DG2010/01/13

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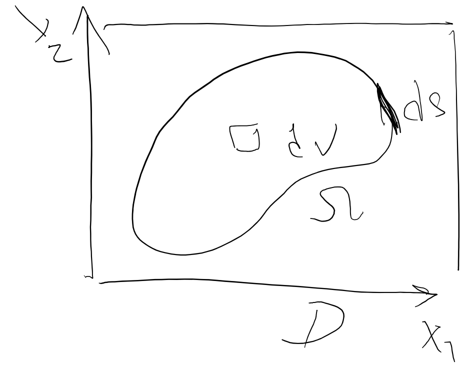
We start by comparing CFEM and DG formulation of thermal heat conduction:



$$\underbrace{\frac{D}{Dt}E}_{\text{energy}} = \int \underbrace{Q}_{\text{heat source term}}\, dV - \underbrace{\int q\cdot n\, ds}_{\text{outward heat flux}}$$

$E = \int e\, dV$ energy density $\qquad e = \underbrace{C_v}_{\text{volumetric heat capacity}} T \qquad$ (other contributions to e are ignored)

$$\frac{D}{Dt}\int_\Omega C T\, dV = \int_\Omega Q\, dV - \int_{\partial\Omega} q\cdot n\, ds \longrightarrow \text{no advection}$$

divergence thm

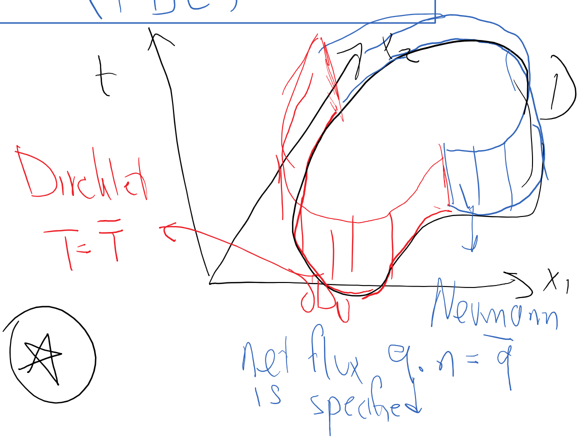
$$\int_\Omega \frac{D}{Dt}(CT)\, dV - \int Q\, dV = \int -\nabla\cdot q\, dV$$

$$\boxed{\begin{array}{l} \forall \Omega \quad \displaystyle\int_\Omega \left[\frac{D}{Dt}(CT) + \nabla\cdot q - Q\right] dV = 0 \\ \text{localization} \rightarrow \quad \displaystyle\frac{D}{Dt}(CT) + \nabla\cdot q - Q = 0 \quad \text{strong form (PDE)} \end{array}}$$

(1)

Now we define all the residuals pertained to PDE and BCs:

$R_i = \frac{D}{Dt}(CT) + \nabla\cdot q - Q$	on D	PDE
$R_u = \bar{T} - T$	on ∂D_u	BCs
$R_f = \bar{q} - q\cdot n$	on ∂D_f	BCs
$T(x,0) = T_0(x)$		IC



Next, we discretize this IBVP with two different methods (discretization is in space only). This is what we call semi-discrete form:

$M\ddot{a} + Ka = F \quad$ discretized in spaced, not discretized in time (semi-discrete form)

$N_i(x) = 0$ on ∂D_u

A) We formulate the problem using continuous FEM:

unknown coefficients

$N_i(x) = 0$ on ∂D_u

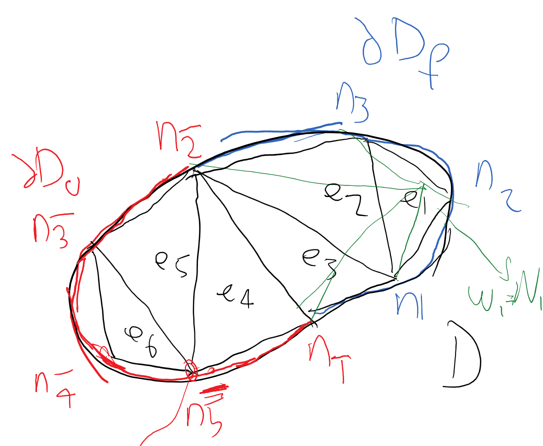
A) We formulate the problem using continuous FEM:

$$T^h(x) = \phi_p(x) + \sum \underbrace{N_i(x)}_{\text{known shape functions}} a_i(t)$$

unknown coefficients that are functions of time

$\phi_p(x)$ satisfies all essential BCs, i.e.

