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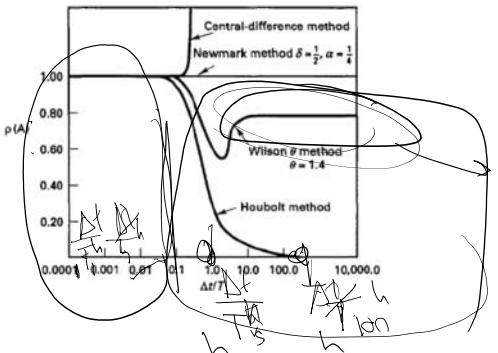
5.4.1 Control of high frequency numerical noise

- In the figure observe spectral radius of different time marching methods versus normalized element size.
- $T = \frac{\omega}{2\pi}$ is the period of a given SDOF.
- Clearly, as expected central-difference method becomes unstable for $\Delta t/T > \frac{1}{\pi}$: As we observed in (357) (also (358)) central difference method is stable if $\Delta t\omega \leq 2$, $T = \frac{\omega}{2\pi} \Rightarrow \Delta t/T \leq \frac{1}{\pi}$
- Other methods in the figure are unconditionally stable.
- One very important aspect of a time marching method in these plots is,

$$\rho_\infty = \lim_{\Delta t/T \to \infty} \rho(\mathbf{A}(\frac{\Delta t}{T})) \qquad (387)$$

for example for Wilson-θ method $\rho_\infty \approx 0.8$



kills high frequency content

MDOF

$M\ddot{U} + C\dot{U} + KU$

add damping later

$\ddot{x}_i + 2\xi\omega_i\dot{x}_i + \omega_i^2 x_i = 0$

$\omega_1 < \omega_2 < \dots < \omega_n$

semi discrete Frequency

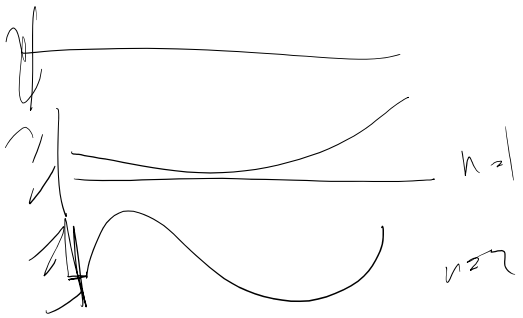
$T_1^h > T_2^h > \dots > T_n^h$

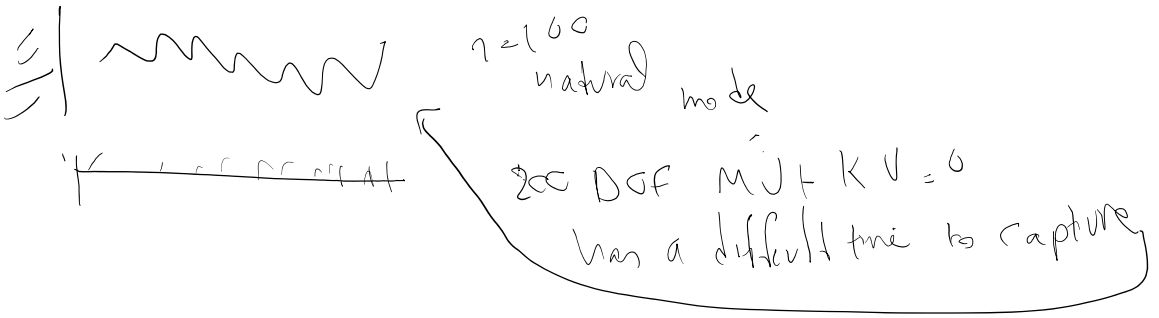
$T_2 \geq \frac{T_1}{2}$

Time discretized eq of $M\ddot{U} + C\dot{U} + KU = 0$

$\frac{\Delta t}{T_1^h} < \frac{\Delta t}{T_2^h} \dots \frac{\Delta t}{T_n^h}$

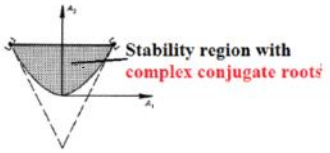
Killing high frequency content is welcomed because we observed before with FEM the error in capturing higher frequency content increases (because they have a higher "wave number")





- $\Delta t/T \to \infty$ for individual SDOFs of a MDOF system (ω is in fact ω_i^h) can happen under two conditions which have important implications:
  1. $\Delta t \to \infty$ (**Too large of a time step**) which means very large time step is taken with respect to T. Often this can be a source of large numerical dissipation if $\Delta t \ll \max_i T_i$ (*i.e.*, time step is much larger than the period of the lowest natural mode) and $\rho_\infty < 1$. Having such high time steps can be afforded in unconditionally stable methods. If this condition occurs, this a sign that too large of a time step from numerical error perspective is taken.
  2. $T \to 0$ (*i.e.*, $\omega \to \infty$ **High frequency modes**): In this case, we are dealing with high frequency modes of the problem. Below, we discuss how by optimizing (having smallest) ρ_∞ we can effectively eliminate high frequency numerical noise.
- Assuming that case one is not of concern (*i.e.*, not too large of a time step is taken to quickly dissipate the solution by the numerical time integration when $\rho_\infty < 1$) a main concern of a numerical integration if the control of high frequency numerical noise.
- High-frequency behavior: "Because the higher modes of semi-discrete structural equations are artifacts of the discretization process and not representative of the behavior of the governing partial differential equations, it is generally viewed as desirable and often is considered absolutely necessary to have some form of algorithmic damping present to remove the participation of the high-frequency modal components." [Hughes, 2012].

