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Stability of

like BSFT

FSFT

Example 3 *Direct proof of stability of*

$$v_m^{n+1} = \alpha v_m^n + \beta v_{m+1}^n \tag{411}$$

is stable if $|\alpha| + |\beta| \le 1$. *(source [Strikwerda, 2004] Example 1.5.1)*

This type of update for example was observed in FTBS scheme applied to advection equation (26a) $u_{,t} + a(x,t)u_{,x} = 0$ for constant $a(x,t) = a$ in (27b): $\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_m^n - v_{m-1}^n}{h} = 0 \Rightarrow v_m^{n+1} = (1-\bar{k})v_m^n + \bar{k}v_{m-1}^n$ (*cf.* (35b)) with $\bar{k} = a\frac{k}{h}$ being the normalized time step. Thus, for FTBS scheme $\alpha = 1 - \bar{k}$ and $\beta = \bar{k}$. The analysis is as follows,

triangle inequality

$$\begin{aligned}
\sum_{m=-\infty}^{\infty} |v_m^{n+1}|^2 &= \sum_{m=-\infty}^{\infty} |\alpha v_m^n + \beta v_{m+1}^n|^2 \\
&\le \sum_{m=-\infty}^{\infty} |\alpha|^2|v_m^n|^2 + 2|\alpha||\beta||v_m^n||v_{m+1}^n| + |\beta|^2|v_{m+1}^n|^2 \\
&\le \sum_{m=-\infty}^{\infty} |\alpha|^2|v_m^n|^2 + |\alpha||\beta|\left(|v_m^n|^2 + |v_{m+1}^n|^2\right) + |\beta|^2|v_{m+1}^n|^2
\end{aligned}$$

$2|v^n_m||v^n_{m+1}| \le |v^n_m|^2 + |v^n_{m+1}|^2$

$2ab \le a^2 + b^2$ $(a-b)^2 \ge 0$

$$\begin{aligned}
&\sum_{m=-\infty}^{\infty} |\alpha|^2|v_m^n|^2 + |\alpha||\beta|\left(|v_m^n|^2 + \boxed{|v_{m+1}^n|^2}\right) + \boxed{|\beta|^2|v_{m+1}^n|^2} \\
&= \sum_{m=-\infty}^{\infty} |\alpha|^2|v_m^n|^2 + |\alpha||\beta||v_m^n|^2 + \sum_{m=-\infty}^{\infty} |\beta|^2|v_{m+1}^n|^2 + |\alpha||\beta||v_{m+1}^n|^2 \longrightarrow m \\
&= \sum_{m=-\infty}^{\infty} |\alpha|^2|v_m^n|^2 + |\alpha||\beta||v_m^n|^2 + \sum_{m=-\infty}^{\infty} |\beta|^2|v_m^n|^2 + |\alpha||\beta||v_m^n|^2 \\
&= \sum_{m=-\infty}^{\infty} \left(|\alpha|^2 + 2|\alpha||\beta| + |\beta|^2\right)|v_m^n|^2 \\
&= (|\alpha| + |\beta|)^2 \sum_{m=-\infty}^{\infty} |v_m^n|^2
\end{aligned}$$

$$h\sum_{m=-\infty}^{\infty} |v_m^{n+1}|^2 \le (|\alpha| + |\beta|)^2 \, h\sum_{m=-\infty}^{\infty} |v_m^n|^2$$

$$\|v^{n+1}\| \le (|\alpha| + |\beta|)^2 \|v^n\|$$

$\longrightarrow$

$$\|v^n\| \le (|\alpha| + |\beta|)^{2n} \|v^0\|$$

$nk = T$

or

$$\boxed{\|v^n\| \le (|\alpha| + |\beta|)^{2T/k} \|v^0\|}$$

when can we claim

$$\|v^n\| \le C_T \|v^0\|$$

$|\alpha|+|\beta|\le 1 \qquad (|\alpha|+|\beta|)^{2n}\le 1 \longrightarrow \|v^n\|\le 1\cdot\|v^0\|$

Stable

$|\alpha|+|\beta|>1$ not being stable can also be proved.

