DC2020/04/20

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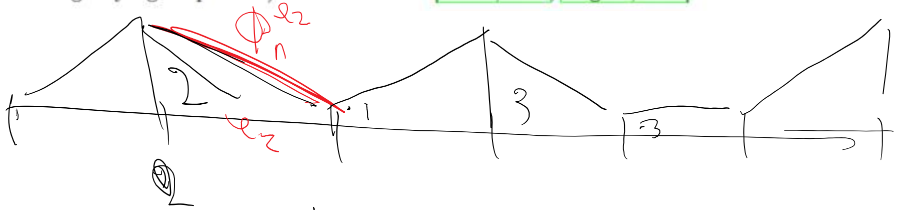
Why the maximum eigenvalue of individual elements is a conservative estimate for the maximum eigenvalue of a structure?

5.5.1 Maximum bound of MDOF eigenvalue by its element eigenvalues

The complete background for this proof (including Rayleigh's quotient) can be found in [Bathe, 2006, Hughes, 2012].

#dofs n



global mode for ω_n

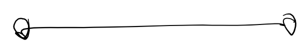
eigenvector of

$\Phi = [\,2 \;\; .1 \;\; 3 \;\; .3 \;\; \ldots\,]$

Max eigenvalue of structure

$$\omega_n = \frac{\phi_n K \phi_n}{\phi_n M \phi_n} = \frac{\sum_e \phi_n^{e_i} K^{e_i} \phi_n^{e_i}\, \omega_n}{\sum_e \phi_n^{e_i} M^{e_i} \phi_n^{e_i}}$$

$$\omega_{min}^{e} \le \frac{\phi^e K^e \phi^e}{\phi^e M^e \phi^e} \le \omega_{max}^{e}$$



arbitrary element shape

$\phi^e = \phi_n^{e_i}$

$$\phi_n^{e_i} K^{e_i} \phi_n^{e_i} \le \omega_{max}^{e_i}\, \phi_n^{e_i} M^{e_i} \phi_n^{e_i} \le \omega_{max}^{all\ e}$$

$$\omega_n \le \frac{\sum_e \omega_{max}^{e_i} M^{e_i} \phi_n^{e_i}}{\sum_e \phi_n^{e_i} M^{e_i} \phi_n^{e_i}} \le \omega_{max}^{all\ e}\, \frac{\sum \phi_n^{e_i} M \phi_n^{e_i}}{\sum \phi_n^{e_i} M \phi_n^{e_i}}$$

1

$$\boxed{\omega_n \le \operatorname{Max}_e\left(\omega_{max}^{e}\right)}$$

max frequency of structure

difficult

max of element max frequency easy to compute

Central Difference

$$\Delta t \le \frac{2}{\omega_n}$$

If $\Delta t \le \frac{2}{\max_e \omega^e_{max}} \longrightarrow \Delta t \le \frac{2}{\omega_n}$

stability

Read the slides on how mass matrix can have a drastic impact on stable time step:

favorable any ways

$$\Delta t \le \frac{2}{\omega_e^m} = \frac{h_{\min}}{c}\begin{cases} 1 & \text{Lumped mass matrix} \\ \sqrt{\frac{2}{3}} \approx 0.667 & \text{High order mass matrix} \\ \frac{1}{\sqrt{3}} \approx 0.577 & \text{Consistent mass matrix} \end{cases}, \quad \text{for central difference method}$$

TABLE 9.5 $\boxed{\textit{Central difference method}}$ *critical time steps for some elements:*

$\Delta t_{cr}^{(m)} = T_n^{(m)}/\pi = 2/\omega_n^{(m)}$

Two-node truss element:

$$\mathbf{K}^{(m)} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \qquad \mathbf{M}^{(m)} = \frac{\rho L}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Delta t_{cr}^{(m)} = \frac{L}{c};$$

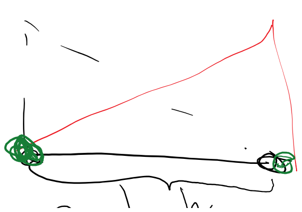
How does the polynomial order affects the maximum stable time step?