- The control of high frequency noise can be achieved by minimizing ρ_∞ to quickly dissipate any high frequency numerical noise that can be introduced in the solution.
- As an example, we optimize parameters for the Newmark method:
  - As was shown in the previous figure $\delta = \frac{1}{2}$ results in $\rho_\infty = 1$, *i.e.*, no dissipation of high frequency noise.
  - We need to choose $d > 0$ to have $\rho_\infty < 1$.
  - For a given δ we can optimize α such that high frequency dissipation is maximized (*i.e.*, ρ_∞ minimized).
  - This condition is created by requiring the eigenvalues of the amplification factor to be complex conjugate values.
  - Remembering for a 2×2 amplification matrix $\mathbf{A}$ such condition is $A_1^2 > A_2$; *cf.* (363).



$A^2 - 2A_1 A + A_2 = 0$

amplification factor

$A = A_1 \pm \sqrt{A_1 - A_2^2}$

To have the roots of Newmark method in the dashed region we have the following:

- To ensure that the amplification factors are complex conjugate parameters α, δ from (386) are restricted as:

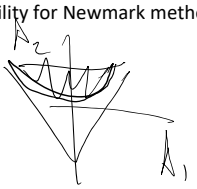
$$\begin{cases} \text{Unconditional stable:} & \begin{bmatrix} 0 \le \xi < 1 \\ \delta \ge \frac{1}{2} \\ \alpha \ge \frac{(\delta+\frac{1}{2})^2}{4} \ge \frac{\delta}{2} \end{bmatrix} \\ \text{Conditional stable:} & \begin{bmatrix} 0 \le \xi < 1 \\ \delta \ge \frac{1}{2} \\ \alpha < \frac{\delta}{2} \\ \overline{\Delta t} = \omega \Delta t \le \Omega_{\text{bif}} \end{bmatrix} \end{cases} \quad \text{where}$$

more restrictive

$$\alpha = \frac{(\delta+\frac{1}{2})^2}{4} \quad (388a)$$

what happens for this α

Complex amplification factor + Stability for Newmark method



(α, δ) and Δt,

$$\begin{cases} \text{Unconditional stable:} & 2\alpha > \delta \ge \frac{1}{2} \\ \text{Conditional stable:} & \begin{bmatrix} \delta \ge \frac{1}{2} \\ \alpha < \frac{\delta}{2} \\ \overline{\Delta t} = \omega \Delta t \le \Omega_{\text{crit}} \end{bmatrix} \end{cases} \quad \text{where} \quad (386a)$$

$\alpha = \frac{\delta}{2}$

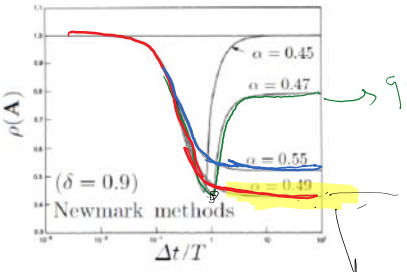
Just stability for Newmark

$$\Omega_{\text{crit}} = \frac{\xi\left(\delta - \frac{1}{2}\right) + \left[\frac{\delta}{2} - \alpha + \xi^2\left(\delta - \frac{1}{2}\right)^2\right]^{\frac{1}{2}}}{\frac{\delta}{2} - \alpha}, \quad \text{critical normalized sampling frequency} \quad (386b)$$

$\alpha = \frac{(\delta+\frac{1}{2})^2}{4} = .49$

- $\alpha < \frac{(\delta+\frac{1}{2})^2}{4}$. For $\delta = 0.9 \Rightarrow \alpha < 0.49$. In this case eigenvalues of $\mathbf{A}$ bifurcate to complex conjugate values, but then past some

- $\alpha < \frac{(\delta+\frac{1}{2})^2}{4}$: For $\delta = 0.9 \Rightarrow \alpha < 0.49$. In this case eigenvalues of $\mathbf{A}$ bifurcate to complex conjugate values, but then past some $\Delta t/T$ value they bifurcate back to higher values. For the minimum $\alpha = \delta/2$ for unconditional stability we even have $\rho_\infty \to 1$ resulting in no dissipation for high frequency oscillations (noises). This can be see in cases $\alpha = 0.45$ and $\alpha = 0.47$.
- $\alpha > \frac{(\delta+\frac{1}{2})^2}{4}$: For $\delta = 0.9 \Rightarrow \alpha > 0.49$. In this case eigenvalues of $\mathbf{A}$ do not bifurcate but have $\rho_\infty < 1$. The case $\alpha = 0.55$ is shown.
- $\alpha = \frac{(\delta+\frac{1}{2})^2}{4}$: For $\delta = 0.9 \Rightarrow \alpha = 0.49$. This the optimum value: In this case eigenvalues bifurcate and result in an optimum (minimum) $\rho_\infty < 1$ for a given δ.



Accordingly, in practice to have the best dissipation of higher frequency noise we often set $\alpha = \frac{(\delta+\frac{1}{2})^2}{4}$

- There are some other approaches to dissipatte high frequency noise:
  1. Artificial damping: Similar to D_h in (114) in the context of FV methods various types of numerical damping operators

can be added to a numerical method to control high frequency oscillations and other numerical artifacts. However, depending on the type of numerical method special case should be taken as at times they only damp an intermediate band of frequencies without significant effect in the all-important high modes.

