text{ (no summation on } i) \quad \Leftrightarrow \tag{348b}$$

$$-a_i^k + \bar{c}_1 a_i^{k-1} + \bar{c}_2 a_i^{k-2} + \cdots + \bar{c}_k = 0 \quad \Leftrightarrow \quad (\bar{c}_i = -\frac{c_i}{c_0}, cf. (353)) \tag{348c}$$

$$\boxed{c_0 a_i^k + c_1 a_i^{k-1} + c_2 a_i^{k-2} + \cdots + c_k = 0} \tag{348d}$$

Eigenvalues satisfy the characteristic equation above. They must satisfy (*)

$c_0 x_{n+1} + c_1 x_n + c_1 x_{n-1} + \cdots + c_k x_{n-k+1} = 0$

$x_n = a^n x_0$ assume

$C_0 (a)^{n+1} + C_1 (a)^n + \cdots + C_k (a)^{n-k+1} = 0$

divide by a^{n-k+1}

$$\boxed{C_0 a^k + C_1 a^{k-1} + \cdots + C_k = 0}$$

matches that

- Accordingly, the stability analysis of LMS scheme is as follows,

$$|a_i| \le 1, \text{ if } a_i \text{ is not repeated } (n_i^A = 1) \text{ otherwise } |a_i| < 1, \text{where} \tag{349a}$$

$$a_i \text{ are eigenvalues of } \mathbf{A}, \ i.e., \text{ roots of } c_0 a_i^k + c_1 a_i^{k-1} + \cdots + c_k = 0 \tag{349b}$$

Example of this process:

5.3.1.1 Stability analysis of LMS methods: Central Difference method

- Consider central difference method update equations,

$$\ddot{x}_n + 2\xi\omega\dot{x}_n + \omega^2 x_n = f(t_n) \tag{350a}$$

$\dot{x}_n = \dfrac{x_{n+1} - x_{n-1}}{2\Delta t}$

$\ddot{x}_n = \dfrac{x_{n+1} + x_{n-1} - 2x_n}{\Delta t^2}$

- By direct plugging (350b) and (350c) into (350a) we obtain the following update equation,

$$\boxed{\underbrace{(1 + \xi\Delta t\omega)}_{C_0} x_{n+1} + \underbrace{(-2 + (\Delta t\omega)^2)}_{C_1} x_n + \underbrace{(1 - \xi\Delta t\omega)}_{C_2} x_{n-1} = 0}$$

$C_0 a^2 + C_1 a + C_2 = 0$

solve

$|a| < 1$

or $a = 1$ simple root

$\rightarrow$ find max $\Delta t(\xi, \omega)$

- We obtain the same equation (351) with a bit longer approach by formally obtaining the values $\alpha_i, \beta_i, \gamma_i$ for LMS method applied to the second order ODE (350a).

- Formally, the expressions of a 2-step LMS method applied to linear second order ODE in (343b) ($\sum_{i=0}^{k}\{\alpha_i x_{n+1-i} - 2\Delta t\beta_i \xi\omega x_{n+1-i} - \Delta t^2\gamma_i\omega^2 x_{n+1-i}\} = 0$ are ($f(t) = 0$),

$$
\begin{array}{llllr}
\alpha_0 = 1 & \alpha_1 = -2 & \alpha_2 = 1 & \ddot{x} \text{ (in (350c))} & (352a)\\
\beta_0 = -\frac{1}{2} & \beta_1 = 0 & \beta_2 = \frac{1}{2} & \dot{x} \text{ (in (350b))} & (352b)\\
\gamma_0 = 0 & \gamma_1 = -1 & \gamma_2 = 0 & x \text{ (inserted for } t(t_n) \text{ in (350a))} & (352c)
\end{array}
$$

  - If both β_0 and γ_0 were zero, this method formally would have been explicit *cf.* [Hughes, 2012] §9.3.2.
  - The method, formally is not fully explicit because $\beta_0 \neq 0$ involves values for t_{n+1}.
  - As we will see the method still is only conditionally stable. The main cause is $\gamma_0 = 0$.

