DC2020/04/13

Monday, April 13, 2020 1:10 PM

Continuing the discussion on absolute stability:

$$a_p\frac{\mathrm{d}^p u}{\mathrm{d}t^p} + a_{p-1}\frac{\mathrm{d}^{p-1} u}{\mathrm{d}t^{p-1}} + \cdots + a_1\frac{\mathrm{d}u}{\mathrm{d}t} + a_0 u(t) = f(t)$$

$u(t) = C_i e^{\lambda_i(t)}$

$$u(t) = e^{\lambda t},\ f(t) = 0,\ \text{so (366)} \quad \Rightarrow \boxed{a_p\lambda^p + a_{p-1}\lambda^{p-1} + \cdots + a_1\lambda + a_0 = 0} \tag{367}$$

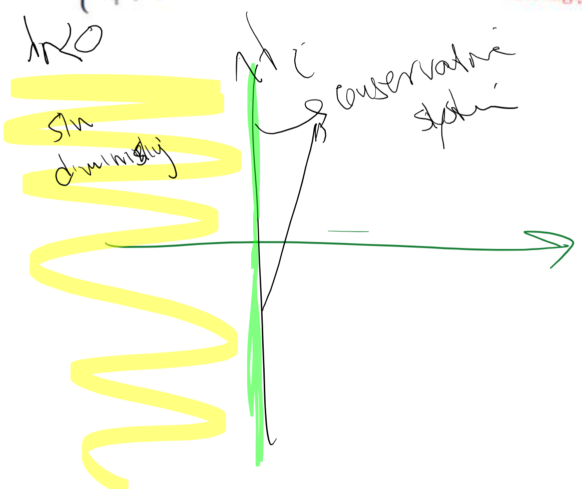
- To study the boundedness of (369) let,

$$\lambda_i = \lambda_i^R + \imath\lambda_i^I \qquad \Rightarrow \qquad u_i(t) = P_i(t)e^{(\lambda_i^R + \imath\lambda_i^I)t} = P_i(t)e^{\imath\lambda_i^I t}e^{\lambda_i^R t} \tag{370}$$

where λ_i^R, λ_i are real and imaginary parts of λ_i and $\imath$ is the imaginary number ($\imath^2 = -1$).

- Now, the boundedness of $u_i(t)$ depends on the sign of λ_i^R given that $|e^{\imath\lambda_i^I t}| = 1$:

$$\begin{cases} \lambda_i^R < 0 & \text{Bounded and diminishing, } (\lim_{t\to\infty} u_i(t) = 0) \\ \lambda_i^R = 0 \ \&\ m_i = 1 (\lambda_i \text{ simple root of (367)}) & \text{Bounded and oscillatory (the solution oscillated but not tending to zero)} \\ \lambda_i^R = 0 \ \&\ m_i > 1 (\lambda_i \text{ repeated root of (367)}) & \text{Weakly (algebraically) unbounded: } u_i(t) \text{ algebraically tends to } \infty \text{ as } t \to \infty \\ \lambda_i^R > 0 & \text{Strongly (exponentially) unbounded, } (\lim_{t\to\infty} u_i(t) = \infty) \end{cases} \tag{371}$$



exact sln $A_i e^{\lambda_i t}$

λ_r

n'th ODE $\equiv$

n 1st order ODE

$\dot{u}_\alpha - \lambda_\alpha u_\alpha = 0 \longrightarrow u_\alpha = A e^{\lambda_\alpha t}$

want to see the region of numerical stability for

$e^{\lambda t} \longrightarrow$ exact

$\lambda_r < 0$

$\boxed{\dot{u} - \lambda u = 0}$

$e^{-\lambda_r t}$

$-\lambda$ $\quad\longleftarrow\quad$ λ

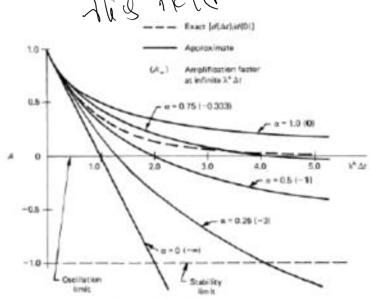
we analyzed this $\quad \dot{u} + \lambda u = 0$

Generalized-α

$|A| < 1 \longrightarrow$ find (λ_r, λ_i) for which this hold

$$\boxed{A = \frac{1 + (1-\alpha)(\Delta t\lambda)}{1 - \alpha(\Delta t\lambda)}}$$

$$\Delta t < \frac{2}{(1-2\alpha)(-\lambda)}$$



Amplification factor for typical one-step methods.

Summary: Stability for the generalized trapezoidal methods

Amplification factor: $\boxed{A = \dfrac{1 - (1-\alpha)\Delta t\lambda^h}{1 + \alpha\Delta t\lambda^h}}$

Stability requirement: $|A| < 1$ for $\lambda^h = \lambda^h_{n_{eq}}$ (= maximum eigenvalue)

Unconditional stability: $\alpha \geq \frac{1}{2}$

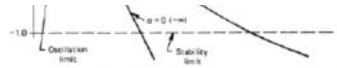
Conditional stability: $\alpha < \frac{1}{2}$, $\boxed{\Delta t < \dfrac{2}{(1-2\alpha)\lambda^h_{n_{eq}}}}$

$\Delta t \geq (1-2\alpha)(-\lambda)$

comes from spectral stability $\rho(A) = |A| \leq 1$ spectral

for asplute stability $\rho(A) = |A| < 1$

Conditional stability: $\alpha < \frac{1}{2}$, $\boxed{\Delta t < \frac{2}{(1-2\alpha)\lambda^h_{\max}}}$



Amplification factor for typical one-step methods.