$\overline{FTBS}$

$\alpha = 1-\bar{k}$

$\beta = \bar{k}$

$\bar{k}=\frac{ak}{h}$

$|1-\bar{k}|+|\bar{k}|\le 1$ happens for $|\bar{k}|\le 1$

$a<0 \qquad \bar{k}<0 \qquad \underbrace{1-\bar{k}}_{\alpha}>0 \longrightarrow |\alpha|+|\beta|>1$

unstable for any k

$$\left.\begin{array}{ll}\text{Consistency} & \alpha+\beta=1\\ \text{Stability} & |\alpha|+|\beta|\le 1\end{array}\right\} \longrightarrow \text{convergent method}$$

Any easier way to prove stability?

6.3.1 Fourier transformation and Fourier series

- We recall the Fourier transform from (203),

$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-\imath\omega t}\mathrm{d}t \quad\Leftrightarrow \tag{412a}$$

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{\imath\omega t}\mathrm{d}\omega \tag{412b}$$

General $f(t)$ is superposition of harmonic waves $e^{i\omega t}$ with amplitude $\hat{f}(\omega)$ obtained from

- Equation (412b) has a profound meaning in that we can write a function $f(t)$ as a "summation" of its frequency modes with frequency ω and amplitude $\hat{f}(\omega)$.
- That is a function $f(t)$ is expressed as a superposition of harmonic waves with different frequencies and different amplitudes.
- A very important identity in Fourier analysis is the Parseval's relation,

$$\int_{-\infty}^{\infty}|f(t)|^2\,\mathrm{d}t=\int_{-\infty}^{\infty}|\hat{f}(\omega)|^2\mathrm{d}\omega \quad\Rightarrow \boxed{\|f\|_2=\|\hat{f}\|_2} \tag{413}$$

the subscript 2 refers to L2 norm (408) $||u||_2 = \sqrt{\int_{\infty}^{\infty} |u(t)|^2 \, dt}$ which for convenience we drop the subscript 2 when there is no confusion in the type of norm employed.

- The Parseval's relation is of utmost importance as it says the norm of a function is equal to the norm of its Fourier transform.
- In stability analysis it is often easier to establish the stability of simple harmonic solutions. Subsequently, we use the Parseval's relation to establish stability for any form of solution by basically decomposing it into its harmonic parts.
- In the stability analysis of FD methods we are interested in how the spatial (discrete) norm of the solution grows. See for example, (407) where $||v^n||_h = \sqrt{h \sum_{m=-\infty}^{\infty} |v_m^n|^2}$ and the stability condition (419) $||v^n||_h^2 \le C_T \sum_{j=0}^{J} ||v^j||_h^2$.
- Accordingly, it is reasonable to apply the Fourier transform to x rather than t.
- By just writing the Fourier transform in x variable rather than t for a function u we express (412) as,

Fourier transform in space

$$\hat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x) e^{-\imath \xi x} \, dx \quad \Leftrightarrow \quad \xi \in (-\infty, \infty) \tag{414a}$$

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\xi) e^{\imath \xi x} d\xi \tag{414b}$$

ξ is wave number

$[\xi] = \left[\frac{1}{L}\right]$

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the parameter ξ is spatial frequency which is also called wavenumber. The symbols k and ξ are often used for it. However, given that k is used for the time step herein, we use the latter notation for the wavenumber.

- Still, we cannot use the Fourier series analysis and Parseval's relation for the stability analysis of FD method as the solution in FD schemes is only provided at discrete points h apart.

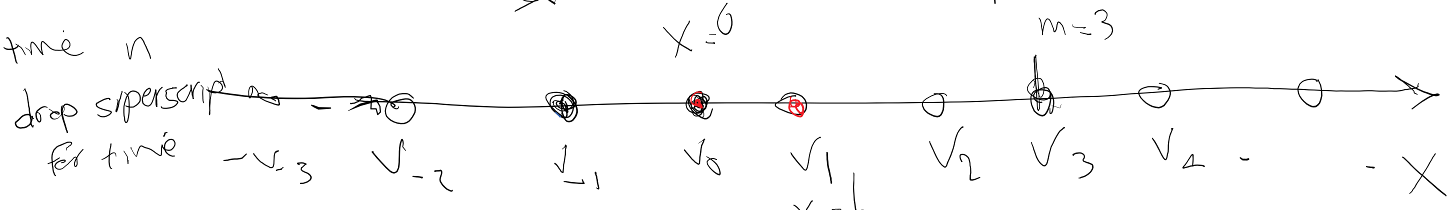
- Instead, we need to use the discrete version of the Fourier transform, ~~which is known as~~ Fourier series,