- Based on the values $\alpha_i, \beta_i, \gamma_i, i = 0, 1, 2$ in (352) the update equation (353) is written as,

$$
c_0 x_{n+1} + c_1 x_n + c_2 x_{n-1} = 0 \qquad \text{where } c_i = \alpha_i - 2\Delta t\beta_i\xi\omega - \Delta t^2\gamma_i\omega^2 \quad \rightarrow \quad \begin{cases} c_0 = 1 + \xi\Delta t\omega \\ c_1 = -2 + (\Delta t\omega)^2 \\ c_2 = 1 - \xi\Delta t\omega \end{cases} \qquad (353)
$$

$$c_0 a^2 + c_1 a + c_2 = 0 \qquad / c_0$$

$$a^2 + \underbrace{\left(\frac{c_1}{c_0}\right)}_{-2A_1} a + \underbrace{\left(\frac{c_2}{c_0}\right)}_{A_2} = 0$$

$$A_1 = \frac{-c_1}{2c_0} = \frac{1 - \frac{1}{2}\overline{\Delta t}^2}{1 + \xi\overline{\Delta t}}$$

$$A_2 = \frac{c_2}{c_0} = \frac{1 - \xi\overline{\Delta t}}{1 + \xi\overline{\Delta t}}$$

$$\boxed{\overline{\Delta t} = \omega\Delta t}$$

normalized time step

$$
a^2 - 2A_1 a + A_2 = 0, \qquad A_1 = \frac{1 - \frac{1}{2}\overline{\Delta t}^2}{1 + \xi\overline{\Delta t}}, A_2 = \frac{1 - \xi\overline{\Delta t}}{1 + \xi\overline{\Delta t}}, \text{where } \overline{\Delta t} := \Delta t\omega \text{ normalized time step} \Rightarrow \qquad (355a)
$$

$$
\boxed{a_{1,2} = A_1 \pm \sqrt{A_1^2 - A_2} = \frac{2 - \overline{\Delta t}^2 \pm \overline{\Delta t}\sqrt{\overline{\Delta t}^2 - 4(1 - \xi^2)}}{2(1 + \xi\overline{\Delta t})}}
$$

$$|a_{1,2}| < 1$$

- That is, $|a_{1,2}| \leq 1$ and if they are equal (*i.e.*, $A_1^2 = A_2$) $|a_1| = |a_2| < 1$.
- Clearly, it is difficult to obtain stability condition by directly checking the absolute values $|A_{1,2}|$.

- A theory, discussed in the next section, shows that for stability we must have,

$$
\underbrace{|A_2| \leq 1, |A_1| \leq \left|\frac{1 + A_2}{2}\right|}_{a^2 - 2A_1 a + A_2 = 0} \Rightarrow
$$

$$|a_{1,2}| \leq 1$$

Consider $a_1 = a_2 = 1$ separately

$$|A_2| = |a_1 a_2| \leq 1$$

$$\left|\frac{1 - \xi\overline{\Delta t}}{1 + \xi\overline{\Delta t}}\right| \leq 1 \quad \text{automatically satisfied}$$

$$
-1 \leq \frac{1 - \xi\overline{\Delta t}}{1 + \xi\overline{\Delta t}} \leq 1 \quad (|A_2| \leq 1 \text{ automatically}
$$

$$
-\frac{1}{1 + \xi\overline{\Delta t}} \leq \frac{1 - \frac{1}{2}(\overline{\Delta t})^2}{1 + \xi\overline{\Delta t}} \leq \frac{1}{1 + \xi\overline{\Delta t}} \qquad \Rightarrow
$$

$$-\left(\frac{1 + A_2}{2}\right) \leq A_1 \leq \frac{1 + A_2}{2}$$

$$\boxed{\overline{\Delta t} \leq 2}$$

$$\Delta t\,\omega \leq 2$$

$$\boxed{\Delta t \leq \frac{2}{\omega}}$$

$-\frac{1+A_2}{2} \leq A_1 \leq \frac{1+A_2}{2}$

$$\boxed{\Delta t \leq \frac{2}{\omega}}$$

- That is,

$$\text{Central difference time integration for SDOF } \ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = 0 \quad \text{is stable if } \Delta t \leq \frac{2}{\omega} \tag{358a}$$

$$\text{Central difference time integration for MDOF } \mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = 0 \quad \text{is stable if} \quad \Delta t \leq \frac{2}{\max_l(\omega_l^h)}$$

$$\text{or more conservatively \& conveniently} \quad \Delta t \leq \frac{2}{\omega_e^m} \tag{358b}$$

- Recall that ω_e^m is the maximum element level natural frequency which can be easily computed.
- $\max_l(\omega_l^h)$ is the maximum frequency of the MDOF problem which often is not computed.
- Since $\omega_e^m > \max_l(\omega_l^h)$ we can conservatively and conveniently use it in estimating stable time step of conditionally stable methods; *cf.* (311) and §5.2.2.
- One very interesting aspect is that the stable time step is not increased by increasing ξ which typically is the case.