- The maximum stable time stable can be chosen as follows

$$\alpha \geq \frac{1}{2} \quad \Rightarrow \quad \text{any } \Delta t \qquad \text{Unconditional stability} \qquad (309a)$$

$\Delta t < \frac{2}{(1-2\alpha)(-\lambda)} < 0$ $\quad \times \quad -\lambda$ $\quad \lambda = \lambda_r + i\lambda_i < 0$, $\lambda_i = 0$

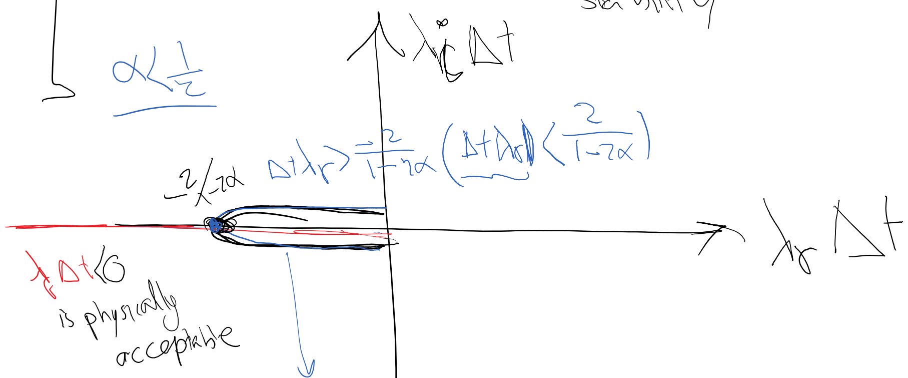
$\boxed{\Delta t \lambda_r > \frac{-2}{1-2\alpha}}$ $\quad \alpha < \frac{1}{2}$

$\Delta t \lambda_r$ free $\quad$ unconditional stability $\quad \alpha \geq \frac{1}{2}$

$\dot{u} - \lambda u = 0$

λ = fixed $\in \mathbb{R}$

$\lambda_r < 0$



$\alpha < \frac{1}{2}$

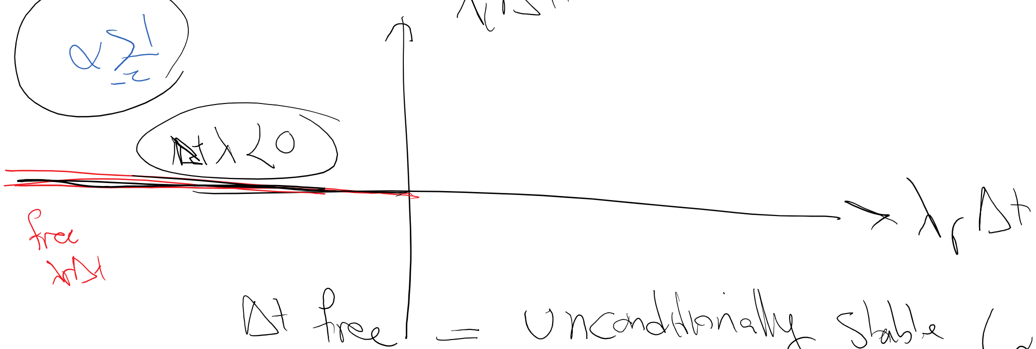
$\lambda_i \Delta t$

$-2/1-2\alpha$ $\quad \Delta t \lambda_r > \frac{-2}{1-2\alpha}$ $(\Delta t \lambda_r) < \frac{2}{1-2\alpha}$

$\lambda_r \Delta t$

$\lambda_r \Delta t < 0$ is physically acceptable

Conditional stability $\Delta t \leq \Delta t_{max}$



$\alpha \geq \frac{1}{2}$

$\lambda_i \Delta t$

$\Delta t \lambda < 0$

free $\lambda_r \Delta t$

$\lambda_r \Delta t$

$\lambda \in \mathbb{R}$

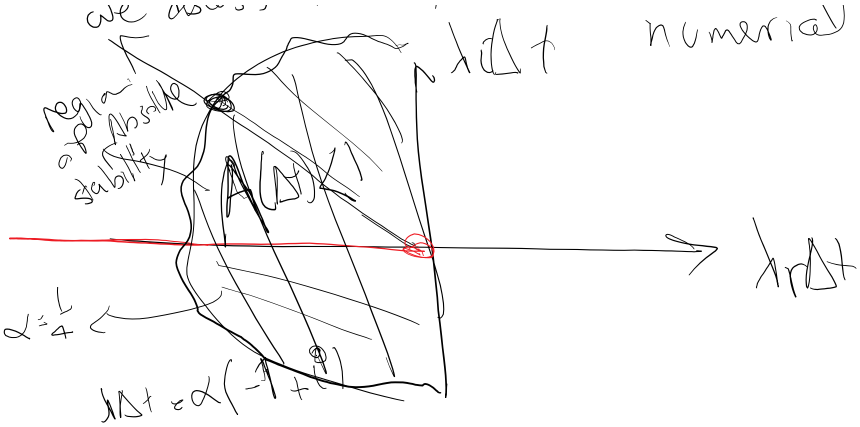
$\lambda < 0$

Δt free = Unconditionally stable ($\alpha = \frac{1}{2}$ trap.. + $\alpha = 1$ FE)

