DG2020/01/15

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CFEM example

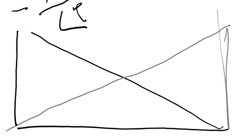
∂D_U $\quad T(t) = \bar{T}(t)$ $\quad e_1 \quad e_2 \quad e_3 \quad \bar{q} = 1$ $\quad \partial D_N$ $\quad \partial D_f$

For all the elements k = 1, C = 1, Le = 1

$L = 3$

Formulas for ke and me:

$$N = [N_1^e, N_2^e] = \left[1 - \frac{x}{L_e}, \frac{x}{L_e}\right]$$

$N_1^e = 1 - \frac{x}{L_e}$  $N_2^e = \frac{x}{L_e}$

$$m^e = \int_e N^{eT} C N^e dx = \int_e \begin{bmatrix} 1 - \frac{x}{L_e} \\ \frac{x}{L_e} \end{bmatrix} C \left[1 - \frac{x}{L_e} \quad \frac{x}{L_e}\right] dx = \frac{CL^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$K^e = \int_e B^{eT} k B^e dx = \int \frac{1}{L_e}\begin{bmatrix} -1 \\ 1 \end{bmatrix}[-1 \quad 1] dx = \frac{k}{L^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(1) $$\boxed{m^e = \frac{C L_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad K^e = \frac{k}{L^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

$M^e = [1 \; 1]$ $\quad M^e = [1 \; 2]$ $\quad M^e = [2 \; 3]$

∂D_U $\quad e_1 \quad 1 \quad e_2 \quad 2 \quad e_3 \quad 3$

$$m^{e_1} = \frac{1}{6}\begin{array}{c} \begin{array}{cc} 1 & 1 \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 1 \end{array} \qquad m^{e_2} = \frac{1}{6}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \end{array} \qquad m^{e_3} = \frac{1}{6}\begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array}\begin{array}{c} 2 \\ 3 \end{array}$$

$$M = \frac{1}{6}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 2+2 & 1 & \\ 1 & 2+2 & 1 \\ & 1 & 2 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

$$K^{e_1} = \begin{array}{c} \begin{array}{cc} 1 & 1 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 1 \end{array} \qquad K^{e_2} = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \end{array} \qquad K^{e_3} = \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} 2 \\ 3 \end{array}$$

$$K = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1+1 & -1 & \\ -1 & 1+1 & -1 \\ & -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

$$\boxed{M = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}}$$ (2)

$$f^e = f_r^e + f_N^e - f_D^e$$

e_1 has $f_D \neq 0$ $\qquad f_D^{e_1} = K^{e_1} a^{e_1} + m^{e} \dot{a}^{e}$ $\qquad a = \begin{bmatrix} \bar{T}(t) \\ 2 \to 0 \end{bmatrix} = \begin{bmatrix} T(t) \\ 0 \end{bmatrix}$

$$T = T_r + T_N - T_D$$

e_1 has $f_D \neq 0$

$$f_D^{e_1} = K^{e_1} a^{e_1} + m^{e} \dot{a}^{e}$$

$$a = \begin{bmatrix} T(t) \\ 2 \to 0 \end{bmatrix} = \begin{bmatrix} T(t) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \bar{T}(t) \\ 0 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \dot{\bar{T}}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{T} + \frac{1}{3}\dot{\bar{T}} \\ -\bar{T} + \frac{1}{6}\dot{\bar{T}} \end{bmatrix} \begin{matrix} 1 \\ 1 \end{matrix}$$

$$\boxed{f^{e_1} = \begin{bmatrix} -\bar{T} - \frac{1}{3}\dot{\bar{T}} \\ \bar{T} - \frac{1}{6}\dot{\bar{T}} \end{bmatrix} \begin{matrix} 7 \\ 1 \end{matrix}}$$

$$f^{e_2} = \underline{0} \qquad f^{e_3} = f_N^{e_3} = \int N^{e^T} \bar{q} \, ds$$

$$= \begin{bmatrix} N_1^{e_3} \\ N_2^{e_3} \end{bmatrix}_{x=3} \bar{q} = \underset{\partial e_N}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

(n_3)

$$\boxed{f^{e_3} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}}$$

$$\boxed{F = \begin{bmatrix} \bar{T} - \frac{1}{6}\dot{\bar{T}} \\ 0 \\ 1 \end{bmatrix}} \quad (3)$$

K & M in eqn 2

need to solve $\quad M\dot{a} + Ka = F$

Discontinuous Galerkin formulation for heat conduction (one of the several DG formulations for this problem).