5.3.1.2 Stability region for a 2 × 2 update equation (with real coefficients)

- Consider the update equation for a size two $\hat{\mathbf{X}}$ with real values (*cf.* (347) for a general size m $\hat{\mathbf{X}}$),

$$\hat{\mathbf{X}}_{n+1} = \mathbf{A}\hat{\mathbf{X}}_n, \qquad \text{where} \qquad \mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{359}$$

- The eigenvalues of $\mathbf{A}$ satisfy,

$$a^2 - 2A_1 a + A_2 = 0 \quad \text{where} \tag{360a}$$

$$A_1 = \frac{1}{2}\text{trace}(\mathbf{A}), \qquad \text{trace}(\mathbf{A}) = A_{11} + A_{22} \qquad \text{first invariant of } \mathbf{A} \tag{360b}$$

$$A_2 = \det(\mathbf{A}), \qquad \det(\mathbf{A}) = A_{11}A_{22} - A_{12}A_{21} \qquad \text{second invariant of } \mathbf{A} \tag{360c}$$

- On the other hand, in many instances we directly reach to a second order polynomial of the form (360a). See for example (355a).
- In this section we provide conditions in which the roots of the second order polynomial in (361) satisfy $|a_{1,2}| \leq 1$ or $|a_{1,2}| < 1$ and provide the full analysis (including when the coefficients A_1 and A_2 are complex in §6.4.6.
- In either case, whether the polynomial is directly derived or is from a size two $\hat{\mathbf{X}}_n$ update vector roots of (360a) must satisfy a stability condition which is of the form (338).
- For the resulting second order polynomial stability condition reduces to,

$$a^2 - 2A_1 a + A_2 = 0 \quad \text{correspond to a stable scheme iff} \quad \begin{cases} |a_{1,2}| = \left|A_1 \pm \sqrt{A_1^2 - A_2}\right| \leq 1 & \text{if } a_1 \neq a_2 \quad \text{that is } A_1^2 \neq A_2 \\ |a_{1,2}| = |A_1| < 1 & \text{if } a_1 = a_2 \quad \text{that is } A_1^2 = A_2 \end{cases} \tag{361a}$$

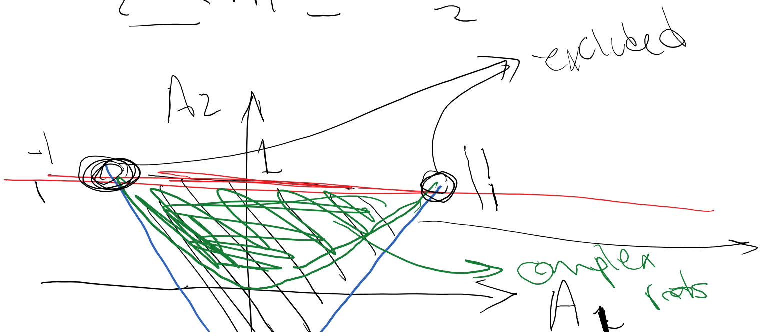
$|A_2| \leq 1$

$-\frac{A_2+1}{2} \leq A_1 \leq \frac{A_2+1}{2}$

$A_1 = \pm 1$

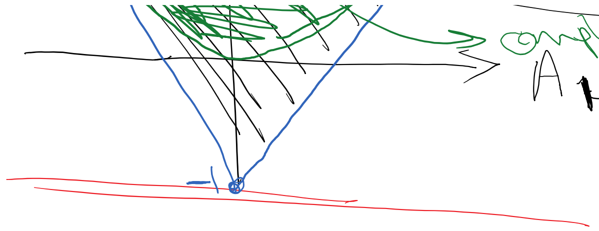
$A_2 = \sqrt{1} = 1$ is not acceptable

$x_n = A^n x_0$



$A_1^2 < A_2$

$A_1^2 - A_2 < 0$



complex A_1 roots → region of stability

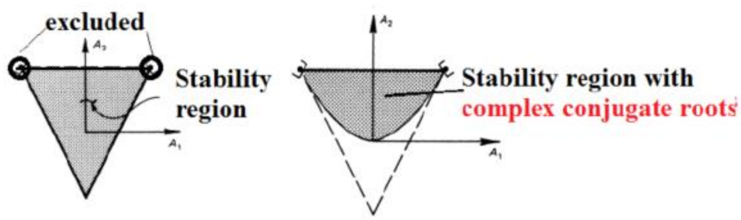
$A_1^2 - A_2 < 0$

complex conjugate roots

- Another important condition is whether the roots are real or complex (in complex conjugate pairs):

$$a_{1,2} \text{ are complex iff } A_1^2 < A_2 \tag{363}$$

- Having complex conjugate roots can damp out high frequency content in various time marching schemes of the solution which is often desirable. This will be discussed further for Newmark methods.
- Overall stability region, and stability region with complex conjugate roots are shown below.