why do we even care beyond negative real axis when we discuss stability

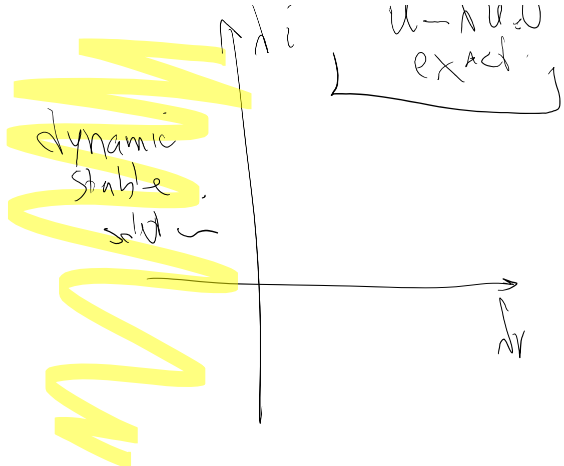
$\lambda_i \Delta t$ $\quad$ numerical

λ_i $\quad \dot{u} - \lambda u = 0$ exact



$\lambda \Delta t = \alpha(-1+i)$

$$\lambda = \frac{\alpha(-1+i)}{\Delta t}$$



$\ddot{x} + x = 0$

$$\begin{cases} \dot{u}_1 + i u_1 = 0 \\ \dot{u}_2 - i u_2 = 0 \end{cases}$$

Continuing the discussion on absolute stability:

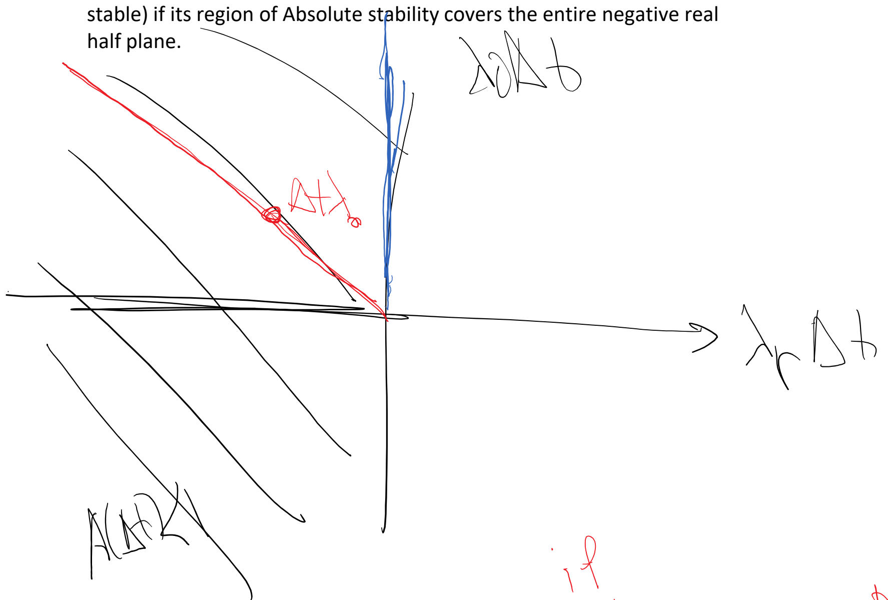
$$a_p \frac{d^p u}{dt^p} + a_{p-1} \frac{d^{p-1} u}{dt^{p-1}} + \cdots + a_1 \frac{du}{dt} + a_0 u(t) = f(t)$$

$u(t) = e^{\lambda t}, f(t) = 0$, so (366) $\Rightarrow$ $\boxed{a_p \lambda^p + a_{p-1} \lambda^{p-1} + \cdots + a_1 \lambda + a_0 = 0}$

$u(t) = \sum B_\alpha e^{\lambda_\alpha t}$

λ_α's are complex

A numerical method (time integration) is called A-stable (absolutely stable) if its region of Absolute stability covers the entire negative real half plane.



$\dot{u} + (-2) u = 0$

unconditionally stable (Δt is free) ← if $\Delta t \lambda_0 \in$ requiring of A-stability

particular ODE

$\dot{u} + (-1+i) u = 0$

$\lambda_0 = -1+i$

A-stability

the solution is unconditionally stable

if Δt is free in

neg. A-stability

$\dot{u} + i u = 0$

$\lambda_1 = i$

For unconditional stability we work with a particular ODE (lambda is fixed) and see if delta t is free.

$\dot{u} + \lambda_0 u = 0$ fixed, is Δt free

A-stability means for ANY lambda, delta t is free

$\lambda_R \le 0$

A-stable: whatever dynamically stable ODE is thrown at the numerical ODE solver, delta T is free.
Unconditional stability only can be discussed in the context of a particular ODE for which we can say delta T is free (this cannot be generalized to other ODEs, for example unconditional stability for heat equation (lambda = -5) does not carry over to unconditional stability for undamped wave equation (lambda = +- i)

- We are more interested in knowing when the exact solution (*i.e.*, physical system) is dynamically stable and afterward knowing when a numerical method can maintain "stability" in a numerical setting, meaning that the solution does not blowup.