Fourier transform of v_i's

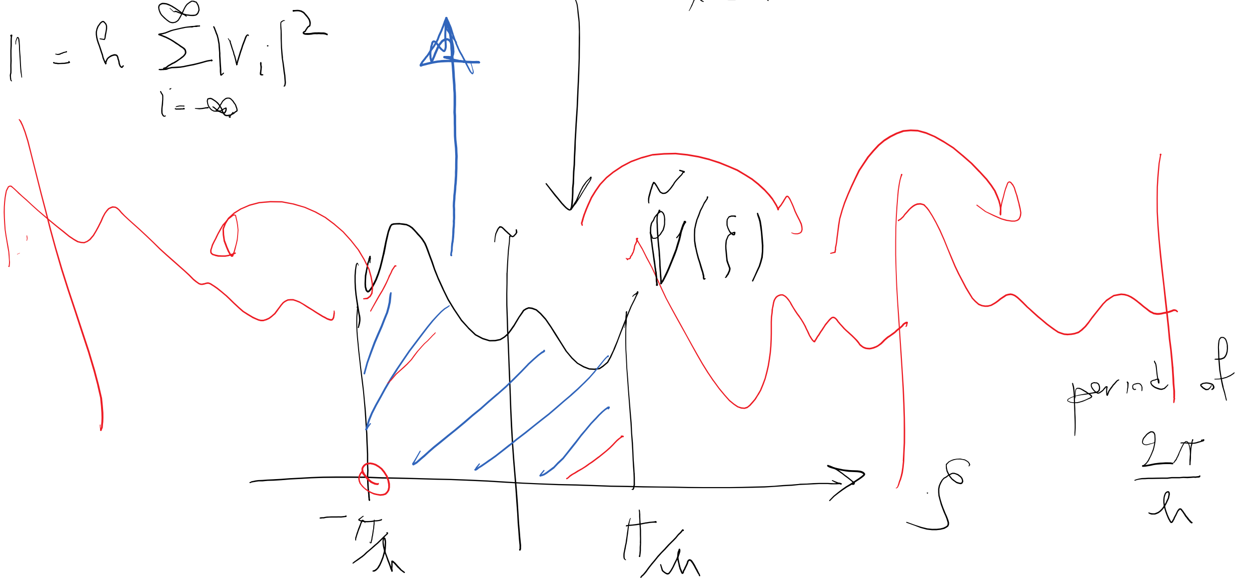
$$\hat{v}(\xi) = \frac{1}{\sqrt{2\pi}} h \sum_{m=-\infty}^{\infty} e^{-\imath m h \xi} v_m \quad \text{for} \quad \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right] \quad \Leftrightarrow \tag{415a}$$

$$v_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{\imath m h \xi} \hat{v}(\xi) d\xi \tag{415b}$$

$\boxed{mh = x}$



$||v|| = h \sum_{i=-\infty}^{\infty} |v_i|^2$



most often we have a periodic funct: $\hat{f}(\xi)$ between $-\frac{\pi}{h}, \frac{\pi}{h}$
and want to expand it as a harmonic series

$$v_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{i m h \xi} \, \hat{v}(\xi) \, d\xi$$

Fourier coefficients

harmonics $(\cos(mh\xi), \sin(mh\xi))$

$$\hat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} e^{-imh\xi} \, v_m$$