  2. α-method: Hilber-Hughes-Taylor (HHT) method:
     - The approach discussed above with choosing $\alpha = \frac{(\delta+\frac{1}{2})^2}{4}$ requires $\delta > \frac{1}{2}$ which results in one order loss of accuracy compared to $\delta = \frac{1}{2}$ (first order compared to second order in Δt). For $\delta = \frac{1}{2}$ and $\alpha = \frac{1}{4}$ from $\frac{(\delta+\frac{1}{2})^2}{4}$ yet the scheme is nondissipative and ρ_∞ rendering it ineffective in dissipative high frequency noise.
     - HHT suggested the following modification $\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$ by splitting the values to t_n and t_{n+1}:

$$\mathbf{M}\ddot{\mathbf{U}}_{n+1} + (1+\alpha)\mathbf{C}\dot{\mathbf{U}}_{n+1} - \alpha\mathbf{C}\dot{\mathbf{U}}_n + (1+\alpha)\mathbf{K}\mathbf{U}_{n+1} - \alpha\mathbf{K}\mathbf{U}_n = \mathbf{R}(t_{n+1+\alpha}), \quad t_{n+1+\alpha} = (1+\alpha)t_{n+1} - \alpha t_n \quad (389)$$

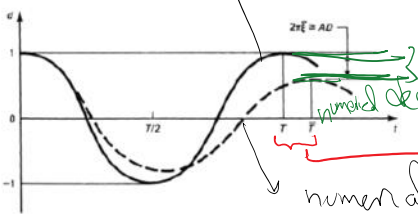
     - While the MDOF is modified by α coefficient the Newmark update is done as usual by employing the following approximation: $\dot{\mathbf{U}}_{n+1} = \dot{\mathbf{U}}_n + \Delta t[(1-\delta)\ddot{\mathbf{U}}_n + \delta\ddot{\mathbf{U}}_{n+1}]$ and $\mathbf{U}_{n+1} = \mathbf{U}_n + \Delta t\dot{\mathbf{U}}_n + \frac{\Delta t^2}{2}[(1-2\bar{\alpha})\ddot{\mathbf{U}}_n + 2\bar{\alpha}\ddot{\mathbf{U}}_{n+1}]$ where $\bar{\alpha}$ is the usual α parameter that is marked by $(\bar{.})$ to distinguish it from HHT α parameter.
     - If $\alpha \in [-\frac{1}{3}\ 0]$, $\delta = \frac{1-2\alpha}{2}$, and $\bar{\alpha} = \frac{(1-\alpha)^2}{4}$ an unconditionally stable, second order accurate scheme results.

5.4.2 Measures of accuracy: L2 error, numerical dissipation and dispersion

- Consider the solution for the undamped second order ODE shown below,

$$\left.\begin{aligned} \ddot{x} + \omega^2 x = 0 \\ {}^0x = 1.0; \quad {}^0\dot{x} = 0.0; \quad {}^0\ddot{x} = -\omega^2 \end{aligned}\right\} \quad (390)$$

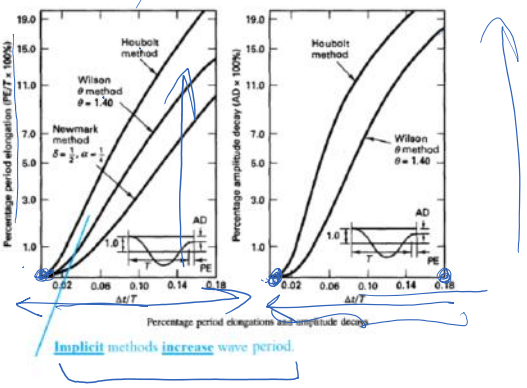
- The exact solution for this ODE is $x(t) = \cos(\omega t)$.
- The numerical solution may not be able to model the exact wave amplitude or period as shown in the figure.



- We have the following definitions,

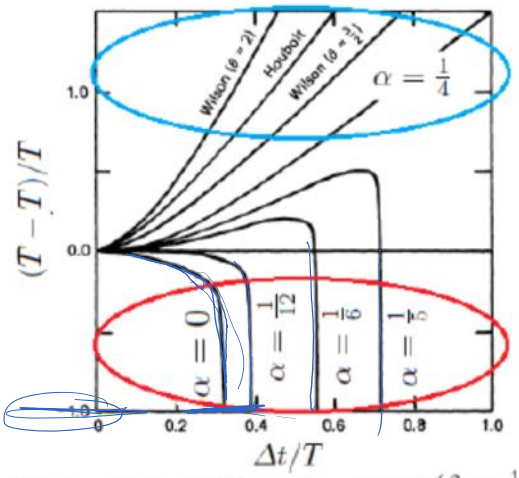
1. Amplitude Decay (AD): The amount the amplitude of the wave decreases relative to the exact solution in one period.

1. Amplitude Decay (AD): The amount the amplitude of the wave decreases relative to the exact solution in one period. Note, the exact solution may actually be dissipative for example when damping is nonzero $\ddot{x}+2\xi\omega\dot{x}+\omega^2 x = 0$, yet we can formalize and separate physical dissipation from numerical one.
2. Period elongation (PE): The difference between numerical period $\bar{T}$ and exact period $T = \frac{2\pi}{\omega}$, that is $\bar{T} - T$. Interestingly, we often have the following trend:
   (a) Implicit integration $\Rightarrow$ PE ≥ 0: With implicit integration methods numerical period is often longer $\bar{T}$ (shorter frequency $\bar{\omega}$) than the exact period T. Examples are Wilson-θ, Houbolt, trapezoidal and unconditional stable Newmark methods.
   (b) Explicit integration $\Rightarrow$ PE ≤ 0: With explicit integration methods numerical period is often shorter $\bar{T}$ (longer frequency $\bar{\omega}$) than the exact period T.



Implicit methods increase wave period.

- More examples are shown in the figure:
  1. For Newmark method $\delta = \frac{1}{2}$, $\alpha < \frac{\delta}{2} = \frac{1}{4}$ correspond to conditional stable regime which as can be seen in the figure shorten the period / elongate frequency.
  2. On the other hand, implicit (unconditionally stable) Newmark method for $\alpha = \frac{1}{4}$ and Wilson and Houbolt methods elongate the period / shorten frequency.



Implicit integrators: Period gets longer

Explicit integrations: Period gets shorter

Comparison of Nemark method ($\delta = \frac{1}{2}$) and Wilson and Houbolt methods.