5.5.3 Effect of element order on maximum time step and other considerations

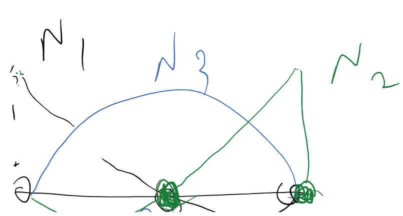
- For a lumped mass matrix and second order ($p = 2$) 1D bar element we obtain,

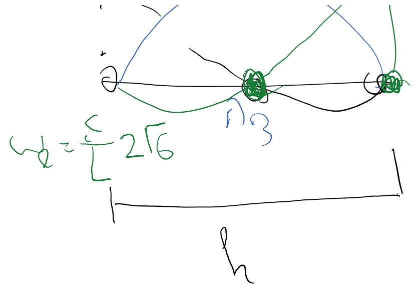
$$\omega_e = 2\sqrt{6}\frac{c}{L}, \qquad p = 2, \text{ lumped mass matrix}$$

- Recalling the maximum frequency from (398a) for $p = 1$ and lumped mass matrix we have the following,



$$\omega_e = \frac{c}{L}\begin{cases} 2 & p = 1 \\ 2\sqrt{6} & p = 2 \end{cases}, \quad \text{lumped mass matrix}$$





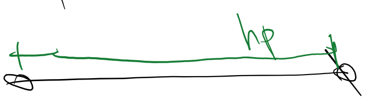
(for lumped mass matrix option) becomes,

$$\Delta t \le \frac{2}{\omega_e^m} = \frac{h_{\min}}{c} \begin{cases} 1 & p = 1 \\ \frac{1}{\sqrt{6}} \approx 0.408 & p = 2 \end{cases}, \quad \text{for central difference method and lumped mass matrix}$$

- So, The time step for $p = 2$ is only 0.408 times of $p = 1$.

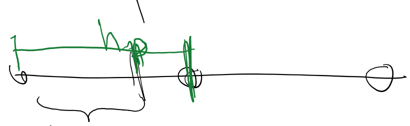
$\Delta t = \frac{h_{eff}}{c}$

p = 1



$h_{eff} = h$

p = 2



$h_{eff} = .408h$

in figure
$h_p = \frac{h}{p}$ ⟶ see below CFEM

1D central difference

$$\Delta t_{max} = \frac{h}{c} = \frac{h_p}{c}$$

can I use this for p=2

$$\Delta t_{max} = \frac{h_p}{c} = \frac{h}{2c} = .5 \frac{h}{c}$$

but the real limit is $.408 \frac{h}{c}$

Often $h_p = \frac{h}{P+1}$ specially in DG

It's a good starting point in terms of the wavelength that the element can capture.

- Another common way to express stability limit for different element orders is as,

$$\Delta t_{\max} = \begin{cases} C_H(p)\frac{h_{p_{\min}}}{c} & \text{Hyperbolic PDE}, c = \text{ wave speed} \\ C_P(p)\frac{h^2_{p_{\min}}}{D} & \text{Parabolic PDE}, D = \text{ diffusion coefficient} \end{cases} \quad \text{where } h_{p_{\min}} = \frac{h_{\min}}{p+1} \tag{405}$$