approximation: replace $-\infty \to -20$, $+\infty \to 20$

Why $\xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$

$\xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$

$\xi h \in [-\pi, \pi]$

harmonics $\{\cos \xi x, \sin \xi x\}$

Does not materialize

$|\xi h| > \pi$

limiting case for stability

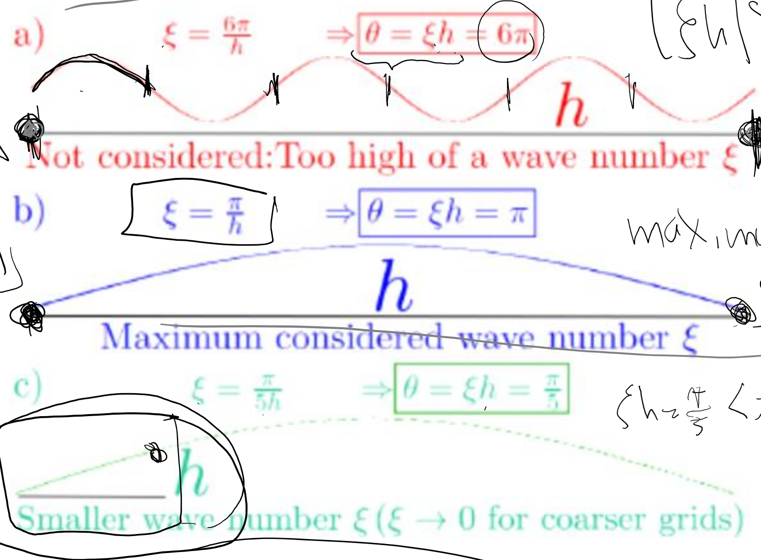
6 half sine waves in h

limiting case

maximum ξ

$\xi h = \frac{\pi}{5} < \pi$

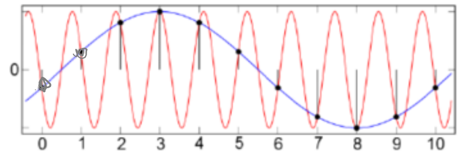
Where we want to be for accuracy



- The definition, $h_{p_{\min}} = \frac{h_{\min}}{p+1}$ and many stability analysis for $p > 1$ are based on having p half a sine wave $0 - \pi$ for an order p element. This is for stability considerations. For accuracy reasons, it is suggested to have <u>at least 10 elements</u> for resolving a wave segment, *e.g.*, half a sine wave; [Shakib and Hughes, 1991].

5 to 10 elements for a half sine wave

* However, what is the physical reasoning of breaking down a discrete solution of spacing h into only wavenumbers in the range $[-\frac{\pi}{h},\frac{\pi}{h}]$ not $[-\infty,\infty]$ as done in Fourier transformation case in (414b) ($u(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{u}(\xi)e^{\imath\xi x}\mathrm{d}\xi$) ?
* There is a very simple explanation of this fact:
* A grid of spacing h cannot distinguish any wavenumbers higher than π/h due to **aliasing effect**. That is for any higher order wavenumber there is a wavenumber in the interval $[-\frac{\pi}{h},\frac{\pi}{h}]$ that can exactly capture the same values at grid points v_m. This is shown in the figure below,



Source: Wikipedia

It is easy to see why aliasing occurs,

$$e^{\imath mh(\xi+\boxed{q(2\pi/h)})}=e^{\imath mh\xi}e^{\imath 2(qm)\pi(h/h)}=e^{\imath mh\xi}e^{\imath 2(qm)}=e^{\imath mh\xi}$$

$e^{\imath mh(\xi+q(2\pi/h))}$ would represent the red line in the figure for $\xi_2=\xi+q(2\pi/h)$ and the blue line the base frequency ξ. That is, a base frequency $\xi\in[-\frac{\pi}{h},\frac{\pi}{h}]$ has the same solution value at all grid points with spacing h for any other larger frequency outside of $[-\frac{\pi}{h},\frac{\pi}{h}]$ in the form $\xi_2=\xi+q(2\pi/h)$ $(q\in\mathbb{Z})$ and this is the physical reasoning of sufficiency of only having frequencies in the range $[-\frac{\pi}{h},\frac{\pi}{h}]$ in harmonic decomposition of the form (415b) of solution with grid values v_m.
In (6.3.1) we used $e^{\imath 2(qm)}=\cos(2(qm))+\imath\sin(2(qm))=1$.

* The material in this section is restricted to 1D analysis, but without much difficulty all the analysis can be extended to 2D and 3D spatial domains, a topic not considered herein.
* The Fourier transform in higher dimensions ($d=2$ in 2D and 3 in 3D) is,

