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We discuss how we can simply plug in simple harmonic solutions with wavenumbers $\xi \in [-\pi/h, \pi/h]$ in a given FD stencil to directly update amplification factor g.
The steps of this argument are as follows,

1. **Harmonic decomposition of the initial condition(s):** First, the IC of the PDE can be written as superposition of waves with wavenumbers $\xi \in [-\pi/h, \pi/h]$ following the Fourier series (415b),

$$v_m^n = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \hat{v}^n(\xi) d\xi \qquad (439)$$

$\cos x\xi + i \sin x\xi$

$$v_m^n = v(x = mh, t = nk) = \int_{-\pi/h}^{\pi/h} e^{ix\xi} \left(\frac{1}{\sqrt{2\pi}} \hat{v}^n(\xi)\right)$$

the solution is a linear superposition of harmonic waves $e^{ix\xi}$ with factors $\left(\frac{1}{\sqrt{2\pi}} \hat{v}^n(\xi)\right)$ for wavenumber ranging from $\left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$

$$v_m^n = \int_{-\pi/h}^{\pi/h} e^{i\xi x} \frac{1}{\sqrt{2\pi}} \hat{v}^n(\xi)\, d\xi$$

$$v_m^0 = \int_{-\pi/h}^{\pi/h} e^{i\xi x} \frac{1}{\sqrt{2\pi}} \hat{v}^0(\xi)\, d\xi$$

general

$$\boxed{\hat{v}^n(\xi) = g^n(\xi h)\, \hat{v}^0(\xi)}$$ if PDE linear

$|g| \leq 1 + Kk$ for $-\pi \leq \theta = h\xi \leq \pi$

The IC can always be written as a sum of harmonic waves (Fourier transform)
The solution at time tn = nk can also be written as a sum of harmonic waves (Fourier transform)

For LINEAR PDEs, the wavenumber ξ component of the solution at time t_n = nk is simply the solution of the problem with IC with a harmonic wave.

Instead of considering all ξ at once study the stability for one wavenumber at a time

PLUG in $e^{i\xi x}$ as the form of IC for fixed $\xi \in [-\frac{\pi}{h}, \frac{\pi}{h}]$

$$v_\xi^n(x) = e^{ix\xi}\,\hat{v}^n(\xi)$$

take a harmonic component of 'H

$$v_m^n = \int_{-\pi/h}^{\pi/h} e^{ix\xi} \frac{1}{\sqrt{2\pi}} \hat{v}^n(\xi)\,d\xi \qquad x = mh$$

we can drop ξ subscript by knowing that we work with fixed ξ.

$$v_m^n = v_\xi^n(x=mh) = e^{imh\xi}\,\hat{v}^n(\xi), \quad \theta = h\xi$$

$$\boxed{v_m^n = e^{im\theta}\,\hat{v}^n}$$

$$\boxed{\hat{v}^n = g^n\,\hat{v}^0}$$

plug sln in the form $e^{ix\xi}$ for IC:

$$(\star)\quad \boxed{v_m^n = g^n e^{im\theta}\,\hat{v}^0}$$

Von Neumann analysis for FTBS $\quad v_m^{n+1} = (1-\bar{k})\,v_m^n + \bar{k}\,v_{m-1}^n$

$$g^{n+1} e^{im\theta}\hat{v}^0 = (1-\bar{k})\,g^n e^{im\theta} + \bar{k}\,g^n e^{i(m-1)\theta}\,\hat{v}^0 \quad \text{using } \star$$

$g^{n+1} e^{im\theta} \hat{v}_0 = (1-\bar{k}) g^n e^{im\theta} \hat{v}_0 + \bar{k} g^n e^{i(m-1)\theta} \hat{v}_0$ Using # $/ g^n e^{im\theta}$

$$\boxed{g = (1-\bar{k}) + \bar{k}e^{-i\theta}}$$

Previously, we obtained this relation as follows:

- which can be rewritten as (35b),

$$v_m^{n+1} = (1-\bar{k})v_m^n + \bar{k}v_{m-1}^n \quad \text{for normalized time step} \quad \bar{k} = a\frac{k}{h} \tag{420}$$