$\ddot{u} - c^2 u_{,xx} = 0$

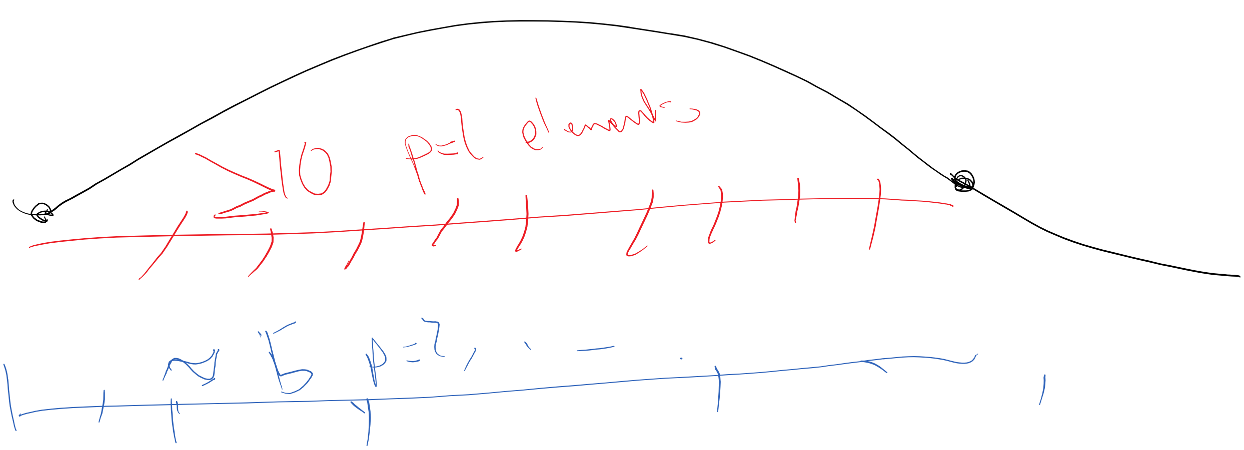
$\dot{u} - D u_{,xx} = 0$

  - $h_{p_{\min}}$ = is an effective element size based on the polynomial order that represent. This size in 1D is the distance between element nodes (if uniformly distributed) and in general represents the length-scale of a "wave", *i.e.*, region with a changed deflection, that an element can model.
  - $C_H(p)$ and $C_P(p)$ are correction factors that depend on,
    * Mass matrix option: *i.e.*, lumped mass, consistent mass, *etc.*.
    * Temporal integration scheme.
    * Underlying numerical method: For example, the same time of estimate can be applied to discontinuous Galerkin methods, *etc.* where for example a "mass matrix" (from item 1 above) may or may not exist, and same with the time integration order (*e.g.*, when spacetime FE methods are used). This can also depend on how many independent fields are interpolated (one-field versus multi-field) and possibly other details of a numerical method.

Summary

- The stable time step of conditionally stable methods depend on mass matrix option, details of the spatial discretization method, time integration method, and **spatial polynomial order** p.
- Instead of the maximum frequency (eigenvalue) of a MDOF system $\max_l(\omega_l^h)$, conservatively the maximum frequency (eigenvalue) of the individual elements ω_e^m is chosen in evaluating stable time step.
- The definition, $h_{p_{\min}} = \frac{h_{\min}}{p+1}$ and many stability analysis for $p > 1$ are based on having p half a sine wave $0 - \pi$ for an order p element. This is for stability considerations. For accuracy reasons, it is suggested to have at least 10 elements for resolving a wave segment, *e.g.*, half a sine wave; [Shakib and Hughes, 1991].

effective h_p



6 Mathematical analysis of finite difference methods

6.2 Convergence, consistency, and stability for FE methods

- The idea of convergence is having the FD solution for a given initial boundary value problem tend to the analytical one for any given time, provided that we let the mesh spatial and temporal resolution to zero.
- Again, the proof of convergence for given initial and boundary conditions and PDE is a challenging task as it involves ϵ, δ type limit analysis.
- Instead, as it's common in numerical solution of dynamic problems, we prove consistency and stability and indirectly prove convergence based on these two conditions.
- The following definitions are for one-step FD schemes applied to temporally first order PDEs taken from [Strikwerda, 2004] §1.4 and §1.5
- Formal definition of convergence, for one-step FD scheme applied to first order PDE, is

Definition 1 *A one-step FD scheme approximating a PDE is convergence if for any solution to the PDE $u(x,t)$ and solution to FD scheme v_m^n such that v_m^0 converges to $u_0(x)$ as mh converges to x, then v_m^n converges to $u(x,t)$ as (mh, nk) converges to (x,t) as h, k converge to zero.*

- Basically, definition 1 asserts that a FD scheme is convergent if for any IC, BC, source term, the numerical solution converges to the exact solution at any point if mesh grid sizes h, k approach zero. This idea is shown in the following figure from [Strikwerda, 2004].