- Unfortunately, the condition of A-stability is extremely demanding. [Dahlquist, 1963] has shown the following results (known as Dahlquist Second Barrier Theorem):
  1. No explicit linear multi-step method (LMS) is A-stable.
  2. No A-stable LMS can have order greater than 2.
  3. The second-order A-stable LMS with the smallest error constant is the trapezoid rule method.

the best we can do is gen. alpha with $\alpha = \frac{1}{2}$

- The result for explicit methods is expected because they are not even unconditionally stable for an equation in the form $\dot{x} - \lambda x = 0$ for real $\lambda < 0$. That is, they do not even cover the negative real axis for $\lambda \Delta t$ let alone the negative complex plane ($\lambda^R < 0$ arbitrary λ^I).
- The result for implicit LMS methods, however, is disappointing implying that if we want a LMS method that 100% preserves the well-posedness region of complex plane ($\lambda^R < 0$) by allowing arbitrary Δt we are at most offered a second order of accuracy, for which trapezoidal rul has the smallest error constant.

Can I do consider other properties than A-stability?

5.3.1 Stability analysis of LMS methods

- General first and second order linear ODEs applied to a vector $\mathbf{y}$ can be represented as follows, (*cf.* (240) for general nonlinear first order expression of an ODE),

$$\dot{\mathbf{y}} = f(\mathbf{y}, t) = \mathbf{G}_0\mathbf{y} + \mathbf{H}(t) \qquad \text{Linear first order ODE} \qquad (340a)$$
$$\ddot{\mathbf{y}} = f(\mathbf{y}, \dot{\mathbf{y}}, t) = \mathbf{G}_1\dot{\mathbf{y}} + \mathbf{G}_0\mathbf{y} + \mathbf{H}(t) \qquad \text{Linear second order ODE} \qquad (340b)$$

- When a k-step LMS method is applied to an ODE $\mathbf{y}_{n+1}$ in terms of $\mathbf{y}_n, \mathbf{y}_{n-1}, \cdots, \mathbf{y}_{n-k+1}$.
- Formally, the expressions of an k-step LMS method applied to linear first and second order ODEs in (340) are,

$\dot{y}(t_{n+1-i}) = f(t_{n+1-i})$

$$\sum_{i=0}^{k} \{\alpha_i \mathbf{y}_{n+1-i} + \Delta t \beta_i [\mathbf{G}_0\mathbf{y}_{n+1-i} + \mathbf{H}(t_{n+1-i})]\} = 0 \qquad \text{LMS applied Linear first order ODE} \qquad (341a)$$

$$\sum_{i=0}^{k} \{\alpha_i \mathbf{y}_{n+1-i} + \Delta t \beta_i \mathbf{G}_1 \mathbf{y}_{n+1-i} + \Delta t^2 \gamma_i [\mathbf{G}_0\mathbf{y}_{n+1-i} + \mathbf{H}(t_{n+1-i})]\} = 0 \qquad \text{LMS applied Linear second order ODE} \qquad (341b)$$

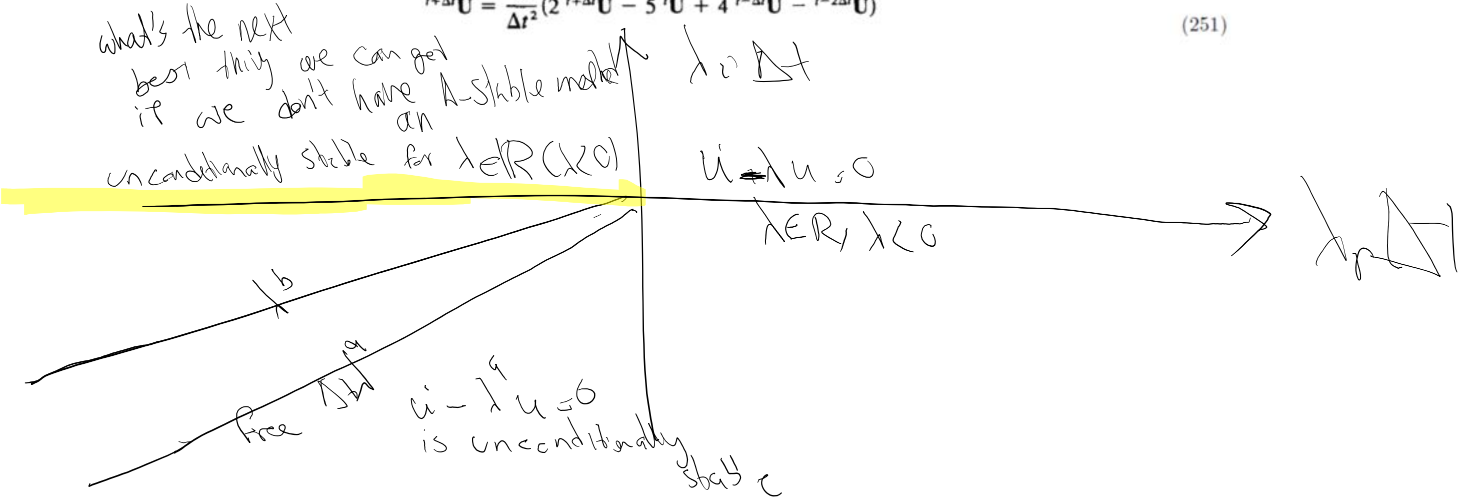
BDF's are a particular group of LMS for which $\beta_i = 0$, $i > 0$

An example is Houbolt method, if we write the following equation in the form (341a) and find alpha_i and beta_is, we'll see beta_i, I > 0 are zero.