5.3.1.3 Stability analysis of LMS methods: Houbolt method

- We previously discussed the 3-step LMS Houbolt method in §4.3.2 with the FD operators:

$$
\begin{aligned}
{}^{t+\Delta t}\ddot{x} + 2\xi\omega\, {}^{t+\Delta t}\dot{x} + \omega^2\, {}^{t+\Delta t}x &= {}^{t+\Delta t}r \\
{}^{t+\Delta t}\ddot{x} &= \frac{1}{\Delta t^2}(2\, {}^{t+\Delta t}x - 5\, {}^{t}x + 4\, {}^{t-\Delta t}x - {}^{t-2\Delta t}x) \\
{}^{t+\Delta t}\dot{x} &= \frac{1}{6\Delta t}(11\, {}^{t+\Delta t}x - 18\, {}^{t}x + 9\, {}^{t-\Delta t}x - 2\, {}^{t-2\Delta t}x)
\end{aligned}
\tag{364}
$$

→ implicit

- which results in the update equation,

$$
\begin{bmatrix} {}^{t+\Delta t}x \\ {}^{t}x \\ {}^{t-\Delta t}x \end{bmatrix} = \mathbf{A} \begin{bmatrix} {}^{t}x \\ {}^{t-\Delta t}x \\ {}^{t-2\Delta t}x \end{bmatrix} + \mathbf{L}\, {}^{t+\Delta t}r \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} \dfrac{5\beta}{\omega^2\Delta t^2} + 6\kappa & -\left(\dfrac{4\beta}{\omega^2\Delta t^2} + 3\kappa\right) & \dfrac{\beta}{\omega^2\Delta t^2} + \dfrac{2\kappa}{3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
$$

$$
\beta = \left(\frac{2}{\omega^2\Delta t^2} + \frac{11\xi}{3\omega\Delta t} + 1\right)^{-1}; \qquad \kappa = \frac{\xi\beta}{\omega\Delta t} \qquad \mathbf{L} = \begin{bmatrix} \dfrac{\beta}{\omega^2} \\ 0 \\ 0 \end{bmatrix}
\tag{365}
$$

- Noticeably $\rho(\mathbf{A}) < 1$ for all Δt meaning that Houbolt method is unconditionally stable, as mentioned before.

5.3.2 Absolute stability, A-stable methods

5.3.2.1 Introduction: Properties of analytical solution

We consider the pth order ODE

$$a_p \frac{d^p u}{dt^p} + a_{p-1}\frac{d^{p-1} u}{dt^{p-1}} + \dots + a_1 \dot{u} + a_0 u(t) = f(t)$$

with ID $u(0) = u_0$, $\dot{u}(0) = \dot{u}_0$, ..., $\dfrac{d^{p-1}u}{dt^{p-1}}(0) = u_0^{(p-1)}$

with (1) $u(0) = u_0(0)$, $u(0) = u_0 \ldots \frac{du}{dt^{p-1}}(0) = u_0^{v}$

$$u(t) = \underbrace{u_p(t)}_{\text{solves inhomog eqn (*) with } f(t)} + \sum_{i=1}^{p} a_i u_i(t)$$

general slns solving ODE with $f(t) = 0$

one of general slns

$$u(t) = e^{\lambda t} \longrightarrow a_p \lambda^p e^{\lambda t} + a_{p-1}\lambda^{p-1} e^{\lambda t} + \cdots + a_1 \lambda e^{\lambda t} + a_0 e^{\lambda t} = 0$$

- We first discuss how the solution is obtained for homogeneous case ($f(t) = 0$) and later comment on the solution of (366).
- Considering $f(t) = 0$ and letting $u(t) = e^{\lambda t}$ we obtain,

$$u(t) = e^{\lambda t}, f(t) = 0, \text{ so (366)} \quad \Rightarrow \boxed{a_p\lambda^p + a_{p-1}\lambda^{p-1} + \cdots + a_1\lambda + a_0 = 0} \tag{367}$$

We obtain p roots λ from the characteristic polynomial.

$$\lambda's = \{-.5, -.5, 1, 2, 2, 2, 5+i, 5-i\} \quad p = 8$$

$$\underbrace{e^{-.5t} \quad te^{-.5t}}_{\lambda_0} \quad \underbrace{e^{t}}_{\lambda_1} \quad \underbrace{e^{2t} \; te^{2t} \; t^2e^{2t}}_{\lambda_2} \quad e^{5t} \quad \lambda_3 \quad \lambda_4$$