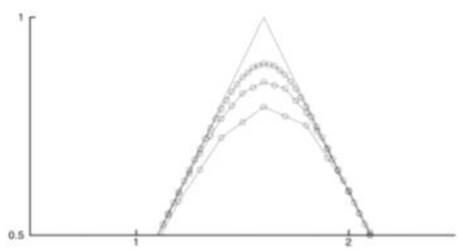


Figure 1.9. *Lax–Friedrichs scheme convergence.*

- As mentioned before, it is easier to prove convergence trough consistency and stability conditions.

We generally don't directly prove convergence.

- Consistency is a local condition which assert the finite difference operation is consistent with the underlying differential operator.
- Two difference between convergence and consistency are
  1. Convergence refers to the closeness of solutions while consistency refers to the closeness of differential operator occurring in the PDE.
  2. Convergence is a global condition by stipulating that the numerical and exact solutions are close at any point while consistency only requires the differential operator at a point to be close to the PDE differential operator.

Definition 2 *Given a partial differential equation $Pu = f$ and a FD scheme $P_{h,k} = f$, the FD scheme is consistent with the PDE if for any smooth function $\phi(x,t)$*

$$P\phi - P_{h,k}\phi \to 0 \quad \text{as} \quad h, k \to 0$$

the convergence is a point-wise condition at any given point (x,t).

the error in PDE (diff op.) goes to zero as $h, k \to 0$

Example 1 *Proof of consistency for the Forward-Time Forward-Space (FTFS) scheme (source [Strikwerda, 2004] Example 1.4.1),*

For the one-wave wave equation (26a) $(u_{,t} + a(x,t)u_{,x} = 0)$, the differential operator P is $\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}$ so that,

exact operator $\leftarrow$ $$P\phi = \phi_t + a\phi_x$$

FTFS $P_{h,k}\phi = \frac{\phi_m^{n+1} - \phi_m^n}{k} + a\frac{\phi_{m+1}^n - \phi_m^n}{h} = 0$

Te Ka

$\bar{k} = \frac{ka}{h}$

$\phi^n_m \xrightarrow{h} \phi^n_{m+1}$

n, index values of $(\ )^n_m$

$P\phi - P_{h,k}\phi \to 0$

$\phi^{n+1}_m = \phi^n_m + k\phi_{,t} + \frac{1}{2}k^2\phi_{,tt} + O(k^3)$

$\phi^n_{m+1} = \phi^n_m + h\phi_{,x} + \frac{1}{2}h^2\phi_{,xx} + O(h^3)$

$$P_{h,k} = \underbrace{\phi_{,t} + a\phi_{,x}}_{P\phi} + \frac{1}{2}k\phi_{,tt} + \frac{1}{2}ah\phi_{,xx} + O(k^2) + O(h^2)$$

where the derivatives on the RHS are all evaluated at (x_m, t_n), and so,

$$P_{h,k} = \phi_{,t} + a\phi_{,x} + \frac{1}{2}k\phi_{,tt} + \frac{1}{2}ah\phi_{,xx} + \mathcal{O}\left(k^2\right) + \mathcal{O}\left(h^2\right)$$

$$P\phi - P_{h,k} = -\frac{1}{2}k\phi_{,tt} - \frac{1}{2}ah\phi_{,xx} + \mathcal{O}\left(k^2\right) + \mathcal{O}\left(h^2\right) \to \quad \text{as} \quad (h,k) \to 0$$

Therefore, the scheme is consistent.

