DG2020/01/22

Wednesday, January 22, 2020 11:35 AM

sample element calculations

$$T = a_1 + a_2 x = Na = [1 \;\; x]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$T_L = T(x=0) = [1 \;\; 0]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = a_1 \qquad n_L = -1$$

$$T_R = [1 \;\; L]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = a_1 + L a_2 \qquad n_R = 1$$

$x = 0$ $\qquad$ $x = L$

$$B = \underbrace{\nabla N}_{\frac{dN}{dx}} = [0 \;\; 1]$$

$$q_L = -k a_2 \qquad q = -kB = -k[0 \;\; 1]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -k a_2 \qquad q_R = -k a_2$$

$$N = [N_1 \; N_2] \quad a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad w = (Na)^T = [a_1' \; a_2']\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$$

$$\int_e \Big(wC\dot{T} - \underbrace{\nabla w\, q}_{\substack{+\nabla w\, k \nabla T \\ B^T k B a}} - wQ\Big)\, dV + \underline{\int_{\partial e} w q^*} + \int_{\partial e} -k\nabla w \cdot n (T^* - T)\, ds = 0$$

$$\to \underbrace{\int_e N^T C N\, dv}_{m^e}\, \dot{a} + \underbrace{\left[\int_e (B^T k B)\, dv\right]}_{K^e} a - \underbrace{\int_e N^t Q\, dv}_{F_r} + (N^T_{x_L} q^*_L) + (N^t_{x_R} q^*_R)$$

$$+ \Big[\underbrace{-k B^T n}_{-k\nabla w \cdot n} (T^* - T) \Big]_{x_L} + \Big[-k B^T n (T^* - T) \Big]_{x_R} = 0$$

For the given choice of shape functions, we insert traces of T and q from left and right sides:

$$m^e \dot{a} + K^e a - F_r + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{N^T_L} q^*_L + \underbrace{\begin{bmatrix} 1 \\ L \end{bmatrix}}_{N^T_R} q^*_R + \left(-k\begin{bmatrix} 0 \\ 1 \end{bmatrix} \underbrace{(-1)}_{n_L}\right)\left(T^*_L - [1 \;\; 0]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right)$$

$$+ \left(-k\begin{bmatrix} 0 \\ 1 \end{bmatrix} \underbrace{(1)}_{n_R}\right)\left(T^*_R - [1 \;\; L]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = 0$$

$$m^e \dot{a} + K^e a - F_r + \begin{bmatrix} q^*_L + q^*_R \\ L q^*_R \end{bmatrix} + \left(+k\begin{bmatrix} 0 \\ 1 \end{bmatrix} T^*_L - k\begin{bmatrix} 0 \\ 1 \end{bmatrix} T^*_R\right)$$

$$+\cancel{k}\begin{bmatrix}0 & 0\\ 1 & 2\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix}-k\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\boxed{m^e\dot{a}+\tilde{K}a^e-F_r+\begin{bmatrix}q_L^*+q_R^*\\ Lq_R^*+K(T_L^*-T_R^*)\end{bmatrix}+\cancel{k}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}} \quad (1)$$

$$\tilde{K}=\int_e B^T \cancel{k} B\,dV,\quad N=[1\ \ x],\quad B=\frac{dN}{dx}=[0\ \ 1]$$

$$=\int_0^L\begin{bmatrix}0\\ 1\end{bmatrix}\cancel{k}[0\ \ 1]\,dx=\cancel{k}L\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

$$m^e=\int N^T c N\,dx=\int_0^L\begin{bmatrix}1\\ x\end{bmatrix}c[1\ \ x]\,dx=cL\begin{bmatrix}1 & L/2\\ L/2 & L^2/3\end{bmatrix}$$