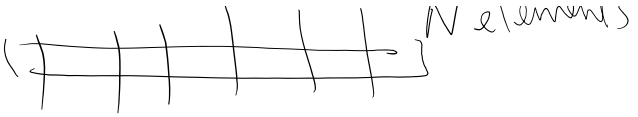
$\rho \ddot{U} - E U_{,xx} = 0$

$\omega_n = n\pi \frac{c}{L}$ exact periods $n = 1 \ldots \infty$

Semi-discretization

N elements

Semi-discretization



can capture $(N+1)$ ndof frequencies

$$M\ddot{U} + KU = 0 \qquad M\Phi_i = \omega_i^h \Phi_{(i)}$$

ω_i^h → semi-discrete frequency

no summation on i

ndof ω_i^h s

Relation between ω_i , ω_i^h $(i = 1 \ldots ndofs)$

ω_i → exact natural frequency i

ω_i^h → semi-discrete natural frequency

- *A priori* error estimates for natural frequencies and natural modes are in the form,

Consistent mass matrix, high enough integration

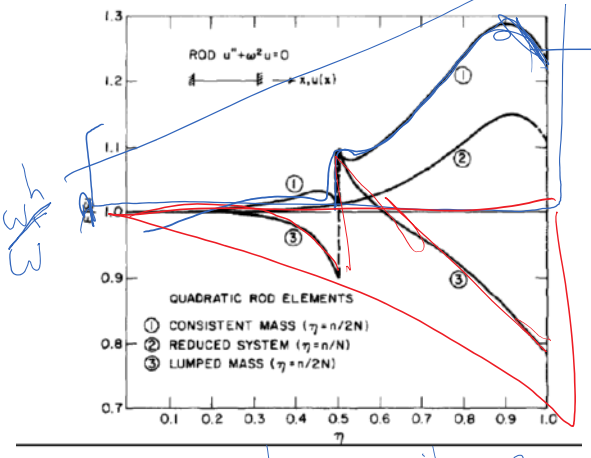
$\omega_i^h \geq \omega_i$

semi-discrete, exact

$$0 \leq \omega_i^h - \omega_i \leq Ch^{2(p+1-m)}\omega_i^{\frac{2p+2-m}{m}}$$

$$\|\Phi_i^h - \Phi_i\|_m \leq Ch^{(p+1-m)}\omega_i^{\frac{p+1}{m}}$$

$\omega = \sqrt{\frac{k}{m}}$ → k stiffness, m mass



ω^h / ω

consistent mass → $\omega_i^h > \omega_i$

lumped mass → $\omega_i^h < \omega_i$

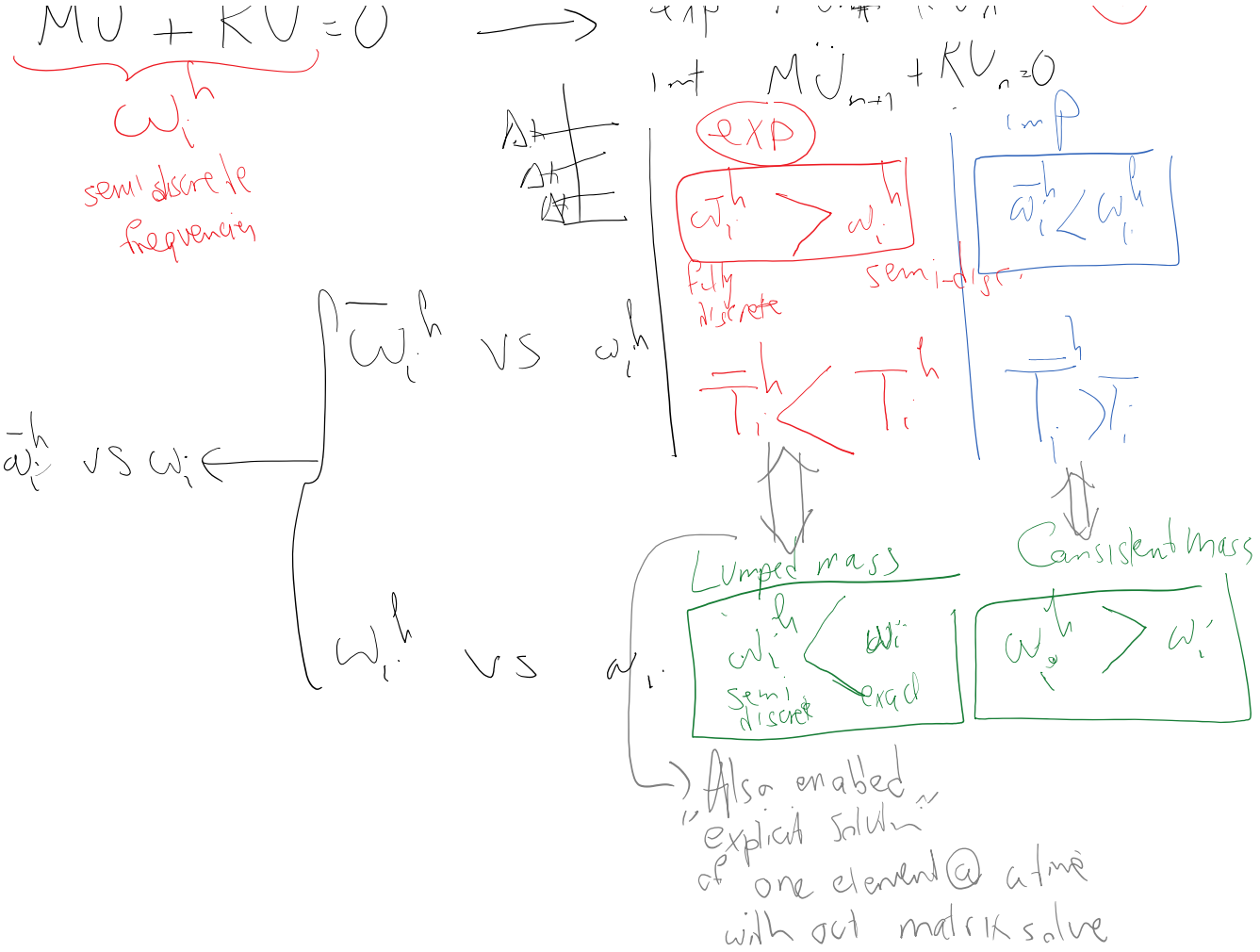
↳ normalized frequency #

Step 2 full discretization : Add time integral.