Lax-Friedrich's method

For the Lax-Friedrichs scheme (27d) the FD differential operator is,

$$P_{h,k} = \frac{\phi^{n+1}_m - \frac{1}{2}\left(\phi^n_{m-1} + \phi^n_{m+1}\right)}{k} + a\frac{\phi^n_{m+1} - \phi^n_{m-1}}{2h}$$

We use the Taylor series,

$$\phi^n_{m\pm1} = \phi^n_m \pm h\phi_{,x} + \frac{1}{2}h^2\phi_{,xx} \pm \frac{1}{6}h^3\phi_{,xxx} + \mathcal{O}\left(h^4\right)$$

where, as before, the derivatives are evaluated at (x_m, t_n) and we have,

$$\frac{1}{2}\left(\phi^n_{m-1} + \phi^n_{m+1}\right) = \phi^n_m + \frac{1}{2}h^2\phi_{,xx} + \mathcal{O}\left(h^4\right) \quad \text{and}$$

$$\frac{\phi^n_{m+1} - \phi^n_{m-1}}{2h} = \phi_{,x} + \frac{1}{6}h^2\phi_{,xxx} + \mathcal{O}\left(h^4\right)$$

$$P_{h,k}\phi = \underbrace{\phi_{,t} + a\phi_{,x}}_{P\phi} + \underbrace{\frac{1}{2}k\phi_{,tt} - \frac{1}{2}k^{-1}h^2\phi_{,xx} + \frac{1}{6}ah^2\phi_{,xxx} + \mathcal{O}\left(h^4 + k^{-1}h^4 + k^2\right)}_{\text{error}}$$

must have

$k^{-1}h^2 \to 0$

k does not go to zero faster than h^2!

most ...

$k^{-1}h^2 \to 0$ k does not go to zero faster than h^2!

wave eqn typically have $\bar{k}$ = cte $\leq$ stability limit

$\frac{ka}{h} = C \to k^{-1}h^2 = \frac{ah}{C} \to 0$

Example 2 *Conditional consistency of the Lax-Friedrichs scheme (source [Strikwerda, 2004] Example 1.4.2),*

So $P_{h,k} - P\phi \to 0$ as $h, k \to 0$; *i.e.*, it is consistent as long as $k^{-1}h^2$ also tends to zero.

- Note that some schemes such as Lax-Friedrichs schemes are conditionally stable meaning that h, k must satisfy certain condition for the consistency pf the method.
- For the Lax-Friedrichs scheme as it is applied to hyperbolic equations we require $h \propto k$ (for stability) so the consistency condition $k^{-1}h^2 \to 0$ requires $h \to 0$ which is $h \to 0$ which is satisfied. Basically, as long as k does not tend to zero faster than h^2 as $h \to 0$ the Lax-Friedrichs scheme is consistent.

Stability:

Definition 4 *Stability of temporally first order PDEs: A finite difference scheme $P_{h,k}v_m^n = 0$ for a temporally first-order PDE is stable in the stability region Λ if there an integer J such that for any positive time T, there is a constant C_T such that,*

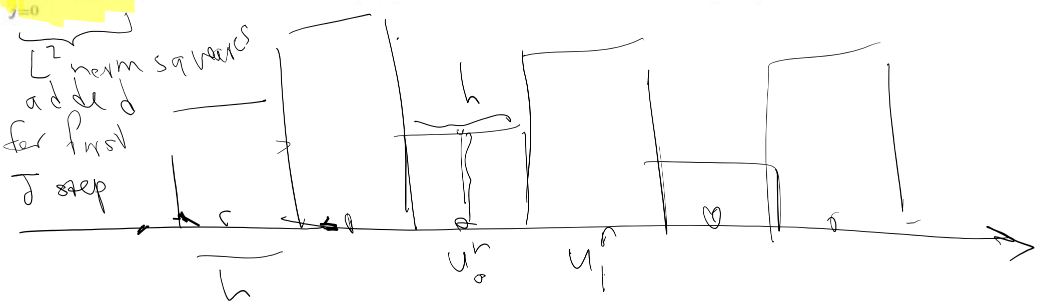
like L^2 of function square

$$h\|v^n\|_h^2 \leq C_T h \underbrace{\sum_{j=0}^{J} \|v^j\|_h^2}_{} \quad \textit{for } 0 \leq nk \leq T \quad \textit{with} \quad (h,k) \in \Lambda. \qquad (406)$$

solution at time t_n

$\sum_{i=-\infty}^{\infty} h\, u_i^{n\,2}$

L^2 norm squares added for first J step



J: # of backward steps (LMS)

Can depend on $T = nk$

Cannot depend on k, h

$k, h, T = nk$

Why should (can) C_T depend on T?