$$\phi_p(x) = \tilde{T}(x) \quad \forall x \in \partial D_u$$



In CFEM, we strongly satisfy Ru and weakly satisfy Ri and Rf:

$$\phi_p(x,t) = \sum_{i=1}^{n_p} N_{\bar{i}}(x) \underbrace{a_{\bar{i}}(t)}_{\text{prescribed Temperature at prescribed nodes}}$$

$n_f = 3$

$n_p = 5$

$a_{\bar{5}}(t)$ is prescribed $\bar{T}$ in time

satisfies essential BC at prescribed nodes & is a linear (for linear elements) approximation of $\bar{T}$ between the nodes

$$\phi_p(n_{\bar{j}}, t) = \sum_{i=1}^{n_p} N_{\bar{i}}(n_{\bar{j}})\, a_{\bar{i}}(t)$$

$$= \sum_{i=1}^{n_p} \delta_{\bar{i}\bar{j}}\, a_{\bar{i}}(t) = a_{\bar{j}}(t)$$

$$T^h = \phi_p(x,t) + \underbrace{\sum_{i=1}^{n_f} a_i(t) N_i(x)}_{\text{free dofs}}$$

$$\textcircled{*} \rightarrow \boxed{T^h = \underbrace{\sum_{i=1}^{n_p} \underbrace{a_{\bar{i}}(t)}_{\text{known}} N_{\bar{i}}(x)}_{\text{prescribed part, } \phi_p} + \underbrace{\sum_{i=1}^{n_f} \underbrace{a_i(t)}_{\text{unknown}} N_i(x)}_{\text{free part}}}$$

R_u is automatically satisfied in $\textcircled{*}$ (on prescribed nodes)

$N_i(x) = 0$ on ∂D_u

WRS: Find $a_i(t)$ $i \in \{1, \dots, n_f\}$ such that for all $w \in \{N_1, \dots, N_f\}$

$$\boxed{\int_D w R_i \, dV + \int_{\partial D_f} w R_f \, ds = 0 \Rightarrow \\ \int_D \underbrace{w}_{\text{0th order}} \left(\frac{D}{Dt}(CT) + \underbrace{\nabla \cdot q}_{\text{since } q = -K\nabla T \text{ (Fourier heat law) 2nd order in T}} - Q \right) dV + \int_{\partial D_f} w(\bar{q} - q \cdot n) \, ds = 0} \quad (2)$$

Use divergence theorem to balance derivatives

$$\nabla \cdot (\underset{\text{scalar}}{w} \underset{\text{vector}}{q}) = \nabla w \cdot q + w \nabla \cdot q$$

$$\int_D w \nabla \cdot q \, dV = \underbrace{\int_D \nabla \cdot (wq) \, dV}_{\text{div. theorem}} - \int_D \nabla w \cdot q \, dV$$

$$'' = \int_{\partial D} w q \cdot n \, dV - \int \nabla w \cdot q \, dV \longrightarrow \text{plug in (2)}$$

$$\int w (C\dot{T} - Q) \, dV + \int_{\partial D_f} w(\bar{q} - q \cdot n) \, ds$$

$$+ \underbrace{\left[- \int \nabla w \, q \, dV + \int_{\partial D} w q \cdot n \, ds \right]}_{\int_D w \nabla \cdot q \, dV} = 0$$

$$\int_D (wC\dot{T} - \nabla w \cdot q - wQ) \, dV + \int_{\partial D_f} w\bar{q} + \left[\int_{\partial D = \partial D_u \cup \partial D_f} w q \cdot n \, ds - \int_{\partial D_f} w q \cdot n \, ds \right] = 0$$

$$'' \qquad '' \qquad '' \qquad + \cancel{\int_{\partial D_u} (w) q \cdot n \, ds} = 0$$

0 because $w = N_i = 0$ on ∂D_u

by noting $q = -K\nabla T$ and eqn above

we get

$$\boxed{\int_D \left(wC\dot{T} + \nabla w \cdot K\nabla T\right) dV = \int_D wQ\, dV - \int_{\partial D_f} w\bar{q}\, ds} \quad (C3)$$

$$\underbrace{}_{M\dot{a}} + \underbrace{}_{Ka} = \underbrace{}_{F \text{ force}}$$