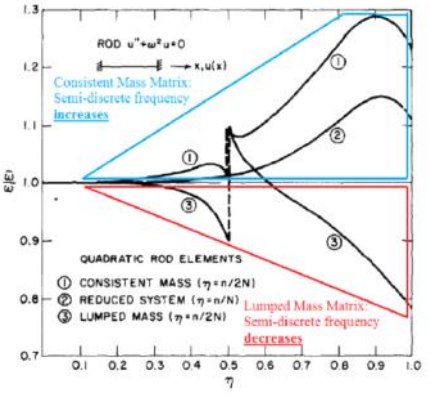
$M\ddot{U} + KU = 0$ → exp $M\ddot{U}_n + KU_n = 0$ → $-\omega_i^4$

int $M\ddot{U}_n + KU_n = 0$

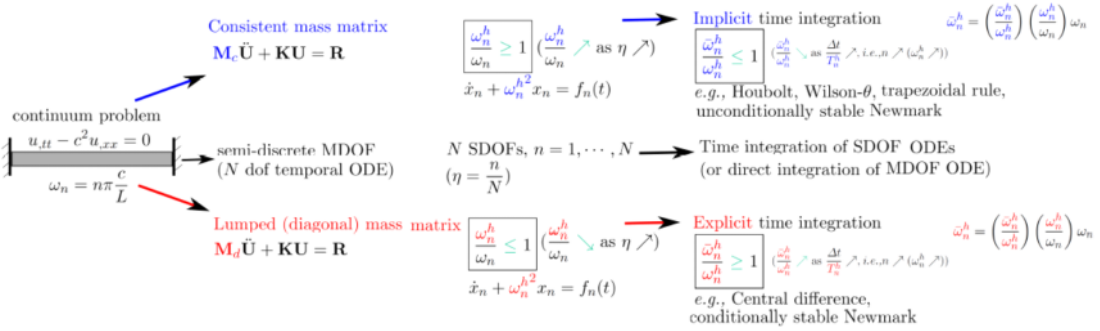
time integral modified frequency



- Consider that we are solving 1D elastodynamic problem $u_{,tt} - c^2 u_{,xx} = 0$ for a double end fixed bar of length L.
- $\Rightarrow$ the exact natural frequencies are $\omega_n = n\pi\frac{c}{L}$.
- Numerical modes ω_n^h for an N dof MDOF spatial discretization $\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\mathbf{U} = 0$ take the following form:
  1. Consistent mass matrix: $\omega_n^h > \omega_n$.
  2. Lumped mass matrix: $\omega_n^h < \omega_n$.
- For this 1D problem, we observe that the relative error $\frac{\omega_n^h}{\omega_n}$ only depends on how far we are from the number of modes the MDOF model can capture through $\eta = \frac{n}{N}$.



- All the way from the continuum level frequencies to semi-discrete MDOF discretization $\mathbf{M}\ddot{\mathbf{U}}+\mathbf{C}\dot{\mathbf{U}}+\mathbf{K}\mathbf{U}=\mathbf{R}$ to its numerical integration we deal with three groups of frequencies:
  1. ω: exact frequencies of the continuum problem.
  2. ω^h: semi-discrete frequencies are natural frequencies of MDOF FEM discretization $\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$.
  3. $\bar{\omega}^h$: frequencies produced (realized) by time integration of semi-discrete MDOF $\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$.
- Eventually accuracy of the numerically integrated ODE in representing periods of waves depends on:
  1. How accurately semi-discrete frequencies are modeled: $\frac{\omega^h}{\omega}$.
  2. How accurately time-integration models frequencies: $\frac{\bar{\omega}^h}{\omega^h}$.
- We want to match time integration methods with appropriate mass matrix option so that the errors from the following two steps are in opposite directions and to some extend cancel each other out.
- This concept is shown in the next figure.



Matching (Consistent mass → Implicit TI) / (Lumped mass → Explicit TI) 472

- Based on the results from previous slide, we make the following conclusions for the choice of mass matrix based on time integration model:
  1. Implicit Integration → Consistent mass matrix: Implicit integration methods often elongate T^h (shorten ω^h: $\frac{\bar{\omega}^h}{\omega^h} \leq 1$) which best is matched with consistent mass matrix as it shortens periods T (elongate ω: $\frac{\omega^h}{\omega} \geq 1$).
  2. Explicit Integration → Lumped mass matrix:
     (a) Implicit integration methods often shorten T^h (elongate ω^h: $\frac{\omega^h}{\bar{\omega}^h} \geq 1$) which best is matched with lumped mass matrix as it elongates periods T (shorten ω: $\frac{\omega^h}{\omega} \leq 1$).
     (b) In addition use of lump mass matrix + damping $\mathbf{C} = \mathbf{0}$ + explicit method enables a local and trivial linear system solve. For example, in (247) for central difference method we had $\hat{\mathbf{M}} = \frac{1}{\Delta t^2}\mathbf{M} + \frac{1}{2\Delta t}\mathbf{C}$ which for $\mathbf{C} = \mathbf{0}$ yielded $\hat{\mathbf{M}} = \frac{1}{\Delta t^2}\mathbf{M}$. If further a lumped mass matrix is used the update equation simply becomes $U_i^{n+1} = \frac{\Delta t^2}{m_{ii}} R_i^n$; *cf.* (248).
- That is, use of lumped mass matrix for explicit integration methods not only can result in a local solution process but also is preferred from numerical error perspective (period elongation error).