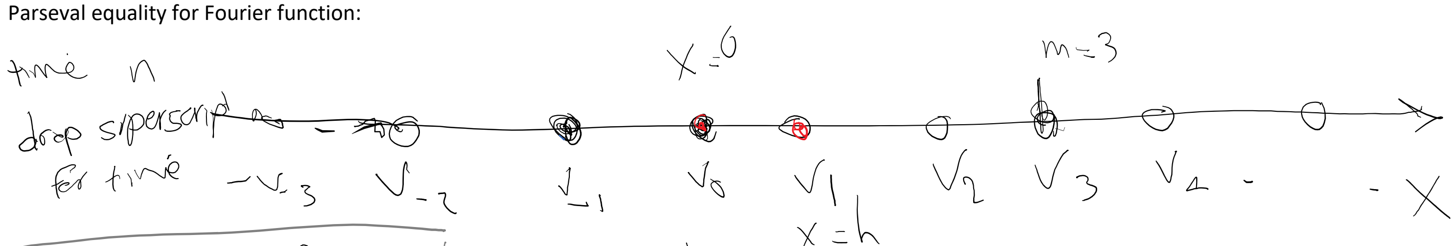
$$\hat{u}(\vec{\xi})=\frac{1}{2\pi^{d/2}}\int_{\mathbb{R}^d}u(\vec{\mathbf{x}})e^{-\imath\vec{\xi}.\vec{\mathbf{x}}}\mathrm{d}\vec{\mathbf{x}}\qquad\Leftrightarrow\tag{416a}$$

$$u(\vec{\mathbf{x}})=\frac{1}{2\pi^{d/2}}\int_{\mathbb{R}^d}\hat{u}(\vec{\xi})e^{\imath\vec{\xi}.\vec{\mathbf{x}}}\mathrm{d}\vec{\xi}\tag{416b}$$

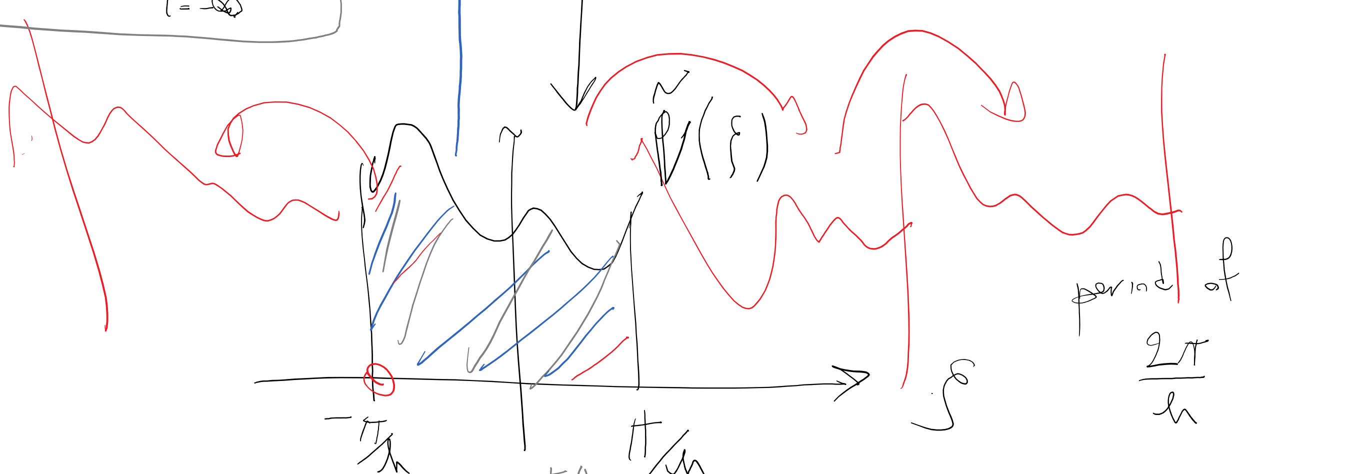
where $\vec{\mathbf{x}}$ and $\vec{\xi}$ are vectors in $\mathbb{R}^d$ and . is the inner-product operator.

* Fourier series (415) can also be easily extended to 2D and 3D for grids with even different spacings h_1,h_2,h_3 which again we do not pursue them given the similarity of the analysis of 2D and 3D FD problems to 1D ones.