4.3.2 Houbolt method (an implicit LMS method for elastodynamics)

- Houbolt method is a LMS ($k = 3$) implicit method where the FD stencils for $\ddot{\mathbf{U}}$ and $\dot{\mathbf{U}}$ are

$$
\begin{aligned}
{}^{t+\Delta t}\dot{\mathbf{U}} &= \frac{1}{6\Delta t}\left(11\,{}^{t+\Delta t}\mathbf{U} - 18\,{}^{t}\mathbf{U} + 9\,{}^{t-\Delta t}\mathbf{U} - 2\,{}^{t-2\Delta t}\mathbf{U}\right)\\
{}^{t+\Delta t}\ddot{\mathbf{U}} &= \frac{1}{\Delta t^2}\left(2\,{}^{t+\Delta t}\mathbf{U} - 5\,{}^{t}\mathbf{U} + 4\,{}^{t-\Delta t}\mathbf{U} - {}^{t-2\Delta t}\mathbf{U}\right)
\end{aligned}
\qquad (251)
$$



- These Backward Differentiation Formulae (BDF), which are LMS schemes, can have **orders of accuracy as high as 6**, yet covering **all negative real axis** in their region of absolute stability.

The coefficients are obtained by requiring that the order of accuracy of the method is as high as possible, *i.e.*, by making the coefficients C_j zero in (12.47) for $j = 0, 1, \ldots, k$. For $k = 1$ this yields the implicit Euler method (BDF1), whose order of accuracy is, of course, 1; the method is A-stable. The choice of $k = 6$ results in the sixth-order, six-step BDF method (BDF6):

$$
147y_{n+6} - 360y_{n+5} + 450y_{n+4} - 400y_{n+3} + 225y_{n+2} - 72y_{n+1} + 10y_n = 60hf_{n+6}\,. \qquad (12.50)
$$

y from all prev & current steps

but $y' = f$ only for the current step

Although the method (12.50) is not A-stable, its region of absolute stability includes the whole of the negative real axis (see Figure 12.5). For

To construct useful methods of higher order we need to relax the condition of A-stability by requiring that the region of absolute stability should include a large part of the negative half-plane, and certainly that it contains the whole of the negative real axis.

The most efficient methods of this kind in current use are the **Backward Differentiation Formulae**, or BDF methods. These are the linear multistep methods (12.35) in which $\beta_j = 0$, $0 \le j \le k-1$, $k \ge 1$, and $\beta_k \ne 0$. Thus,

$$
\alpha_k y_{n+k} + \cdots + \alpha_0 y_n = h\beta_k f_{n+k}\,.
$$

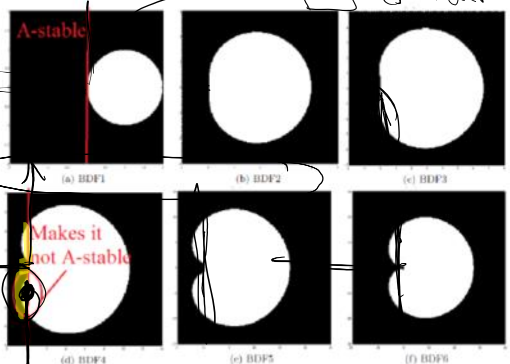
the intermediate values, $k = 2, 3, 4, 5$, we have the following kth-order, k-step BDF methods, respectively:

$$\begin{aligned} 3y_{n+2} - 4y_{n+1} + y_n &= 2hf_{n+2}\,, \\ 11y_{n+3} - 18y_{n+2} + 9y_{n+1} - 2y_n &= 6hf_{n+3}\,, \\ 25y_{n+4} - 48y_{n+3} + 36y_{n+2} - 16y_{n+1} + 3y_n &= 12hf_{n+4}\,, \\ 137y_{n+5} - 300y_{n+4} + 300y_{n+3} - 200y_{n+2} + 75y_{n+1} - 12y_n &= 60hf_{n+5}\,, \end{aligned}$$

referred to as BDF2, BDF3, BDF4 and BDF5. Their regions of absolute stability are also shown in Figure 12.5. In each case the region of absolute stability includes the negative real axis. Higher-order methods of this type cannot be used, as all BDF methods, with $k > 6$, are zero-unstable.

good is A stable $|A| \le 1$
is bad $|A| > 1$

regions of absolute - stability for BDF1 - BDF6



Absolute stability regions: the region of absolute stability is the set of points in the complex plane outside the white region

The region of absolute stability contains all the negative real axis.

5.3.2.4 Uses of region of absolute stability plots in practice

- The question that may arise from the discussion in (5.3.2.3) is that why worry absolute stability of a method in the whole left complex half plane ($\lambda\Delta t$, $\lambda^R < 0$) rather than only the negative real axis ($\lambda^R < 0, \lambda^I = 0$).
- For example, we solve equations of the form,

$$\dot{x} + 2x = 0, \qquad (\lambda = -2)$$

and not equations of the form

$$\dot{x} + \imath x = 0, \qquad (\lambda = -\imath), \qquad \text{or} \qquad \dot{x} + (1+\imath)x = 0, \qquad (\lambda = -1-\imath)$$

- However, let us consider the following second order ODEs,

undamped oscillator

damped oscillator

$$\ddot{x} + x = 0 \tag{375a}$$
$$\ddot{x} + 2\dot{x} + 2x = 0 \tag{375b}$$

$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = 0$

- However, let us consider the following second order ODEs,

$\xi = 0 \quad \omega = 1$

$$\ddot{x} + x = 0 \tag{375a}$$

Underdamped

$$\ddot{x} + 2\dot{x} + 2x = 0 \tag{375b}$$

$\xi\sqrt{2} = 1 \quad \xi = \frac{1}{\sqrt{2}} \quad \omega = \sqrt{2}$

< 1

- These can be for example SDOFs (229a) ($\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = f(t)$) that are obtained from model decomposition of (225) $\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$ with $f(t) = 0$).