- If the root i is repeated m_i times, the general solution corresponding to this root is $P_i(t)e^{\lambda_i t}$ where P_i is an $m_i - 1$ order polynomial.
- So, the solution is,

$$a_p \frac{d^p u}{dt^p} + a_{p-1}\frac{d^{p-1}u}{dt^{p-1}} + \cdots + a_1 \frac{du}{dt} + a_0 u(t) = 0 \quad \Rightarrow \quad \underbrace{u(t) = \sum_{i=1}^{\bar{p}} P_i(t) e^{\lambda_i t}}_{\text{polynomials of order } m_i - 1} \tag{368}$$

where λ_i are distinct roots of (367) with multiplicities m_i.

$$u(t) = \underbrace{P_0(t)}_{\text{order 1}} e^{-.5t} + \underbrace{P_1}_{\text{constant}} e^{1t} + \underbrace{P_2(t)}_{\text{order 2}} e^{2t} + P_3 e^{(5+i)t} + P_4 e^{(5-i)t}$$

individual terms are

$$\boxed{P_i(t)\, e^{\lambda_i t}}$$

order $m_i - 1$

when is this term bounded

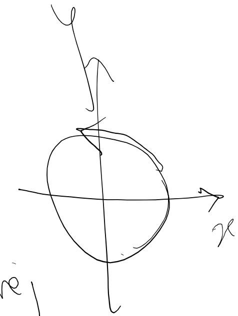
m_i is algebraic multiplicity of λ_i

.

drop i

$$P(t)\, e^{(\lambda_r + i\lambda_i)t} = \boxed{P(t)\, e^{\lambda_r t}\, e^{i\lambda_i t}}$$

$$|e^{i\lambda_i t}| = 1$$



- $\lambda_r < 0$ decays
- $\lambda_r = 0$
  - $m = 1 \rightarrow P(t) = P_{\text{constant}}$ Constant (conservative mode)
  - $m > 1$ weakly (algebraically) becoming unbounded
  - $t\ (m=2)$, $t^2\ (m=3)$, . . .
- $\lambda_r > 0$ $e^{\lambda_r t}$ Strongly (exponentially) unbounded

- To study the boundedness of (369) let,

$$\lambda_i = \lambda_i^R + \imath\lambda_i^I \quad \Rightarrow \quad u_i(t) = P_i(t)e^{(\lambda_i^R+\imath\lambda_i^I)t} = P_i(t)e^{\imath\lambda_i^I t}e^{\lambda_i^R t} \tag{370}$$

where λ_i^R, λ_i are real and imaginary parts of λ_i and $\imath$ is the imaginary number ($\imath^2 = -1$).

- Now, the boundedness of $u_i(t)$ depends on the sign of λ_i^R given that $|e^{\imath\lambda_i^I t}| = 1$:

$$\begin{cases} \lambda_i^R < 0 & \text{Bounded and diminishing, } (\lim_{t\to\infty} u_i(t) = 0) \\ \lambda_i^R = 0 \;\&\; m_i = 1 (\lambda_i \text{ simple root of (367)}) & \text{Bounded and oscillatory (the solution oscillated but not tending to zero)} \\ \lambda_i^R = 0 \;\&\; m_i > 1 (\lambda_i \text{ repeated root of (367)}) & \text{Weakly (algebraically) unbounded: } u_i(t) \text{ algebraically tends to } \infty \text{ as } t \to \infty \\ \lambda_i^R > 0 & \text{Strongly (exponentially) unbounded, } (\lim_{t\to\infty} u_i(t) = \infty) \end{cases} \tag{371}$$

- So for the (stability) boundedness of the solution to (368) with $f(t) = 0$ we have,

$$\begin{cases} \text{Exponentially unbounded} & \text{If } \underline{\text{any}} \text{ root has positive real part} \\ \text{Algebraically unbounded} & \text{If all } \lambda_i \le 0 \text{, but roots with } \lambda_i^R = 0 \text{ are repeated } m_i > 1 \\ \text{Bounded} & \text{If all } \lambda_i \le 0 \text{, and roots (if any) with } \lambda_i^R = 0 \text{ are simple } m_i = 1 \end{cases} \tag{372}$$

- Obviously, unstable modes may not get activated for particular ICs if their $P_i(t)$ is identically zero.
- An important question is if the IC is perturbed a bit, whether the perturbation result in unbounded changes in the solution as $t \to \infty$. This property is called dynamic-stability and for an p^{th} order linear ODE, it requires the satisfaction of (372).
- The solution of (366) for $f(t) \neq 0$ can be obtained by using Laplace transform. The solution will include convolutions of the kernels of the form (369) and $f(t)$. The details of the process can be found in any introductory ODE book.
- We are more interested in knowing when the exact solution (*i.e.*, physical system) is dynamically stable and afterward knowing when a numerical method can maintain "stability" in a numerical setting, meaning that the solution does not blowup.