a) $R_i = C\dot{T} + \nabla \cdot q - Q$ inside

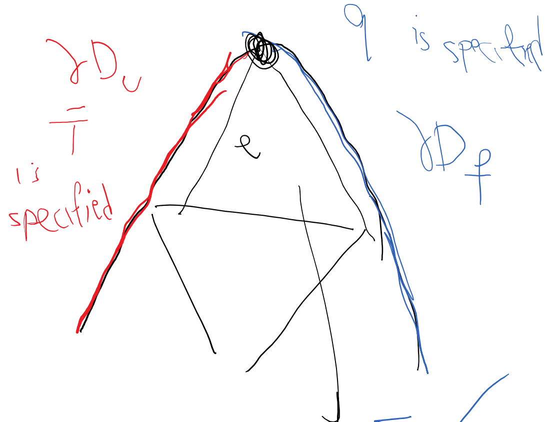
b) $R_f = \bar{q} - q \cdot n$ on ∂D_f

c) $\underbrace{R_u = \bar{T} - T}$ on ∂D_u

CFEM this is enforced strongly

CFEM

CFEM

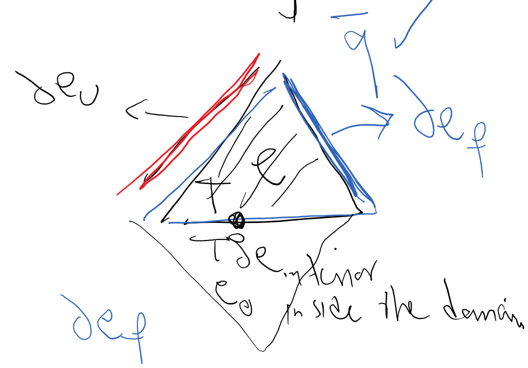


Generalize b to a DG formulation:
DG formulations are element centered. So, we write the residuals for **elements**.

DG

a) $R_i = C\dot{T} + \nabla \cdot q - Q$ in e

b) $R_f = q_n^* - q \cdot n$ on ∂e



net flux with spatial normal n $\leftarrow q_n^* = \begin{cases} \bar{q} & \partial e_f \\ q \cdot n & \partial e_u \\ q_n^*(q, q_0, T, T_0, \vec{n}, \ldots) & \end{cases}$

$\longleftrightarrow$ $q_n^* - q \cdot n = 0$ identically on ∂e_u

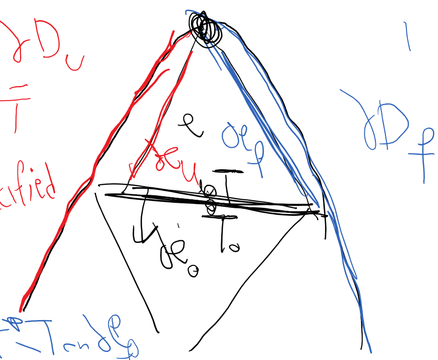
normal n $\quad (q_n^*(q, q_0, T, T_0, \vec{n}, \dots))$

∂e_u

↳ One choice $q_n^* = (q + q_0)\cdot n$
average flux

what we really want: keep my q completely free on ∂D_u

c) $R_U = \cancel{\bar{T}} - T \rightarrow R_U = \bar{T}^* - T$



$$\bar{T}^* = \begin{cases} T & \partial e_f \rightarrow \text{like previous case } T^* - T \text{ on } \partial e_f \text{ is identically zero, leaving } T \text{ free} \\ \bar{T} & \partial e_U \\ T^*(T, T_0, q, q_0, n) & \text{on } \partial e_I \end{cases}$$

an example average flux $T^* = \frac{T + T_0}{2} \rightarrow T^* - T = \frac{T_0 - T}{2}$

Now that we have all the residuals we can formulate the DG weak form:

Find the solution in all elements e, subject to the following weak statement:

$$\forall w \in W \quad \int_e w R_i + \int_{\partial e} w \underbrace{(q_n^* - q \cdot n)}_{R_f} ds$$

$\partial e \rightarrow$ Unlike CFEM, this is done over the whole boundary of e (∂e_f CFEM)

$$+ \int_{\partial e} f(w) \underbrace{(T^* - T)}_{R_U} ds = 0$$

in CFEMs we don't have this term altogether because by using nodal dofs & ϕ_p concept T is continuous between elements & $T = \bar{T}$ on ∂D_U (at nodes)

$f(w)$ is the value or a derivative of w.