weak statement

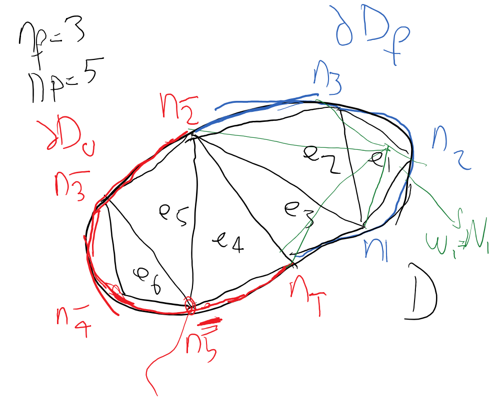
$$(C4)\begin{cases} T = N_f a_f + N_p a_p \\ w = \cancel{N_f} \quad \text{or} \quad \underbrace{N_f^T} \end{cases}$$

because we want to satisfy (C3) for $N_{f_1}, \dots, N_{f_{nf}}$

$$N_f = [N_{f_1}(x), \dots, N_{f_{nf}}(x)] \quad a_f = \begin{bmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_{nf}(t) \end{bmatrix}$$

$$N_p(x) = [N_{p_1}(x), \dots, N_{p_{np}}(x)] \quad a_p = \begin{bmatrix} a_1(t) \\ \vdots \\ a_{np}(t) \end{bmatrix}$$

$n_p = 3$
$n_f = 5$



plug T, w from (C4) in (C3) to get:

$$\nabla T = \nabla\left[N_f(x) a_f(t) + N_p(x) a_p(t)\right] = \underbrace{\nabla N_f(x)}_{B_f} a_f(t) + \underbrace{\nabla N_p(x)}_{B_p} a_p(t)$$

$$\nabla T = B_f a_f + B_p a_p$$

$$w = N_f^T \Rightarrow \nabla w = B_f^T$$

$$\int_D \left(wC\dot{T} + \nabla w \cdot K\nabla T\right) dV = \int_D wQ\, dV - \int_{\partial D_f} w\bar{q}\, ds$$

go to F_D

$$\int_D N_f^T C \left(\overbrace{\underbrace{N_f \dot{a}_f}_{M_{ff}} + N_p \dot{a}_p}^{\dot{T}}\right) + B_f^T \cdot k \left(\underbrace{B_f a_f}_{K_{ff}} + B_p a_p\right) dV = \underbrace{\int_D N_f^T Q\, dV}_{F_r} - \underbrace{\int_{\partial D_f} N_f^T \bar{q}\, ds}_{F_N}$$

note
$$\int_D N_f^T C \widehat{N_f(x) a_f(t)}\, dV = \left[\int N_f^T(x) C N_f(x)\, dV\right] \dot{a}_f(t)$$

$$\boxed{\begin{aligned} &M_{ff}\, \dot{a}_f + K_{ff}\, a_f = \underset{\text{source term}}{F_r} + \underset{\text{Neumann}}{F_N} - \underset{\text{Dirichlet}}{F_D} \\ &M_{ff} = \int (N_f^T C N_f)\, dV \end{aligned}}$$

semi-discrete CFEM equation
time: continuum
space: discrete

space; discrete

$$M_{ff} = \int_{D} (N_f^T C N_f)\, dV$$

$$K_{ff} = \int_{D} B_f^T k B_f \, dV$$

$$F_r = \int N_f^T Q \, dV \qquad F_N = -\int_{\partial D_f} N_f^T \bar{q} \, ds$$

$$F_D = M_{fp} \dot{a}_p + K_{fp} a_p$$

$$M_{fp} = \int N_f^T C N_p \, dV \qquad K_{fp} = \int B_f^T k B_p \, dV$$

C4

Global formulation

Not preferable because it needs K_{fp} & M_{fp}

Local (element-level) calculations circumvent the calculation of global Mfp and Kfp:

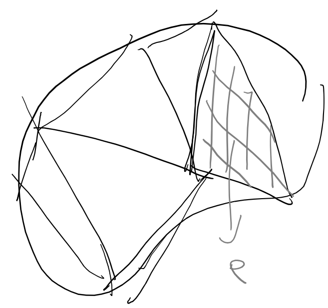
$$M^e = \int_e N^{e^T} C N^e \, de \qquad K^e = \int B^{e^T} k B^e \, dV$$

$$f^e = f_r^e + f_N^e - f_D^e$$

$$f_r^e = \int_e N^{e^T} Q \, dV \qquad f_N^e = -\int N^{e^T} \bar{q} \, ds$$

$$f_D^e = M^e \dot{a}^e + K^e a^e$$

in a^e known prescribed values are used & free ones that are unknown take the value zero in this calculation



Assemble this to global system