We even have this property for the exact solution when it's dynamically stable

$\dot{u} + a u_x = \beta u$

$u(x,t) = u_0(x - at)\, e^{\beta t}$

$\|u(\cdot,T)\| = \|u(\cdot,0)\|\, e^{\beta T}$

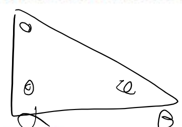
exact sln

- The number J refers to the number of steps required in a multi-step method. For example for a 1-step method that only requires t_n for updating t_{n+1} J will be 0, that is only initial data will be used in eq:FD:Stability:FirstOrder.

$$||v^n||_h^2 \le C_T ||v^0||_h^2, \qquad (J=0) \text{ in (419) for single-step methods} \tag{410}$$

- Comments on the value C_T
  - The most important aspect is that C_T **only depends on T not k nor h**: This means that no matter what grid size is used the solution at time T does not blow-up by for example letting $k \to 0$.
  - For unstable FD methods by letting $k \to 0$ the FD grid can represent higher frequency content (as will be discussed in §6.3) and the limit C_T will grow as $k \to 0$. That is, there is no constant C_T only dependent on T for unstable methods.
  - Note that C_T can be larger than one and in fact norm $||v^n||_h \to \infty$ as $n \to \infty$. That is, the solution can tend to infinity. This type of stability limit ($C_T > 1$) can arise if the spatial norm of the underlying exact physical solution also tend to infinity.
  - If the solution of the underlying solution is in fact bounded or decaying (as in many physical problems called dynamically stable; *cf.* §6.5) the spatial norm of physical solution does not grow and the FD scheme may have a $C_T \le 1$.
  - The stability condition of a numerical method is closely related to the concept of well-posedness or dynamic stability of a physical system which will be discussed in §6.5.

  - Stability is rarely directly checked. As will be discussed in §6.3 stability of a FD scheme is often investigated in the frequency domain. The example shows how stability can be checked directly.



Example 3 *Direct proof of stability of*

$$\boxed{v_m^{n+1} = \alpha v_m^n + \beta v_{m+1}^n} \tag{411}$$

$\alpha + \beta = 1$
consistency

is stable if $|\alpha| + |\beta| \le 1$. *(source [Strikwerda, 2004] Example 1.5.1)*

This type of update for example was observed in FTBS scheme applied to advection equation (26a) $u_{,t} + a(x,t)u_{,x} = 0$ for constant $a(x,t) = a$ in (27b): $\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_m^n - v_{m-1}^n}{h} = 0 \Rightarrow v_m^{n+1} = (1 - \bar{k})v_m^n + \bar{k}v_{m-1}^n$ (*cf.* (35b)) with $\bar{k} = a\frac{k}{h}$ being the normalized time step. Thus, for FTBS scheme $\alpha = 1 - \bar{k}$ and $\beta = \bar{k}$. The analysis is as follows,

$$\begin{aligned}
\sum_{m=-\infty}^{\infty} |v_m^{n+1}|^2 &= \sum_{m=-\infty}^{\infty} |\alpha v_m^n + \beta v_{m+1}^n|^2 \\
&\le \sum_{m=-\infty}^{\infty} |\alpha|^2 |v_m^n|^2 + 2|\alpha||\beta||v_m^n||v_{m+1}^n| + |\beta|^2 |v_{m+1}^n|^2 \\
&\le \sum_{m=-\infty}^{\infty} |\alpha|^2 |v_m^n|^2 + |\alpha||\beta| \left(|v_m^n|^2 + |v_{m+1}^n|^2\right) + |\beta|^2 |v_{m+1}^n|^2
\end{aligned}$$