- The mass matrix is parameterized with r:

$$K^e = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad M^e = \frac{A\rho}{L}\begin{bmatrix} \frac{1}{2} - r & r \\ r & \frac{1}{2} - r \end{bmatrix} \tag{391}$$

- For $r = \frac{1}{6}, 0, \frac{1}{12}$ we recover consistent mass, lumped mass, and a higher order mass matrix (resulting in higher convergence rates for natural modes / frequencies), respectively,

Consistent mass($r = \frac{1}{6}$)	Lumped mass($r = 0$)	High order mass($r = \frac{1}{12}$)
$M^e = \frac{A\rho}{6L}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$	$M^e = \frac{A\rho}{2L}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$M^e = \frac{A\rho}{12L}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$ (392a)

Mass matrix option: semi-discrete

Time integration — fully discrete

- The goal is to optimize r based on $\frac{\Delta t}{h}$ and α(Newmark integration parameter).

$$\sin^2\frac{\bar{\omega}h\Delta t}{2} = \frac{\sin^2(\omega h/2)}{4[\alpha - r(h/\Delta t)^2]\sin^2(\omega h/2) + (h/\Delta t)^2}$$

fully discrete frequency

Newmark

exactly frequency

Δt

h

wave speed $=1$

$\sqrt{\frac{E}{\rho}} = 1$

$= 1 \;\&\; \bar{\omega}^h = \omega$

$\alpha = r\left(\frac{h}{\Delta t}\right)$

$h/\Delta t = 1$

$\alpha = r$

if exp

$h/\Delta t = 1 \longrightarrow \Delta t = h$ $(c=1)$

$\Delta t \leq \frac{h}{c} = h$

CFL $=1$

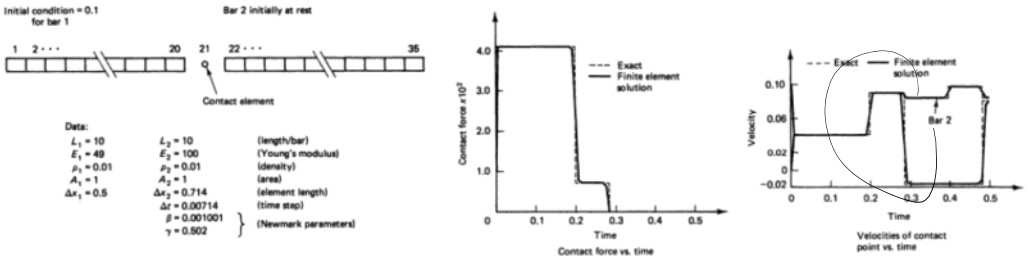
$\alpha = r$

Newmark

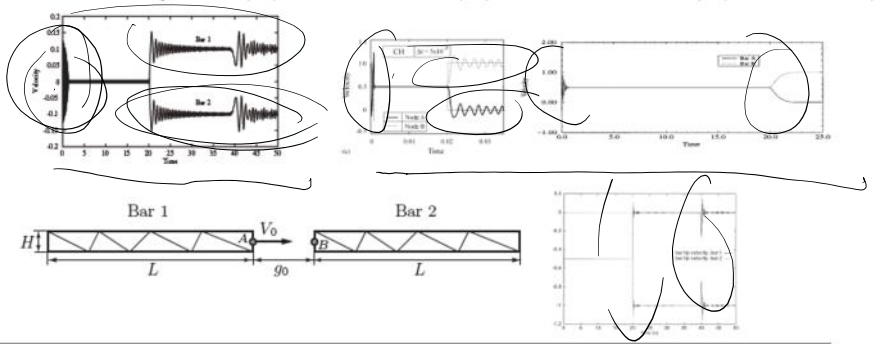
Mass matrix parameter

- In this case the errors introduced by finite element spatial discretization, the particular mass matrix and temporal algorithm all cancel to yield exact results.
- Time step $\Delta t = h/c$ is called the characteristic time step.
- It is interesting to note that reducing the size of the time step Δt while holding the mesh length h fixed can only worsen the results.

- In this case we converge to the exact solution of the spatially discrete, temporally continuous system (i.e., "mass points and springs") rather than the exact solution of $u_{,tt} - c^2 u_{,xx} = 0$.
- In more general settings (*e.g.*,, unequal element lengths, variable material properties, multidimensional problems, etc.), results obtained by matched methods, such as central differences and lumped mass, will not be exact. However, it is felt that results obtained by matched methods will generally be superior to inappropriate combinations, such as consistent mass and central differences [Hughes, 2012].
- For example, results below from [Hughes et al., 1976b] show a perfect example of how optimizing the numerical model parameters and matching (optimizing) time integration method and mass matrix can result in excellent results.
- This problem demonstrates a contact problem between two dissimilar bars.
  - Upon contact there is a sharp transition from traction free state to compressive stress state in the bars.
  - Also, when the compressive waves reflect from the free end of the bar(s) they result in transition of the bars from contact to separation mode which reverses the stress state.
  - This is a benchmark problem for checking contact algorithms and solving this problem is not trivial.
  - The transitions often result in widespread numerical artifacts and noise in both bars, but we observe very good solutions herein with just a few elements.



Benchmark problem [Hughes (76);Laursen, Chawla (97); Czekanski, Meguid (01); Cirak, West (05); *etc.*]



* Application to second order temporal ODEs: RK methods are naturally formulated for first order ODEs. Although one can reformulate the update equations for a second order ODE (*e.g.*, either by directly using Taylor series expansion of the exact solution or employing the first order ODE update equations) the direct use of RK methods to second order ODEs, *e.g.*, elastodynamic problem, is very limited. However, one can express second order (or higher order) PDEs in time as a system of first order PDEs. In which case, RK method can be directly used. Unlike continuous FEMs where RK methods are rarely used, their use is common with discontinuous Galerkin methods; *cf.* Local DG (LDG) and RKDG methods [Cockburn and Shu, 1998a, Cockburn and Shu, 1998b].