- By taking the Fourier series transform on both sides of (420) and recalling (415b) $v_m = \frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h} e^{\imath m h\xi}\hat{v}(\xi)\mathrm{d}\xi$ we obtain,

$$\begin{aligned} v_m^{n+1} &= (1-\bar{k})v_m^n + \bar{k}v_{m-1}^n \\ &= (1-\bar{k})\left\{\frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h} e^{\imath m h\xi}\hat{v}^n(\xi)\mathrm{d}\xi\right\} + \bar{k}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h} e^{\imath(m-1)h\xi}\hat{v}^n(\xi)\mathrm{d}\xi\right\} \quad \Rightarrow \\ v_m^{n+1} &= \frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h} e^{\imath m h\xi}\left[(1-\bar{k}) + \bar{k}e^{-\imath h\xi}\right]\hat{v}^n(\xi)\mathrm{d}\xi \end{aligned} \tag{421}$$

note that the dependence to time step t_n for values v_m^n is shown as superscript for the grid point values and Fourier functions $\hat{v}^n$.

- On the other hand, again from the definition of Fourier transform we have,

$$v_m^{n+1} = \frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h} e^{\imath m h\xi}\hat{v}^{n+1}(\xi)\mathrm{d}\xi \tag{422}$$

- By comparing (421) and (422) we obtain,

$$\hat{v}^{n+1}(\xi) = g(h\xi)\hat{v}^n(\xi) \quad \text{where} \tag{423a}$$

$$g(h\xi) := \left[(1-\bar{k}) + \bar{k}e^{-\imath h\xi}\right] \quad \text{amplification factor (for FTBS method)} \tag{423b}$$

Two more von Neumann examples:

Example 4 *Stability of the Lax-Friedrichs scheme (source [Strikwerda, 2004] Example 2.2.4),*

- Consider the Lax-Friedrichs FD equation for the advection equation $u_{,t} + au_{,x} = 0$ from (27d),

$$\frac{v_m^{n+1} - \frac{1}{2}\left(v_{m-1}^n + v_{m+1}^n\right)}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0$$

- The update equation for v_m^{n+1} is as follows,

$$v_m^{n+1} = -\frac{\bar{k}}{2}\left(v_{m+1}^n - v_{m-1}^n\right) + \frac{1}{2}\left(v_{m+1}^n + v_{m-1}^n\right) \qquad \text{for} \quad \bar{k} = a\frac{k}{h}$$

$g = -\frac{\bar{k}}{2}\left(e^{i\theta} - e^{-i\theta}\right) + \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ extra to $g^n e^{im\theta}\hat{v}_0$

$2i\sin\theta$ $2\cos\theta$

$$\boxed{g(\theta) = \cos\theta - i\bar{k}\sin\theta}$$

$|g(\theta)| = \cos^2\theta + \bar{k}^2\sin^2\theta$

$|g(\theta)| = \cos^2\theta + \bar{k}^2 \sin\theta$

$\bar{k} > 1$ unstable

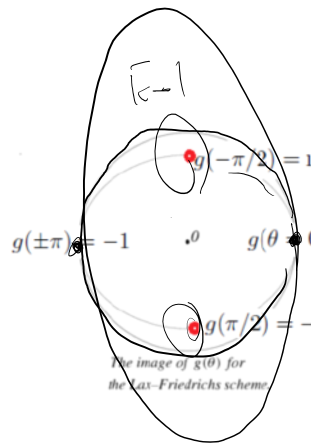
$$\cos\phi = \frac{e^{\imath\phi} + e^{-\imath\phi}}{2} \tag{452b}$$

- In (451) for $\phi = \theta$ we obtain,

$$g(\theta) = \cos\theta - \imath\bar{k}\sin\theta \quad \text{which gives} \tag{453}$$

$$|g(\theta)|^2 = \cos^2\theta + \bar{k}^2\sin^2\theta \tag{454}$$

- Since $g(\theta)$ is explicitly independent from k we need to use the stability condition (433) ($|g(\theta, k, h)| \leq 1$) rather than (432).
- From (454) $|g(\theta)|^2 \leq 1$ for all $\bar{k}$ if any only if $|\bar{k}| = |a\frac{k}{h}| \leq 1$.
- Irrespective of sign of a the Lax-Friedrichs method is conditionally stable for $k \leq k/|a|$
- The figure shows the image of g in the complex plane as θ spans $[-\pi, \pi]$ when the image remains in the unit circle, that is for the case $k \leq k/|a|$.
- The points corresponding to $\theta = 0, \pm\pi/2, \pm\pi$ are given in the equation below and marked in the figure,