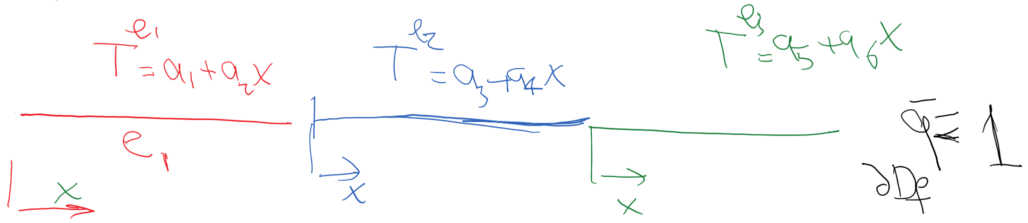
Plugging in 1 we get

$$\boxed{\underbrace{cL\begin{bmatrix}1 & L/2\\ L/2 & L^2/3\end{bmatrix}}_{m_a^e}\dot{a}+2\cancel{k}L\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}a+\begin{bmatrix}q_L^*+q_R^*\\ Lq_R^*+\cancel{k}(T_L^*-T_R^*)\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}} \quad (2)$$

Equation for 1 element.

$T=\bar{T}$

γD_u



$L_{e_i}=1$ because Global $L=3$

Using (2) the system of eqns is

$$\boxed{\begin{aligned}
&m^{e_1}\dot{a}_{e_1}+\begin{bmatrix}0 & 0\\ 0 & 2\end{bmatrix}\underbrace{\begin{bmatrix}a_1\\ a_2\end{bmatrix}}_{a_{e_1}}+\begin{bmatrix}q^{*L}_{e_1}+q^{*R}_{e_1}\\ q^{*R}_{e_1}+T^{*L}_{e_1}-T^{*R}_{e_1}\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\\
&m^{e_2}\dot{a}_{e_2}+\begin{bmatrix}0 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}a_3\\ a_4\end{bmatrix}+\begin{bmatrix}q^{*L}_{e_2}+q^{*R}_{e_2}\\ q^{*R}_{e_2}+T^{*L}_{e_2}-T^{*R}_{e_2}\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\\
&m^{e_3}\dot{a}_{e_3}+\begin{bmatrix}0 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}a_5\\ a_6\end{bmatrix}+\begin{bmatrix}q^{*L}_{e_3}+q^{*R}_{e_3}\\ q^{*R}_{e_3}+T^{*L}_{e_3}-T^{*R}_{e_3}\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}
\end{aligned}} \quad (3)$$

$\cancel{k}=1$

$C=1$

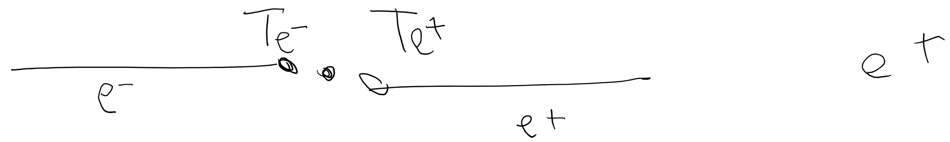
$F_r=0\ (Q=0)$

Next step is finding star values for T and q.