Besides RK method (which by the way is restricted because of butcher effect):

– Cauchy-Kovalewski (CK) / Lax-Wendroff (LW): Achieving high temporal orders of accuracy in time is much more challenging than in space for time marching methods since the solution is only given at discrete time values as opposed to spatial representation of solution by basis functions. This is the main source of difficulty in achieving arbitrary high temporal orders of accuracy with the aforementioned methods.

A successful approach to circumvent discrete representation of solution in time is Cauchy-Kovalewski (CK) or Lax-Wendroff approach which involves the following steps:

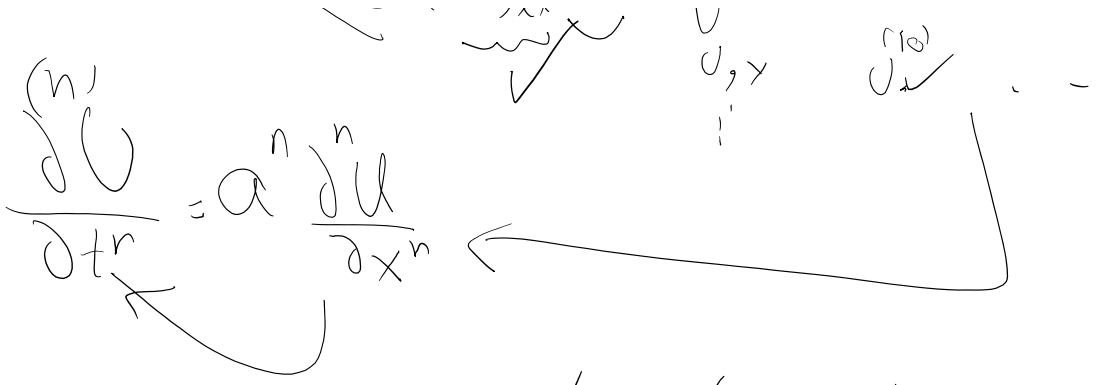
* Expanding the solution in time using the Taylor series:

$$u(x,t+\Delta t) \approx u(x,t) + \Delta t u_{,t}(x,t) + \frac{\Delta t^2}{2} u_{,tt}(x,t) + \cdots + \frac{\Delta t^s}{s!} u^{(s)}(x,t) + \mathcal{O}\left(\Delta t^{s+1}\right) \tag{394}$$

* Replacing temporal derivatives with spatial derivatives using the underlying PDE; *e.g.*,, for the advection equation

$$u_{,t} - a u_{,x} = 0 \quad \text{(advection equation)} \quad \Rightarrow \quad \frac{\mathrm{d}^s u}{\mathrm{d}t^s} = a^s \frac{\mathrm{d}^s u}{\mathrm{d}x^s} \tag{395}$$

* For a given position x (which can correspond to a nodal position in an FEM mesh) obtain $\frac{\mathrm{d}^s u}{\mathrm{d}x^s}$ from FE spatial discretization. Since space is discretize with the more flexible FE method and elements with any desired shape



$$u(x,t+\Delta t) \approx u(x,t) + \Delta t u_{,t}(x,t) + \frac{\Delta t^2}{2} u_{,tt}(x,t) + \cdots + \frac{\Delta t^s}{s!} u^{(s)}(x,t) + \mathcal{O}\left(\Delta t^{s+1}\right) \tag{394}$$

time marching can be as high order as we want.

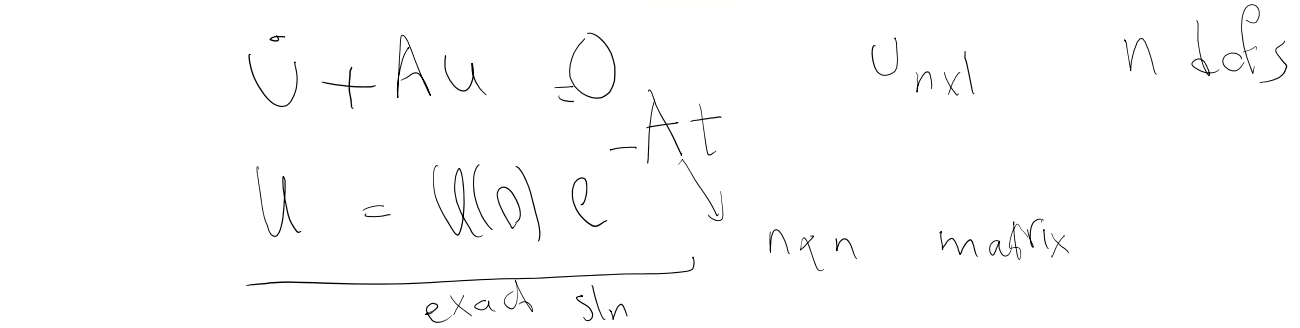
function orders can be formulated, the ability of having high order $\frac{d^s u}{dx^s}$ is relatively trivial compared to having high temporal orders of accuracy with time marching schemes.