The image of $g(\theta)$ for the Lax-Friedrichs scheme

$\bar{k} \leq 1$

Example 5 ***Numerical Stability** of the Lax-Friedrichs scheme applied to a dynamically unstable problem (source [Strikwerda, 2004] Example 2.2.3),*

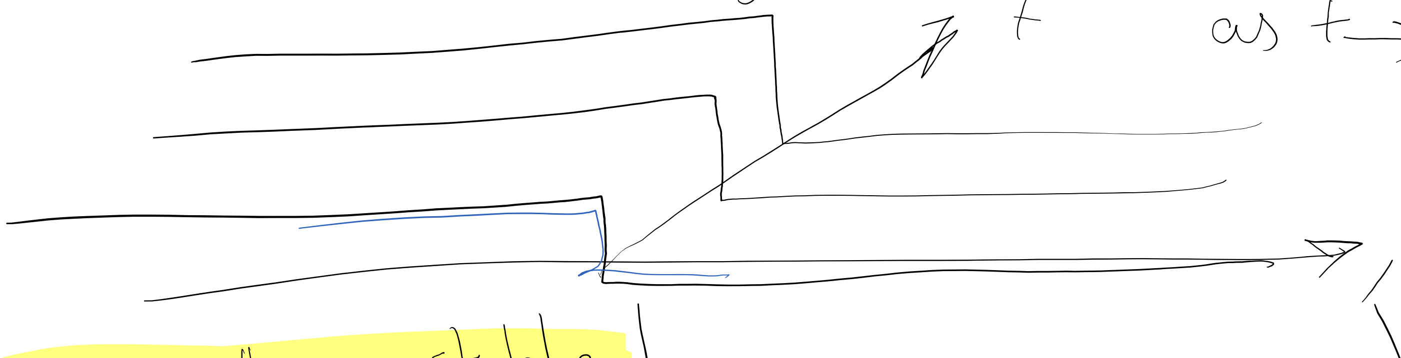
- Consider the following problem,

advect. → react

$$u_{,t} + au_{,x} - u = 0 \tag{456}$$

which is an advection reaction equation $u_{,t} + au_{,x} + \beta u = 0$ with $\beta = -1$ where β is the reaction coefficient.

exact sln to this $u = U_0(x - at)e^t \to \infty$ as $t \to \infty$



Dynamically unstable!

for the exact sln $(\|u(\cdot, t)\| \to \infty$ as $t \to \infty)$

exact sln blows up.

But this is a Well-posed problem (stability for exact sln)

$\exists C_T \ni \|u(\cdot, T)\| \leq C_T \|u(\cdot, t=0)\|$ (IC)

here $u(x, T) = U_0(x - aT)e^T$

here $u(x,T) = u_0(x-aT)e^T$ IC

$$\|u(\cdot,T)\| \le e^T \|u(\cdot,0)\|$$

$C_T = e^T$ only depends on time

Well-posedness: No matter what initial condition is, the solution at time T is bounded by a factor depending ONLY on T (not the form of IC function) multiplying the IC norm.

The problem above is well-posed (the growth is bounded by a factor depending only on Time) BUT not dynamically stable (it grows to inifinity)

Exact solution

well-posedness

$$\|u(\cdot,T)\| < C_T \|u(\cdot,0)\|$$

Numerical

$h, k \quad u \to u^h$

$$\|u^h(\cdot,T)\| \le C_T \|u^h(\cdot,0)\|$$

STABILITY MEANS PRESERVING THE WELL-POSEDNESS OF THE ORIGINAL PDE

IF the PDE is well-posed, we want to maintain that property in Numerical setting.

Dynamic stability

$$\|u(\cdot,T)\| \le C \|u(\cdot,0)\|$$

Constant independent of t

Dynamically stable (doesn't blow up) $\Longrightarrow$ well posed

well-posed $\not\Longrightarrow$ Dynamically stable

Example

$$u_t + a u_{,x} - u = 0$$

$u = u_0(x-at)e^t$ → well-posed, $\not\Rightarrow$ not dynamically stable

$u_{,tt} - u_{,xx} = 0$ is Not well posed

$u_{,tt} - u_{,x} = 0$ 〃 〃

$U_t + a u_{,x} - u = 0$

want to find $\bar{k}$ such that the solution is stable (preserving well posedness in numerical setting) with LF method:

- Now, going back to the FD discretization of (457), the FD update equation is,

$$\frac{v_m^{n+1} - \frac{1}{2}\left(v_{m-1}^n + v_{m_1}^n\right)}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} - v_m^n = 0$$

- The rest of the analysis is very similar to that of example 4 on the stability of Lax-Friedrichs method without reaction term.
- The update equation (451) for v_m^{n+1} is slightly modified to,

$$v_m^{n+1} = -\frac{\bar{k}}{2}\left(v_{m+1}^n - v_{m-1}^n\right) + \frac{1}{2}\left(v_{m+1}^n + k v_{m-1}^n\right) + k v_m^n$$

$g = \cos\theta - i\bar{k}\sin\theta + k$

$|g(\theta)| \le 1$ X

$g(\theta = 0) = 1 + k$

$|g(\theta)|^2 = (\cos\theta + k)^2 + (\bar{k}\sin\theta)^2$

for $\bar{k} \le 1$

$\le \cos^2\theta + \sin^2\theta + 2\cos\theta\, k + k^2$

$\le 1 + 2k + k^2$

$\le (1+k)^2$

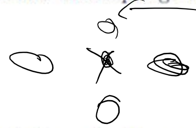
For kbar <= 1, the numerical solution is stable (numerically we preserve the well-posedness).

6.4 von Neumann analysis for multi-step FD schemes

- Multi-step methods refer to those requiring beyond (earlier) time step values than t_n to obtain values for t_{n+1} solutions.
- Multi-step methods can be encountered,
  1. Higher than 1[st] temporal oder for the PDE.
  2. Higher order stencils in time that require beyond (earlier) than time step t_n for updating values for t_{n+1}.
- Below, we provide examples from each category and discuss von Neumann stability analysis using these examples.

6.4.1 von Neumann analysis for leapfrog scheme

- Consider the leapfrog scheme (27e) for the advection equation $u_{,t} + a u_{,x} = 0$,



$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0 \quad \Rightarrow \tag{464a}$$

$$v_m^{n+1} = \bar{k}(v_{m-1}^n - v_{m+1}^n) + v_m^{n-1} \tag{464b}$$

$g = \bar{k}(e^{-i\theta} - e^{i\theta}) + g^{-1}$) $\times\, \hat{V}\, e^{im}$ g^n

$$\left(g = \bar{k}(e^{-i\theta} - e^{i\theta}) + g^{-1} \right) \times \hat{V}_0 e^{im} g^n$$

multiply by g: ← ($e^{-i\theta} - e^{i\theta} = -2i\sin\theta$)

$$g^2 + 2i\bar{k}\sin\theta\, g - 1 = 0$$

$$g = g_1, g_2 = -i\sin\theta\,\bar{k} \pm \sqrt{-\bar{k}^2\sin\theta + 1}$$

- The solutions to (468) are,

$$g_+ = -\imath\bar{k}\sin\theta + \sqrt{1 - \bar{k}^2\sin^2\theta} \quad (469a)$$

$$g_- = -\imath\bar{k}\sin\theta - \sqrt{1 - \bar{k}^2\sin^2\theta} \quad (469b)$$

1-g we had $|g| \leq 1 + K_k$

for dynamically stable PDE we often get $|g| \leq 1$

$|g_-| \leq 1$

$|g_+| \leq 1$

is this sufficient?

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for the moment

$\bar{k} \leq 1$ for $\theta \in [-\pi, \pi]$ $\sin\theta \in [-1, 1]$ $1 - \bar{k}^2\sin^2\theta \geq 0$

$(g_\pm)_{Real} = \pm\sqrt{1 - \bar{k}^2\sin^2\theta}$

$(g_\pm)_{Im} = -\bar{k}\sin\theta$

$|g|^2 = g_{real}^2 + g_{Im}^2 =$

$1 - \bar{k}^2\sin^2\theta + \bar{k}^2\sin^2\theta = 1$

independent of θ

VC $e^{i\xi x}$

$\|V^n\| = g^n \|V^0\|$ $g = 1$ for all ξ

$\|v^n\| = g^n \|v^0\|$ $g = 1$ for all ξ

$\rightarrow$ for all ξ | IC $e^{i\xi x}$ $\|v^n\| = \|v^0\|$

$\rightarrow$ FOR any IC $\|v^n\| = \|v^0\|$

Leap frog is a conservative method, that for the conservative underlying advection equation the norm of the solution does not decay in time also under numerical setting.