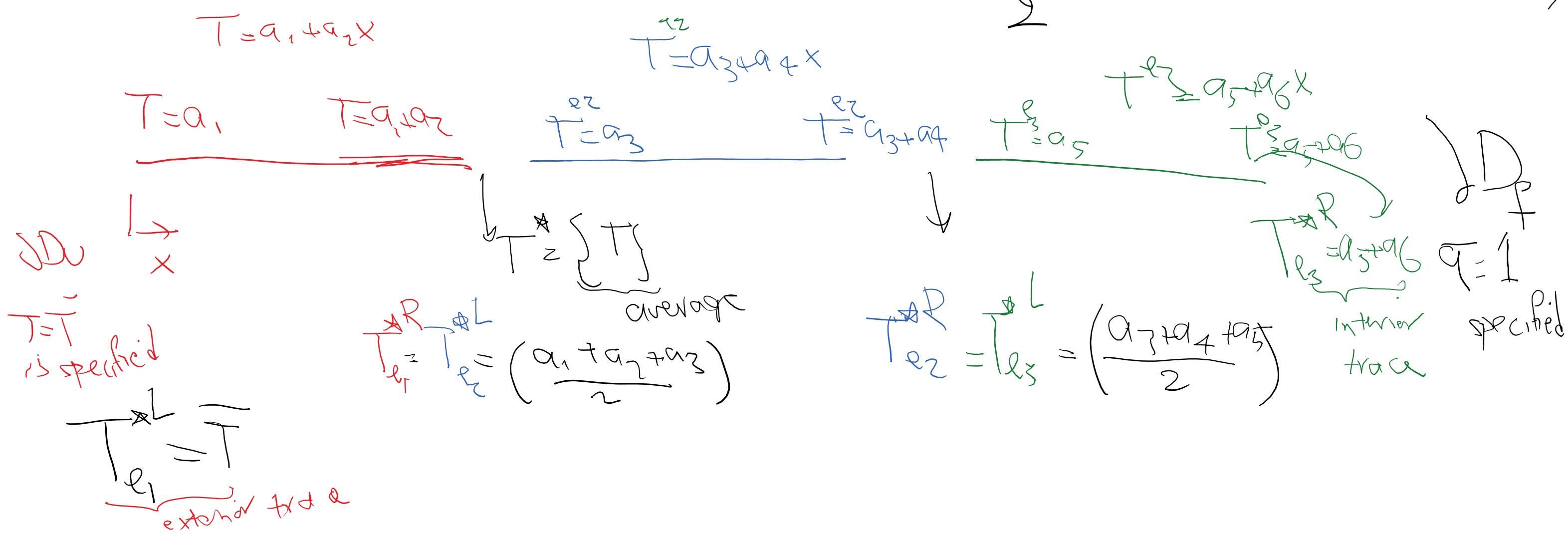
There are many options for star values.

For the moment, we are going to use **average** star values (Need to be careful with this, specially only when T is interpolated).

Star values for T:

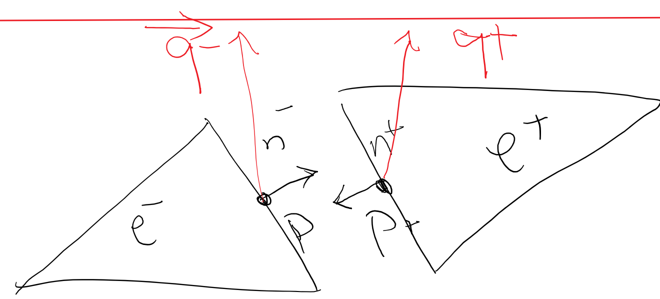


$$T^* = \frac{T_{e^-} + T_{e^+}}{2} \qquad \left(T^* - T = \frac{T_{e^-} - T_{e^+}}{2} \right)$$



q_n^* values

$$\underline{q^- n^- = q^+ n^-}$$



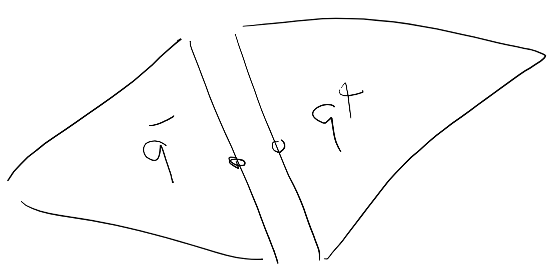
$$\int_{\partial\Omega} \underline{q \cdot n \, ds} \ldots = 0 \rightarrow \text{Jump condition} \qquad \underline{[q] \cdot n = 0}$$