$q = -k\nabla T$

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$q = -k\nabla T$

Some comments on the choices for f(w):

1. f(w) = w

$w = T$ weight of temperature

add this factor in front of integral for dimensional consistency

$$\frac{k}{h}\int_{\partial e} w\cdot\underbrace{(T^* - T)}_{[T]^2}\,ds + \int_{\partial e} w(q_n^* - q\cdot n)\,ds \quad [T][q]$$

factor $\frac{k}{h}$ ⟶ conductivity

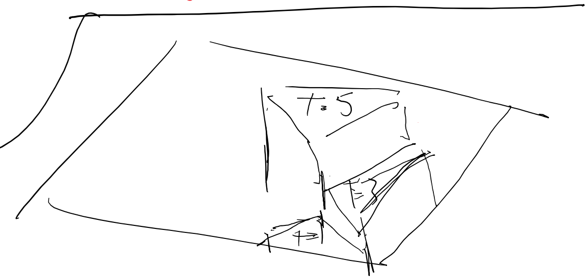
2. Make the formulation "energy symmetric"

$$\int_{\partial e} \hat{T}(q_n^* - q\cdot n)\,ds \qquad w \leftarrow \hat{T}$$

$$\int_{\partial e} q\cdot n\,(T^* - T)\,ds = \int_{\partial e} (-k\nabla w)\cdot n\,(T^* - T)\,ds$$

Potential problem
If we interpolate the element with 0th order polynomial (that is the solution is a constant in each element)

$T_{e_i} = a_i \cdot 1 \qquad w_{e_i} = 1$



$\nabla w = 0$ cannot enforce continuity of T.

Not a good formulation.

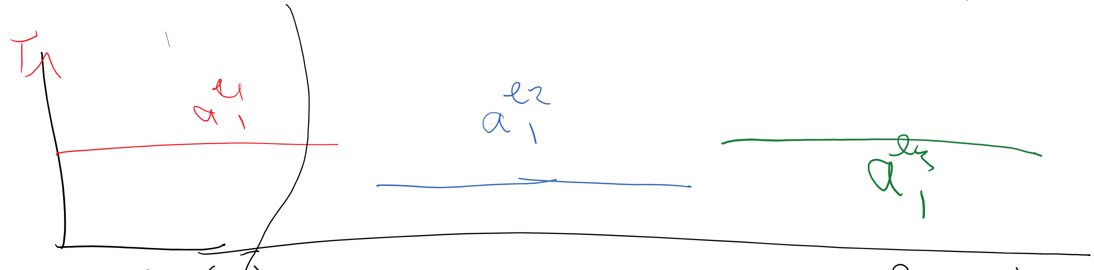
We need at least $p \geq 1$ for this option of $f(w)$

Example: DG matrices for the same 3 element problem

Constant sol'n per element

$$T^{e_1} = a_1^{e_1}\,\mathbb{1} \qquad T^{e_2} = a_1^{e_2}\,\mathbb{1} \qquad T^{e_3} = a_1^{e_3}\,\mathbb{1}$$

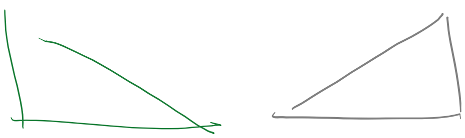
Sample sol'n



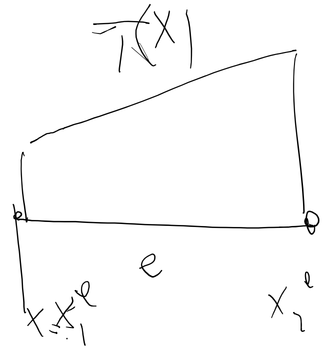
$$\omega^{e_1} = \{1\} \qquad \omega^{e_2} = \{1\} \qquad \omega^{e_3} = \{1\}$$

Since the CFEM solution was order 1 (using linear shape functions), we want to solve this problem with linear DG elements

$$T(x) = a_1^e N^1(x) + a_2^e N^2(x)$$



$$N_1^e = \frac{x_2^e - x}{L} \qquad N_2 = \frac{x - x_1^e}{L}$$



N^1 & N^2 are 1 option for basis for this p=1 element.

Instead, I'll use

$$T(x) = a_1^e \phi_1(x) + a_2^e \phi_2(x) = a_1^e \cdot \mathbb{1} + a_2^e \cdot x$$

$$\vec{\phi}: \begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix} = \begin{bmatrix}1 \\ x\end{bmatrix}$$