2. $\bar{k} > 1$: In this case $1 - \bar{k}^2 \sin^2\theta < 0$ for $\pi/2 \ge \theta > \sin^{-1}(1/\bar{k})$ (*e.g.*, $\theta = \pi/2$). So $\sqrt{1 - \bar{k}^2 \sin^2\theta} = \imath\sqrt{\bar{k}^2 \sin^2\theta - 1}$ for such θ. In this case, $|g_-| > 1$ some θ so the scheme is not stable. For example, for $\theta = \pi$ we have,

$$g_-(\pi) = -\imath\bar{k}\sin(\pi/2) - \sqrt{1 - \bar{k}^2\sin^2(\pi/2)} = -\imath\left(\bar{k} + \sqrt{\bar{k}^2 - 1}\right) > 1 \qquad (\text{since } \bar{k} > 1)$$

$\bar{k} = 1$ $g_- = g_+ = -i\sin\theta$ for $\theta = \frac{\pi}{2}$

$g_-(\theta = \frac{\pi}{2}) = g_+(\theta = \frac{\pi}{2}) = i$

We encountered this situation before:

$$-a_i^k + \bar{c}_1 a_i^{k-1} + \bar{c}_2 a_i^{k-2} + \cdots + \bar{c}_k = 0 \quad \Leftrightarrow \quad (\bar{c}_i = -\frac{c_i}{c_0}, \textit{cf.}\ (353)) \tag{348c}$$

$$\boxed{c_0 a_i^k + c_1 a_i^{k-1} + c_2 a_i^{k-2} + \cdots + c_k = 0} \tag{348d}$$

- It is easy to verify that all eigenvectors of $\mathbf{A}$ have geometric multiplicity of one ($n_i^G = 1$). Why?
- That is, if any eigenvalue a_i is repeated ($n_i^A > 1$) it corresponds to the case $n_i^G < n_i^A$ and it must be smaller than one for stability of the LMS scheme a_i according to stability statement (338).
- Accordingly, the stability analysis of LMS scheme is as follows,

$$|a_i| \le 1, \text{ if } a_i \text{ is not repeated } (n_i^A = 1) \text{ otherwise } |a_i| < 1, \text{where} \tag{349a}$$

$$a_i \text{ are eigenvalues of } \mathbf{A},\ i.e., \text{ roots of } c_0 a_i^k + c_1 a_i^{k-1} + \cdots + c_k = 0 \tag{349b}$$

Leap frog $\bar{k} = 1 \rightarrow g_- = g_+ = -i$ for $\theta = \frac{\pi}{2}$

- The form of general recursive relations will be further discussed in §6.4.3.
- For the moment, we observe beside $\hat{v}^n = A(\xi)g^n = A(\xi)(-1)^n$ there is another solution of the form,

$$\hat{v}^n = B(\xi)ng^n, \qquad \text{for } g = -1 \quad \text{(obtained from (473) for } \bar{k} = 1 \text{ and } \theta = \pm\pi/2) \tag{474}$$

- This can be easily verified by plugging this solution in (475) for $\bar{k} = 1, \theta = \pm\pi/2$. That is, for $\alpha_2 = 1$, $\alpha_1 = 2\imath\bar{k}\sin\theta = 2\imath$, $\alpha_0 = -1$:

$$\begin{aligned}\alpha_2\hat{v}^{n+1} + \alpha_1\hat{v}^n + \alpha_0\hat{v}^{n-1} &= B(\xi)(n+1)(-1)^{n+1} + 2\imath B(\xi)n(-1)^n + (-1)B(\xi)(n-1)(-1)^{n-1} \\ &= B(\xi)(-1)^{n-1}\left\{(n+1)(-1)^2 + 2\imath n(-1)^1 - (n-1)\right\} = B(\xi)\imath^{n-1}\{(-n-1) + (2n) + (-n+1)\} = 0\end{aligned} \tag{475}$$