Parseval equality for Fourier function:



$$\boxed{\|v\|=h\sum_{i=-\infty}^{\infty}|v_i|^2}$$



$$\|\hat{v}\|=\int_{-\pi/h}^{\pi/h}|\hat{v}(\xi)|^2\,d\xi$$

$$\|\hat{v}^n\| = \int_{-\pi/h}^{\pi/h} |\hat{v}(\xi)|^2 \, d\xi$$

$\|v\| = \|\hat{v}\|$ very much like Fourier transform

What's the use sequences

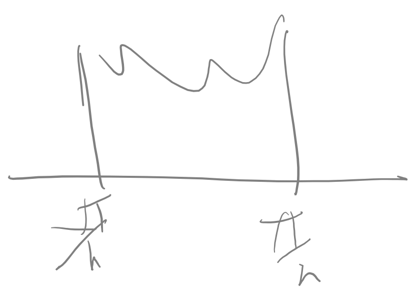
$$\|v^n\| \le C_T \|v^0\| \quad \leftarrow \text{single step method}$$

we prove this

$$\|$$

$$\|\hat{v}^n\| \le C_T \|\hat{v}^0\|$$

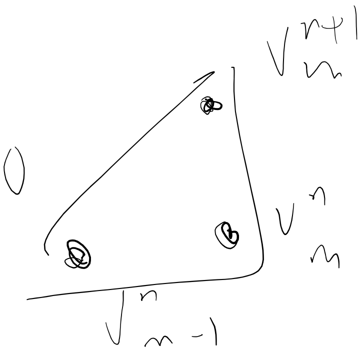
if we prove this (MUCH EASIER)



Example:

FTBS:

$$\frac{v_m^{n+1} - v_m^n}{k} + a\,\frac{v_m^n - v_{m-1}^n}{h} = 0$$



$$\boxed{v_m^{n+1} = (1-\bar{k})\, v_m^n + \bar{k}\, v_{m-1}^n} \qquad \bar{k} = a\frac{k}{h}$$

$$v_m^{n+1} = (1-\bar{k})\left(\frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h} e^{imh\xi}\,\hat{v}^n(\xi)\,d\xi\right) + \frac{\bar{k}}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h} e^{i(m-1)h\xi}\,\hat{v}^n(\xi)\,d\xi$$

$$v_m^{n+1} = (1-\bar{k})\left(\frac{1}{\sqrt{2\pi}}\int_{-\pi/h} e^{\ldots}\, \hat{v}(\xi)\,d\xi\right) + \frac{\bar{k}}{\sqrt{2\pi}}\int_{-\pi/h} e^{i\ldots}\, \hat{v}^n(\xi)\,d\xi$$

$$v_m^{n+1} = \quad // \quad + \frac{\bar{k}}{\sqrt{2\pi}}\int \left(e^{-ih\xi}\right) e^{imh\xi}\, \hat{v}^n(\xi)\,d\xi$$

$$v_m^{n+1} = \frac{1}{\sqrt{2\pi}}\int e^{imh\xi}\left[(1-\bar{k}) + \bar{k}e^{-ih\xi}\right]\hat{v}^n(\xi)\,d\xi$$

But we have

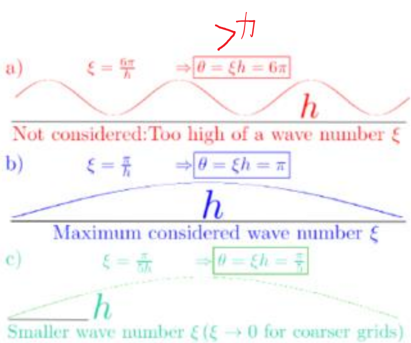
$$v_m^{n+1} = \frac{1}{\sqrt{2\pi}}\int e^{imh\xi}\, \hat{v}^{n+1}(\xi)\,d\xi$$

$$\boxed{\hat{v}^{n+1}(\xi) = \left[(1-\bar{k}) + \bar{k}e^{-ih\xi}\right]\hat{v}^n(\xi)}$$

∴ Fourier functi update is a simple algebraic eqn :)

$$\theta = h\underbrace{\xi}_{[1/L]}^{\text{wave \#}}, \quad h \to [L]$$

nondimensional wave #



$>\pi$ X

maximum θ

$$\hat{v}^{n+1}(\xi) = g(\theta)\,\hat{v}^n(\xi)$$

$$g(\theta) = (1-\bar{k}) + \bar{k}e^{-i\theta}$$

$$\boxed{\hat{v}^n(\xi) = g(\theta)^n\, \hat{v}^0(\xi)}$$

Very simple relation between time n

Very simple relation between time n & time 0 Fourier functions

(No simple relation between v^n_m & v^0_m's that's what made stability proof difficult)

$$\|v^n\|^2 = h\sum_{m=-\infty}^{\infty} \|v^n_m\|^2 \underset{\text{Parseval}}{=} \|\hat{v}^n\|^2 = \int_{-\pi/h}^{\pi/h} |g(\theta)|^{2n} |\hat{v}^0(\xi)|^2 d\xi$$