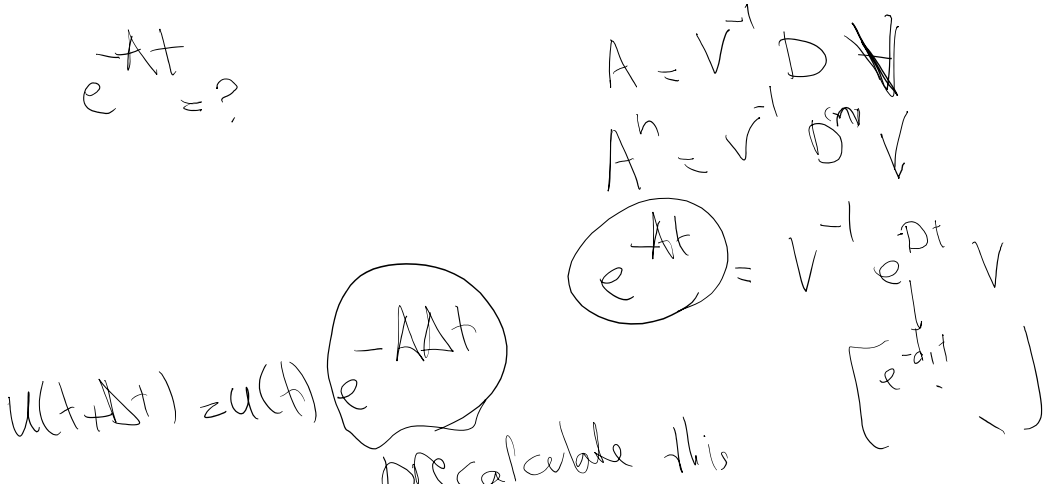
* Finally, by plugging spatial derivatives $\frac{d^s u}{dx^s}$ in (395) and corresponding $\frac{d^s u}{dt^s}$ (which are PDE dependent) in (394) we formulate a method with arbitrary high order of accuracy in time.

Refer to [De Basabe and Sen, 2010] for the application of CK method to second order elastodynamic problem, and [Dumbser and Munz, 2005, Dumbser and Munz, 2006] for the application of CK method in the context of discontinuous Galerkin methods.

– Other high order temporal integration methods. Some notable methods are:

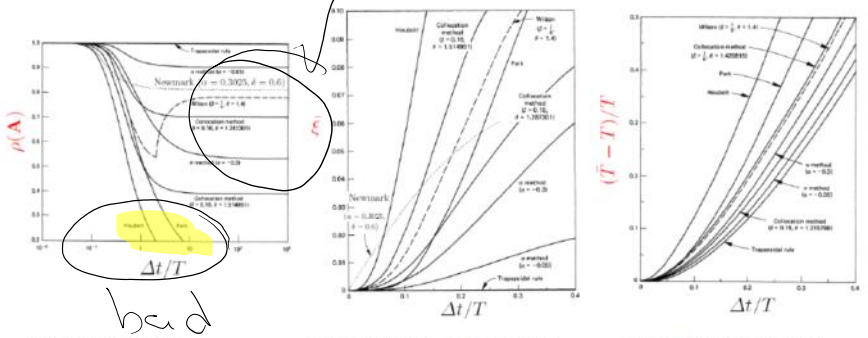
* Spacetime FEMs: While spacetime finite element methods are not using a time marching scheme to advance the solution in time (they directly solve space and time with FEM), the expression of the solution in time with FE shape functions means that arbitrary high temporal orders of accuracy can be achieved with these methods.
* Methods based on analytical expansion of solution: Consider the problem $\dot{\mathbf{u}} + \mathbf{A}\mathbf{u} = \mathbf{0}$ where $\mathbf{u}$ is a vector and $\mathbf{A}$ a matrix. The solution of this ODE is $\mathbf{u}(t) = \mathbf{u}(t=0)e^{-\mathbf{A}t}$. Herein, $\mathbf{u}$ can represent the vector of unknowns that can be obtain by FEM discretization; *cf.* (226b) ($\mathbf{M}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$) or (227) for temporal first order representation of (226a) ($\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$). Such approaches and exponential of a matrix, are the basis of achieving arbitrary high temporal order of accuracy in some methods, *e.g.*, [Fahs, 2012].





precalculate this with desired order of accuracy.

- Control of high frequency numerical artifacts, amplitude damping (AD), and period elongation (PE)



a) Spectral radius b) Algorithmic damping ratio c) Relative period error

In the figure spectral radius, AD, and PE of various time marching methods is displayed versus the normalized time step $\Delta t/T$ ($T = \frac{2\pi}{\omega}$ is the period for a SDOF). We make the following observation (for more details, especially on some other methods displayed in the figure, refer to [Hughes, 2012]): Control of high frequency artifacts and damping of the solution: The spectral radii of the Houbolt and Park methods approach zero as $\Delta t/T \to \infty$ as is typical of backward-difference schemes. While $\rho_\infty = 0$ implies that high frequency artifacts are dissipated, the quick transition of ρ to 0 has adverse effects. These two methods are seen to affect the low modes (*i.e.*,, $\Delta t/T = 0.1$) too strongly, which means even at moderate time step sizes all components of the solution including low frequency content can be severely dissipated. The quick approach of $\rho(\mathbf{A})$ to zero also manifests itself in high amplitude decay for these methods.

On the other hand, for trapezoidal rule we have $\rho(\mathbf{A}) = 1$ which is due to the fact that the method is nondissipative. While the method is second order accurate, having no ability to dissipate high frequency content (numerical artifacts) is an undesirable property. As mentioned before, methods such as Hilber-Hughes-Taylor (HHT) (α-method) not only are temporally second order accurate but provide some dissipation for high frequency content.

Finally, we observe that the Newmark method ($\alpha = 0.3025, \delta = 0.6$) and α method (HHT) have a good high frequency content dissipation behavior while not being too much dissipative for the lower frequency content; we observe $\rho(\mathbf{A}) \approx 1$ for $\Delta t/T$ is small; *i.e.*, for low frequency ω (high period T modes) and having $\rho_\infty < 1$ ensures high frequency numerical noise gets dissipated.

In short, the following is a list of properties are deemed desirable for structural dynamic problems (*i.e.*, when first few modes

Read about Rayleigh quotient concept before the class