- So a general solution for $\bar{k} = 1$ and $\theta = \pm\pi/2$ takes the form,

$$\hat{v}^n = A(\xi)g^n + B(\xi)ng^n = A(\xi)(-1)^n + B(\xi)n(-1)^n \tag{476}$$

- The appearance of the the factor n in the solution and noting that $|g| = 1$ (so $|g|^n = 1$) mean that leapfrog scheme for $\bar{k} = 1$ is not stable. For $\theta = \pm\pi$ the solution **linearly (not exponentially)** is unstable. This type of instability called weak instability as opposed to strong (exponential) instability that would arise when $|g| > 1$. An example of weak instability is shown in the figure.
- Note that for a fixed time T by letting $k \to 0$ and choosing $t_n = nk \approx T$ we observe $n \approx T/k \to \infty$ for a fixed T and we have n grows in (476) and multiplies $B(\xi)$. The proof of instability for $\bar{k} = 1$ can be formally done through Exercise 4.1.5 in [Strikwerda, 2004].
- In this case, as opposed to one-step schemes considered the limiting value of instability for $\bar{k}$ itself is not included in stability zone.
- Rather, stability of leapfrog method requires $\bar{k} = ak/h < 1$ and the method is **NOT stable** for $\bar{k} = 1$

$$\bar{k} = \frac{ka}{h} < 1 \qquad \text{for the stability of leapfrog method} \tag{477}$$

Leapfrog weak (algebraic) instability for $\bar{k} = 1$.

That is, $\hat{v}^n(\xi) = g^n(h\xi)\hat{v}^0(\xi)$. Plugging this in (488) yields,

$$\alpha_q g^{n+1} + \alpha_{q-1}g^n + \cdots + \alpha_0 g^{n-q+1} = 0 \qquad \Rightarrow \qquad \boxed{\alpha_q g^q + \alpha_{q-1}g^{q-1} + \cdots + \alpha_1 g + \alpha_0 = 0} \tag{489}$$

- For a moment, assume that all q roots of q order polynomial in g are distinct. Since for any root g_j $\hat{v}^n = g_j^n\hat{v}^0$ is a solution any linear combination of these solutions with factors independent on n (*e.g.*, they can depend on ξ for example) can be a solution. So, a solution to (489) can be written as,

$$\hat{v}^n = \sum_{j=1^q} A_j g_j^n \tag{490}$$

where again A_j can depend on any parameters that appear in α coefficients in (488) such as $\bar{k}$, θ that were present in (466) and (483).

- Basically, the q recursive relation (488) will have q unknowns A_j that will be obtained from the first q steps of the solution $\hat{v}^0, \cdots, \hat{v}^{q-1}$.
- However, when some roots of (489) are repeated we do not have all dofs A_j $1 \le j \le q$ present in (490).

- In such cases, there are some other nontrivial solutions to (488).
- Looking back at the two-step problems in §6.4.1 and §6.4.2 when (489) had repeated roots for $\bar{k} = 1$ (*e.g.*, $g_{1,2} = -1$ when $\theta = \pm\pi/2$ for the leapfrog method (473) and $g_{1,2} = 1$ or $g_{1,2} = -1$ $\theta = 0, \pm\pi$ for the central time central space scheme in (486)) we also had a secondary solution of the form $\hat{v}^n = Ang^n$; *cf.* (476) ($\hat{v}^n = A(\xi)g^n + A_*(\xi)ng^n = A(\xi)(-1)^n + A_*(\xi)n(-1)^n$) and (487) ($\hat{v}^n = A(\xi)g^n + A_*(\xi)ng^n = A(\xi)(\pm1)^n + A_*(\xi)n(\pm1)^n$), respectively.
- This suggests, that if a root g to (489) is repeated m times then $\hat{v}^n p(n) g^n$ will be a solution to (488) for an arbitrary polynomial $p(n)$ of order $m-1$.
- This in fact is through and is formalized in the following theorem.

Theorem 3 *Solution to a recursive equation: Consider the q-order homogeneous linear recursive (recurrence) relation,*

$$\alpha_q \hat{v}^{n+1} + \alpha_{q-1}\hat{v}^n + \cdots + \alpha_0 \hat{v}^{n-q+1} = 0, \qquad n = 0, 1, \ldots \tag{491}$$

with $\alpha_q \neq 0$, $\alpha_0 \neq 0$ (if the end point coefficients are zero, the relation can be case in a recursive relation with smaller number of steps) and $\alpha_j \in \mathbb{R}$ and ***not dependent on*** *n (they can depend on any parameter other than n).*

We define the q-order ***characteristic polynomial****,*

$$\rho(g) = \alpha_q g^q + \alpha_{q-1} g^{q-1} + \cdots + \alpha_1 g + \alpha_0 \tag{492}$$