IF $|g(\theta)| \le 1$

$$\le \int_{-\pi/h}^{\pi/h} \|\hat{v}^0(\xi)\|^2 d\xi$$

$$\underset{\text{Parseval}}{=} h\sum_{m=-\infty}^{\infty} |v^0_m|^2 = \|v^0\|^2$$

$$\boxed{\text{if } |g(\theta)| \le 1 \longrightarrow \|v^n\|^2 \le \underbrace{1}_{C_T} \|v^0\|^2}$$

Find out when $g(\theta) \le 1$

$e^{-i\theta} = \cos\theta + i\sin(-\theta)$

waves that the discrete FD grid can represent.

- With this notation (427) $g(h\xi) = g(\theta)$ in (423b) becomes,

$$g(\theta) := (1-\bar{k}) + \bar{k}e^{-\imath\theta} \quad \Rightarrow \text{FTBS}$$

$$g(\theta) = g_R + \imath g_I, \quad \text{where} \quad g_R = (1-\bar{k}) + \bar{k}\cos\theta, \quad g_I = -\bar{k}\sin\theta \tag{428a}$$

- In (428a) g is written in terms of its real g_R and imaginary g_I components.
- Now, since we are seeking conditions where $|g(\theta)| \le 1$ we use its square value,

$$|g(\theta)| = \sqrt{g_R^2 + g_I^2} \le 1 \quad \Leftrightarrow \quad |g(\theta)|^2 = g_R^2 + g_I^2 \le 1 \tag{429a}$$

- Now using the identities,

$$1 - \cos\phi = 2\sin^2\frac{1}{2}\phi, \qquad \sin\phi = 2\sin\frac{1}{2}\phi\cos\frac{1}{2}\phi$$

and letting $\phi = \theta$ we obtain,

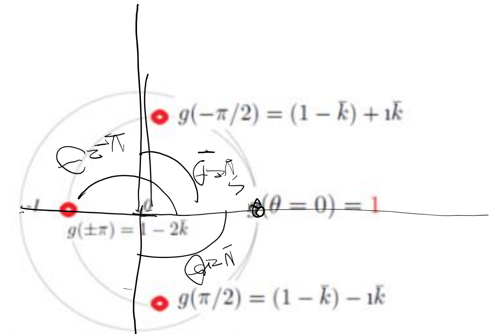
$$\begin{aligned}
|g(\theta)|^2 = g_R^2 + g_I^2 &= (1 - \bar{k} + \bar{k}\cos\theta)^2 + \bar{k}^2\sin^2\theta \\
&= (1 - 2\bar{k}\sin^2\frac{1}{2}\theta)^2 + 4\bar{k}^2\sin^2\frac{1}{2}\theta\cos^2\frac{1}{2}\theta \\
&= 1 - 4\bar{k}\sin^2\frac{1}{2}\theta + 4\bar{k}^2\sin^4\frac{1}{2}\theta + 4\bar{k}^2\sin^2\frac{1}{2}\theta\cos^2\frac{1}{2}\theta \quad \Rightarrow
\end{aligned}$$

$$\boxed{|g(\theta)|^2 = 1 - 4\bar{k}(1-\bar{k})\sin^2\frac{1}{2}\theta} \tag{430}$$

$|g(\theta)|^2 \le 1 \qquad 0 \le |\theta| \le \pi$

$$\boxed{|g(\theta)|^2 = 1 - 4\bar{k}(1-\bar{k})\sin^2\frac{1}{2}\theta} \qquad (430)$$

$|g(\theta)|^2 \le 1 \qquad 0 \le |\theta| \le \pi$



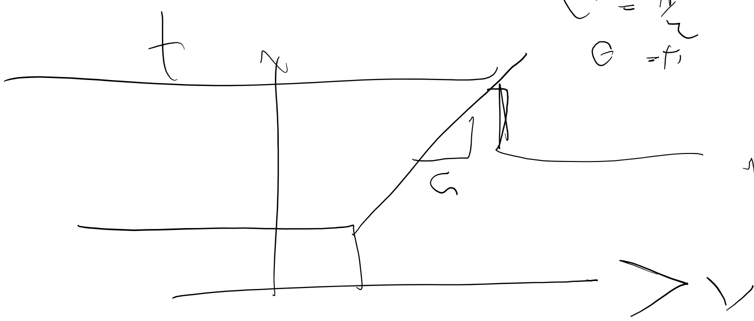
The image of $g(\theta)$ for the forward-time backward-space scheme.