$\dot{d} + \lambda d = 0$ $\quad d(t) = e^{-\lambda t}$ $\quad \lambda > 0 \longrightarrow$ dynamic stability

T^{final}

Gen-trap.

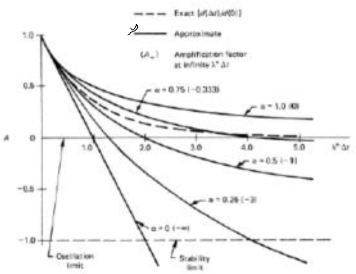
$\alpha + \lambda\alpha = \sim$ sensitivity

Gen-trap.

t_{n+1}

$\alpha \Delta t$

t_n



Amplification factor for typical one-step methods.

Summary: Stability for the generalized trapezoidal methods

Amplification factor: $A = \dfrac{1-(1-\alpha)\Delta t \lambda^h}{1+\alpha \Delta t \lambda^h}$

Stability requirement: $|A| < 1$ for $\lambda^h = \lambda^h_{max}$ (= maximum eigenvalue)

Unconditional stability: $\alpha \geq \frac{1}{2}$

Conditional stability: $\alpha < \frac{1}{2}$, $\Delta t < \dfrac{2}{(1-2\alpha)\lambda^h_{max}}$

- The maximum stable time stable can be chosen as follows

$$\alpha \geq \frac{1}{2} \quad \Rightarrow \quad \text{any } \Delta t \qquad \text{Unconditional stability} \qquad (309a)$$