$$(q^+ - q^-) \cdot n^- = 0$$

$$q^+ n^- = q^- n^-$$

$$-q^+ n^+ = q^- n^-$$

$$\leftarrow \boxed{q_{n^+} = -q_{n^-}}$$



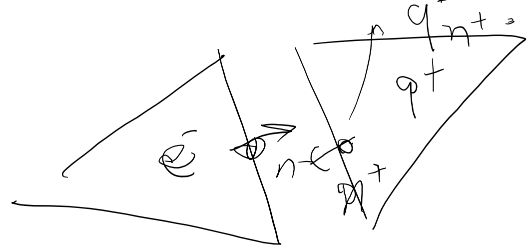
$$\boxed{q^* = \frac{q^- + q^+}{2}} \quad \text{average flux} \checkmark$$

$$\boxed{q^* = \frac{q^- + q^+}{2}}$$ average flux ✓

$$q_n^* = \cancel{\frac{q_{n^-} + q_{n^+}}{2}} = 0$$

→ Use this to compute net fluxes for each side

$q^*_{n^+} = q^* \cdot n^+$



$q^*_{n^-} = q^* \cdot n^-$

∂D_u

$T = \bar{T}$

$T^{e_1} = a_1 + a_2 x$ $\quad$ $T^{e_2} = a_3 + a_4 x$ $\quad$ $T^{e_3} = a_5 + a_6 x$

$n = -1$ $\quad$ $q = -k a_2 = -a_2$ $\quad$ $n = 1$ $\quad$ $q = -a_4$ $\quad$ n $\quad$ $q = -a_6$

$\vec{q}$ $\quad$ $-a_2$ $\quad$ $-a_2$ $\quad$ $n = -1$ $\quad$ $-a_4$ $\quad$ $-a_4$ $\quad$ $n = -1$ $\quad$ $-a_6$ $\quad$ $-a_6$ $\quad$ ∂D_f $\quad$ $\bar{q} = 1$

$$\{q\} = \frac{q^- + q^+}{2} \qquad \left(\frac{-a_2 - a_4}{2}\right) \qquad \left(\frac{-a_4 - a_6}{2}\right)$$

q^*_n net star

$q^{*e_1}_{n_L} = (q \cdot n) = (-a_2)\cdot(-1) = a_2$

$q^{*e_1}_{n_R} = \{q\} \cdot n = -\left(\frac{a_2 + a_4}{2}\right)$

$q^{*e_2}_{n_L} = \{q\} \cdot n = \left(\frac{a_2 + a_4}{2}\right)$

$\left(q^{*e_2}_{n}\right)_R = \{q\} \cdot n = \left(\frac{-a_4 - a_6}{2}\right)$

$\left(q^{*e_3}_{n}\right)_L = \{q\} \cdot n = \left(\frac{a_4 + a_6}{2}\right)$

$\left(q^{*e_3}_{n}\right)_R = \bar{q} = 1$

Plug these in equation (3) to get:

$$M_{6\times 6}\,\dot{a} + K a = F$$

weights sln

$$K = \begin{array}{c|cc|cc|cc}
 & e_1 & & e_2 & & e_3 & \\
\hline
e_1 & 0 & \tfrac{1}{2} & 0 & -\tfrac{1}{2} & 0 & 0 \\
 & -\tfrac{1}{2} & 1 & -\tfrac{1}{2} & -\tfrac{1}{2} & 0 & 0 \\
\hline
e_2 & 0 & \tfrac{1}{2} & 0 & 0 & 0 & -\tfrac{1}{2} \\
 & \tfrac{1}{2} & \tfrac{1}{2} & 0 & 1 & -\tfrac{1}{2} & -\tfrac{1}{2} \\
\hline
e_3 & 0 & 0 & 0 & \tfrac{1}{2} & 0 & \tfrac{1}{2} \\
 & 0 & 0 & \tfrac{1}{2} & \tfrac{1}{2} & -\tfrac{1}{2} & 1
\end{array}
\qquad
F = \begin{bmatrix} 0 \\ -1 \\ \hline 0 \\ 0 \\ \hline -1 \\ -1 \end{bmatrix}$$