$$g(\theta = 0) = 1 \qquad (431a)$$
$$g(\pi/2) = (1-\bar{k}) - \imath\bar{k} \qquad (431b)$$
$$g(-\pi/2) = (1-\bar{k}) + \imath\bar{k} \qquad (431c)$$
$$\boxed{g(\pm\pi) = |1 - 2\bar{k}|} \le 1 \quad \boxed{\bar{k} \le 1} \qquad (431d)$$

$\theta = 0$ approaching the exact s'n

$\theta = \frac{\pi}{2}$ half way max wave#

$\theta = \pi$ limiting wave#



$u_t + a u_x = 0$

- When $\bar{k} > 1$ the loci of $g(\theta)$ of goes beyond the unit circle in the complex plane around $\theta = \pm\pi$ and around those θ. This is for this problem the instabilities initiate from the highest frequency modes $\bar{k}$. This corresponds to instable values for k.
- The fact that $g(\theta) = 1$ for $\theta = 0$ is always given if the FD scheme is consistent, as otherwise in the limit of $h, k \to 0$ (relative to wavenumber ξ) the FD scheme is not consistent with exact PDE.
- Recalling $\bar{k} = a\frac{k}{h}$ from (28) FTBS scheme is conditionally stable for the range $k \le \frac{h}{a}$. This clearly matches our numerical example from §2.1.8 and example 3 earlier in this section.
- Clearly, in neither of these approaches / examples we have proven that the scheme is unstable for $k > \frac{h}{a}$.
- The next theorem demonstrates that $|g(\theta)|^2 > 1$ for some θ for this case where g is independent of k corresponds to instability.

6.3.4 Theorem on stability of FD methods

Theorem 2 *Stability analysis in frequency domain for one-step FD schemes: A one-step FD scheme (with constant coefficients) is stable in a stability region Λ if and only if there is a constant K (independent of θ, k and h) such that,*

$$|g(\theta, k, h)| \le 1 + Kk \tag{432}$$

with $(h, k) \in \Lambda$. If $g(\theta, k, h)$ is independent of h and k, the stability condition (432) can be replaced with the restricted stability condition,

$$|g(\theta, k, h)| \le 1 \tag{433}$$

Proof: We have the Parseval's relation (417) and the definition of g, that

$$||v^n||_h^2 = ||\hat{v}^n||_h^2 = \int_{-\pi/h}^{\pi/h} |\hat{v}^n(\xi)|^2 \mathrm{d}\xi \tag{434a}$$

$$= \int_{-\pi/h}^{\pi/h} |g^n(h\xi, k, h)\hat{v}^0(\xi)|^2 \mathrm{d}\xi = \int_{-\pi/h}^{\pi/h} |g^{2n}(h\xi, k, h)||\hat{v}^0(\xi)|^2 \mathrm{d}\xi \qquad \text{from (424)} \tag{434b}$$

Now from (432) ($|g(\theta, k, h)| \le 1 + Kk$) we have,

$$||\hat{v}^n||_h^2 \le \int_{-\pi/h}^{\pi/h} (1 + Kk)^{2n} |\hat{v}^0(\xi)|^2 \mathrm{d}\xi = (1 + Kk)^{2n} ||\hat{v}^0||_h^2 \tag{435}$$

now for $t = nk \le T$ we have $n \le T/k$. Thus from $(1 + \alpha)^\beta \le e^{\alpha\beta}$ for $\alpha, \beta \ge 0$ we have,

$$(1 + Kk)^{2n} \le (1 + Kk)^{2T/k} \le e^{2KT} \tag{436}$$

and from (435) and (436) we obtain,

$$||\hat{v}^n||_h^2 \le e^{2KT} ||\hat{v}^0||_h^2 \qquad \Rightarrow \qquad \boxed{||v^n||_h^2 \le e^{2KT} ||v^0||_h^2} \quad \text{(using Parseval's relation (417))} \tag{437}$$

$C_T = e^{2KT}$ does not depend on time step