$\lambda = \lambda i \Delta t$

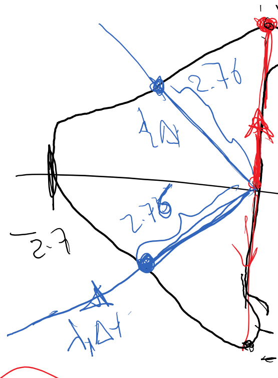
2.5

2

2.76

RK-4

$\dot{u} - \lambda u = 0$



$\lambda\Delta t$

RK-4 $\qquad \dot u - \lambda u = 0$

analyze the numerical method stability

$|A(\Delta t)| < 1$

Consider case (−) ?

$|\lambda_1 \Delta t| \le 2.5 \to |i|\Delta t \le 2.5 \to |\Delta t| \le 2.5$

$\lambda_2 \Delta t \to |-i||\Delta t| \le 2.5 \to |\Delta t| \le 2.5$

$\lambda_1 = i \qquad \lambda_2 = -i$

$\boxed{\ddot x + x = 0} \qquad x = e^{\lambda t} \to \lambda^2 + 1 = 0 \qquad \lambda = \pm i \qquad x = \underbrace{\alpha_1 e^{-it}} + \underbrace{\alpha_2 e^{it}}$

$\Delta t \le \min\{2.5, 2.5\} = 2.5$ ☺

$\ddot x + 2\dot x + 2x = 0 \qquad x = e^{\lambda t} \to \lambda^2 + 2\lambda + 2 = 0 \to \lambda = \dfrac{-1 \pm \sqrt{1-2}}{1} = -1 \pm i$

$\lambda_1 = -1 - i \qquad \underbrace{\quad}_{\sqrt{2}}$

$|\lambda_1 \Delta t| \le 2.76 \to |-1-i|\Delta t_1 \le 2.76 \to \Delta t_1 \le 1.94$

$|-1+i|\Delta t_2 \le 2.76 \qquad \Delta t_2 \le 1.94 \to \boxed{\Delta t \le 1.94}$

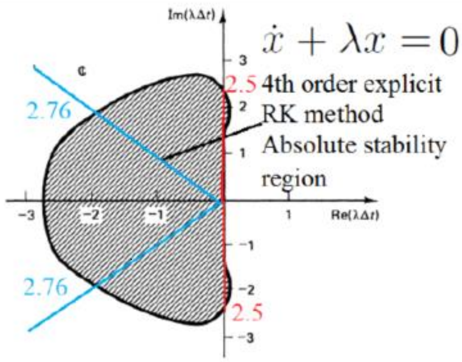
- Back to stability analysis of (5.3.2.4), we observe that in (376a) $\lambda = \pm i$ and the maximum time step is obtained by finding the location that a ray with angles $\pi/2$ and $3\pi/2$ intersects region of absolute stability first.
- Similarly for (376b), the roots are $-1 \pm \imath$ which make lines of angle $3\pi/4$ and $5\pi/4$. In this case, we need to find the intersections of the region of absolute stability with rays of these angles.
- Let us consider that we are using a RK4 method whose region of absolute stability versus $\lambda\Delta t$ is shown in the next figure.
- We first obtain RK4 stable time step for (376a) $\ddot x + x = 0$, decoded with color red. The intersections of the roots $\pm\imath$ with region of absolute stability are both 2.5 (shown in the figure). So,

$$|\lambda \Delta t| \le 2.5, \;\; \lambda = \pm\imath \qquad \Rightarrow |1|\Delta t \le 2.5 \qquad \to \boxed{\Delta t \le 2.5} \tag{377}$$

- and for the damped equation (376b) $\ddot x + 2\dot x + 2x = 0$, the intersection point of rays with angles $3\pi/4, 5\pi/4$ shown in blue in the figure are both 2.76. So,

$$|\lambda \Delta t| \le 2.76, \;\; \lambda = -1 \pm \imath \qquad \Rightarrow |\sqrt{2}|\Delta t \le 2.76 \qquad \to \boxed{\Delta t \le 1.94} \tag{378}$$

$\dot{x} - \lambda x = 0$
λ complex



Note: Having a more stringent time step for the damped system is not due to having damping, rather mainly due to having larger frequency ($\omega = \sqrt{2}$ compared to the damped case).

RK $\quad \dot{y} = f(y, t)$ with multiple stages

How can we solve $\ddot{x} + 2\dot{x} + 2x = 0$

$q = \begin{bmatrix} \dot{x} \\ x \end{bmatrix}$ $\quad \dot{q}_1 + 2q_1 + 2q_2 = 0$
$\quad \dot{q}_2 - q_1 = 0$

$\dot{q} = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix} q$

1st order ODE
solve with RK

- Plots of regions of absolute stability are commonly used to determine stability limits for problems of the type discussed above, *e.g.*, damped SDOF oscillator that shows up in the modal decomposition of many MDOFs.