5.3.2.2 Absolute stability

- Consider the first order ODE,

$$\dot{x} - \lambda x = 0 \qquad (373)$$

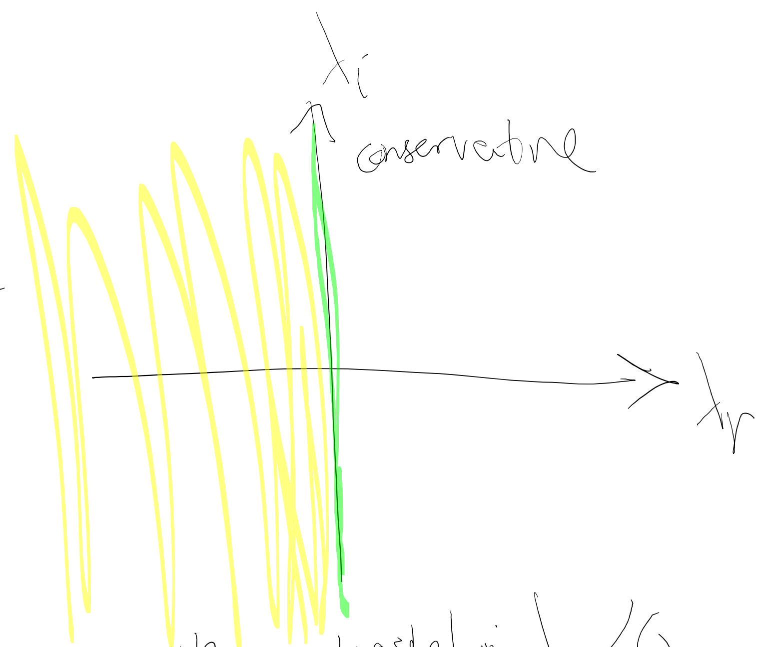
$x = e^{\lambda t}$

like pth order ODE we discussed before

$= \cancel{P(\lambda)}\, e^{\lambda_r t} \, \underbrace{e^{i\lambda_i t}}$

repeated ence

$|\ | = 1$



$\lambda_r = 0$ Conservative

$\lambda_r < 0$ decaying

We are interested in solution decays $\lambda_r < 0$

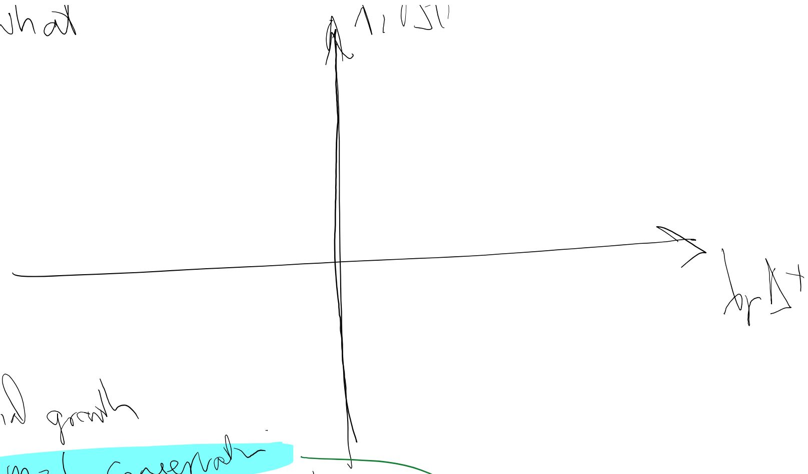
Numerical solution of:

$\dot{x} - \lambda x = 0$ & see for what

$\lambda_i \Delta t$

$\dot{x} - \lambda x = 0$ ⌀ see for what

for what Δt's the numerical method is stable.



$X^{(n)} = A^n X^{(0)}$

$$\rho(A) = \begin{cases} >1 \longrightarrow \text{exponential growth} \\ =1 \longrightarrow \lambda=1 \quad m=1 \quad \text{conservation} \\ \longrightarrow \lambda=1 \quad m>1 \quad \text{algebraic growth} \\ \boxed{<1 \quad \text{decaying}} \end{cases}$$

Spectral stable

// doesn't blow up //

decaying property

Absolute stability

- An ODE solver whose update equation can be cast in the form $\hat{\mathbf{X}}_n = \mathbf{A}^n \hat{\mathbf{X}}_0$ ($\mathbf{A}$ is the amplification factor), *e.g.*, RK or any LMS scheme, is said to be absolutely stable at a fixed $\lambda \Delta t$ if the spectral radius of A is strictly less than 1: $\rho(\mathbf{A}) < 1$ for the solution of $\dot{x} - \lambda x = 0$ is stable with the time step Δt.
- Recall that spectral stability (338) in fact allows $\rho(\mathbf{A}) = 1$, whereas for absolute stability $\rho(\mathbf{A}) < 1$.
- To reiterate, spectral stability (338) is repeated here,
  - $\rho(\mathbf{A}) \leq 1$ and if $\mathbf{A}$ is not diagonalizable ,
  - eigenvalues a_i with $n_i^A > n_i^G$ satisfy $|a_i| < 1$.

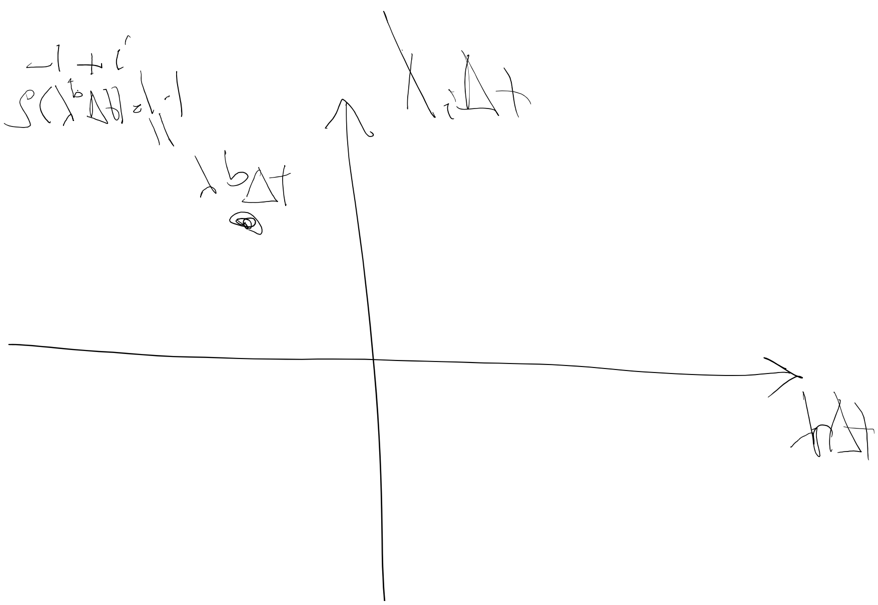
  which clearly allows $\rho(\mathbf{A}) = 1$.

- This means that for an absolutely stable point $\lambda \Delta t$: $\rho(\mathbf{A}) < 1$, the numerical solution definitely dissipates and approaches zero as $n \to 0$ (n is the time step).