Reading source for A-stability

This section briefly discussed the following concept:

- Dahlquist's theorems that discuss the existence and properties of explicit and implicit LMS methods.
- Concept of region of absolute stability.
- Concepts of zero-stable, A-stable, and stiffly-stable

The following is a list of resources that provide more details on these topics:

- [Süli and Mayers, 2003] pages 329-341: Sections 12.6 Linear multi-step methods; §12.7 Zero-stability; §12.8 Consistency; §12.9 Dahlquist's theorems; §12.10 Systems of equations.
- [Hughes, 2012] section 9.3 (only §9.3.1 and 9.3.2)

5.3.3 Stability analysis of one-step multivariate methods

- As mentioned in (5.3) two of the cases that the amplification factor takes a matrix form $\mathbf{A}$ are,
  1. Value and previous step values of x in (329b): ${}^t\hat{X} = [{}^{t+\Delta t}x \; {}^t x \; {}^{t-\Delta t}x \; \cdots]$. This will be the form of ${}^t\hat{X}$ for LMS methods as we observed in §5.3.1.
  2. Value and subsequent time derivatives of x in (329b): ${}^t\hat{X} = [{}^t x \; {}^t\dot{x} \; {}^t\ddot{x}]$. Examples be from Newmark and θ-Wilson methods which will be discussed subsequently.

5.3.3.1 Stability analysis of one-step multivariate methods: Wilson-θ method

- The assumption of Wilson-θ method is that acceleration varies linearly over the interval t to $t+\theta\Delta t$, where $\theta \geq 1$ whose range of having a stable method will be obtained by the stability analysis below.
- Linear acceleration and its first and second integration yields,

$$
\begin{aligned}
{}^{t+\tau}\ddot{x} &= {}^{t}\ddot{x} + \left({}^{t+\Delta t}\ddot{x} - {}^{t}\ddot{x}\right)\frac{\tau}{\Delta t}\\
{}^{t+\tau}\dot{x} &= {}^{t}\dot{x} + {}^{t}\ddot{x}\,\tau + \left({}^{t+\Delta t}\ddot{x} - {}^{t}\ddot{x}\right)\frac{\tau^2}{2\Delta t}\\
{}^{t+\tau}x &= {}^{t}x + {}^{t}\dot{x}\,\tau + \tfrac{1}{2}\,{}^{t}\ddot{x}\,\tau^2 + \left({}^{t+\Delta t}\ddot{x} - {}^{t}\ddot{x}\right)\frac{\tau^3}{6\Delta t}
\end{aligned}
\tag{379}
$$

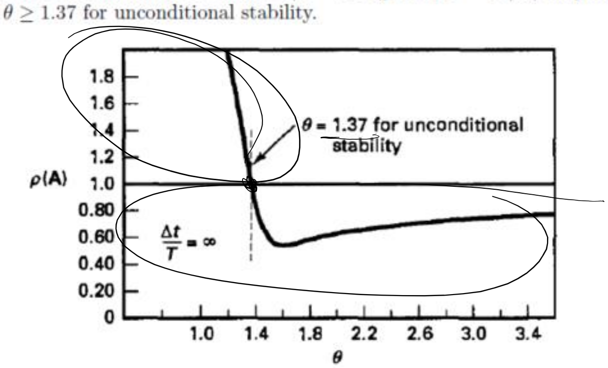
in §4.4.1.

$$
\begin{bmatrix} {}^{t+\Delta t}\ddot{x} \\ {}^{t+\Delta t}\dot{x} \\ {}^{t+\Delta t}x \end{bmatrix} = \mathbf{A}\begin{bmatrix} {}^{t}\ddot{x} \\ {}^{t}\dot{x} \\ {}^{t}x \end{bmatrix} + \mathbf{L}\,{}^{t+\theta\Delta t}r \qquad
\mathbf{A} = \begin{bmatrix}
1 - \frac{\beta\theta^2}{3} - \frac{1}{\theta} - \kappa\theta & \frac{1}{\Delta t}(-\beta\theta - 2\kappa) & \frac{1}{\Delta t^2}(-\beta) \\
\Delta t\left(1 - \frac{1}{2\theta} - \frac{\beta\theta^2}{6} - \frac{\kappa\theta}{2}\right) & 1 - \frac{\beta\theta}{2} - \kappa & \frac{1}{\Delta t}\left(-\frac{\beta}{2}\right) \\
\Delta t^2\left(\frac{1}{2} - \frac{1}{6\theta} - \frac{\beta\theta^2}{18} - \frac{\kappa\theta}{6}\right) & \Delta t\left(1 - \frac{\beta\theta}{6} - \frac{\kappa}{3}\right) & 1 - \frac{\beta}{6}
\end{bmatrix}
$$

$$
\beta = \left(\frac{\theta}{\omega^2\,\Delta t^2} + \frac{\xi\,\theta^2}{\omega\,\Delta t} + \frac{\theta^3}{6}\right)^{-1}; \qquad \kappa = \frac{\xi\beta}{\omega\,\Delta t} \qquad \mathbf{L} = \begin{bmatrix} \frac{\beta}{\omega^2\,\Delta t^2} \\ \frac{\beta}{2\omega^2\,\Delta t} \\ \frac{\beta}{6\omega^2} \end{bmatrix}
\tag{382}
$$

- Stability requires eigenvalues of $\mathbf{A}$ to satisfy $|a_i| \leq 1$ and if they have lower geometric multiplicity than algebraic multiplicity $(n_i^G < n_i^A)$ satisfying $|a_i| < 1$ as discussed in (338).

- For example in the figure below it is shown that in the limit $\Delta t/T \to \infty$ (period $T = 2\pi/\omega$) amplification factor is larger than one for $\theta > 1.37$ necessitating $\theta \geq 1.37$ for unconditional stability.