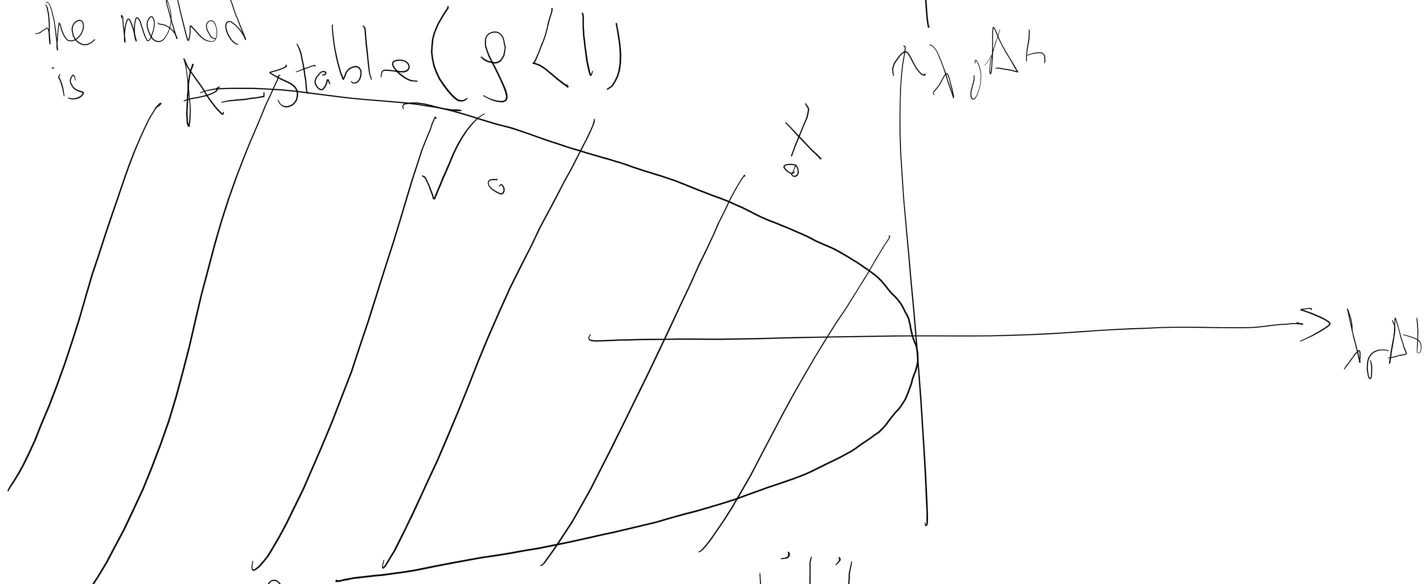
Region of absolute stability:

$\Delta t \lambda^a = -5 + i$

$\rho(\lambda^a \Delta t) = 5$



Collect all $\lambda \Delta t$'s for which for which the method is A-stable ($\rho < 1$)



region of absolute stability

$\lambda \Delta t$

$\{\lambda \Delta t \mid \lambda_r < 0\} \subseteq$ Region of absolute stability

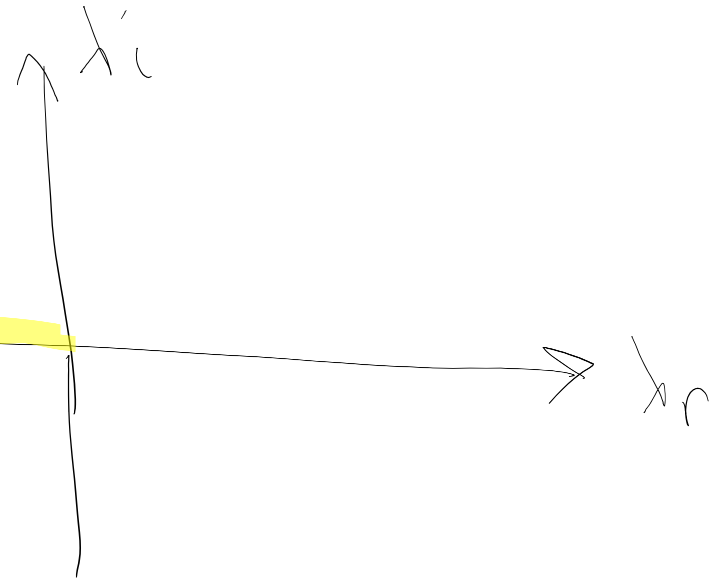
A-Stable

We are doing as good as the physics is doing! That's as close as we can get to the nice properties of the underlying ODE

We are doing as good as the physics is doing! That's as close as we can get to the nice properties of the underlying ODE

$\lambda < 0$ real (*)

$\dot{x} \; \lambda x = 0$



What does unconditional stability mean for (*)