$\theta = 1.4$

Newmark method:

(α, δ) and Δt,

$$
\begin{cases}
\text{Unconditional stable:} \quad 2\alpha > \delta \geq \frac{1}{2} \\
\text{Conditional stable:} \quad \begin{bmatrix} \delta \geq \frac{1}{2} \\ \alpha < \frac{\delta}{2} \\ \overline{\Delta t} = \omega\Delta t \leq \Omega_{\text{crit}} \end{bmatrix}
\end{cases}
\qquad \text{where}
\tag{386a}
$$

this was not the case for central difference

$\rho \nearrow \rightarrow \Delta t \nearrow$

$$
\Omega_{\text{crit}} = \frac{\xi\left(\delta - \frac{1}{2}\right) + \left[\frac{\delta}{2} - \alpha + \xi^2\left(\delta - \frac{1}{2}\right)^2\right]^{\frac{1}{2}}}{\frac{\delta}{2} - \alpha}, \quad \text{critical normalized sampling frequency}
\tag{386b}
$$

- $\overline{\Delta t}$ is normalized time step (also called normalized frequency).
- As usual when the method is conditionally stable ω in (386a) will be the worst (*i.e.*, maximum) frequency that the MDOF discrete problem $(\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = 0)$ can model $\max_l(\omega_l^h)$.
- In practice we replace this with more convenient and conservative value ω_e^m, *i.e.*, the highest frequency of individual elements.

5.4 Practical considerations in using time marching methods

5.4.1 Control of high frequency numerical noise

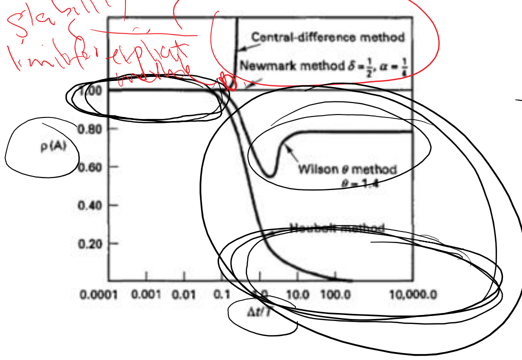
$\ddot{x} + \omega^2 x = 0$ $\qquad T = \frac{\omega}{2\pi}$

- In the figure observe spectral radius of different time marching methods versus normalized element size.
- $T = \frac{\omega}{2\pi}$ is the period of a given SDOF.
- Clearly, as expected central-difference method becomes unstable for $\Delta t/T > \frac{1}{\pi}$: As we observed in (357) (also (358)) central difference method is stable if $\Delta t\omega \le 2$, $T = \frac{\omega}{2\pi} \Rightarrow \Delta t/T \le \frac{1}{\pi}$
- Other methods in the figure are unconditionally stable.
- One very important aspect of a time marching method in these plots is,

$$\rho_\infty = \lim_{\Delta t/T \to \infty} \rho(\mathbf{A}(\frac{\Delta t}{T})) \tag{387}$$

for example for Wilson-θ method $\rho_\infty \approx 0.8$

stability limit for explicit method



$\frac{\Delta t}{T} = 2\pi(\omega \Delta t)$

$\Delta t / T \longrightarrow \infty$

Δt Very large w.r.t T

OR

T is very small / ω very large

$\Delta t / T$

- $\Delta t/T \to \infty$ for individual SDOFs of a MDOF system (ω is in fact ω_i^h) can happen under two conditions which have important implications:
  1. $\Delta t \to \infty$ (Too large of a time step) which means very large time step is taken with respect to T. Often this can be a source of large numerical dissipation if $\Delta t \ll \max_i T_i$ (*i.e.*, time step is much larger than the period of the lowest natural mode) and $\rho_\infty < 1$. Having such high time steps can be afforded in unconditionally stable methods. If this condition occurs, this a sign that too large of a time step from numerical error perspective is taken.
  2. $T \to 0$ (*i.e.*, $\omega \to \infty$ High frequency modes): In this case, we are dealing with high frequency modes of the problem. Below, we discuss how by optimizing (having smallest) ρ_∞ we can effectively eliminate high frequency numerical noise.
- Assuming that case one is not of concern (*i.e.*, not too large of a time step is taken to quickly dissipate the solution by the numerical time integration when $\rho_\infty < 1$) a main concern of a numerical integration if the control of high frequency numerical noise.
- High-frequency behavior: "Because the higher modes of semi-discrete structural equations are artifacts of the discretization process and not representative of the behavior of the governing partial differential equations, it is generally viewed as desirable and often is considered absolutely necessary to have some form of algorithmic damping present to remove the participation of the high-frequency modal components." [Hughes, 2012].

- Figure below shows how low frequency part of the solution does not damp out much ($\Delta t/T_1 = 0.01, 0.1$) as for these low values of $\Delta t/T$ $\rho(\mathbf{A}) \lessapprox 1$. On the other hand, for high(er) frequency content (low(er) T) $\Delta t/T_i = 1, 10, 1001000$ $\rho(\mathbf{A}) \to \rho_\infty$ and these waves are almost entirely dissipated. This is the desired response as we want to maintain the physical part of the solution and dissipate / filter numerical noise.

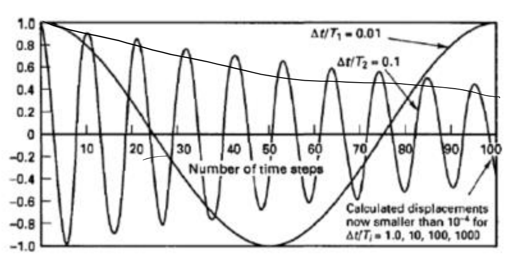


Figure 9.7 Displacement response predicted with increasing $\Delta t/T$ ratio; Wilson θ method, $\theta = 1.4$ [Bathe, 2006]