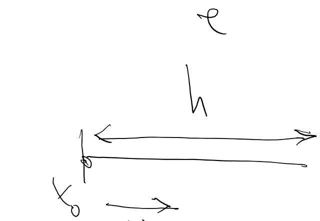
# DG2020/01/29

Wednesday, January 29, 2020 11:41 AM

Recall

$$m^e = \int_e N^{e^T} C N^e \, dv$$



$$\xi = \frac{x - x_0}{h}$$

$$N = [1 \quad \xi]$$

$$m^e = \int_0^1 \begin{bmatrix} 1 \\ \xi \end{bmatrix} \underset{\text{const}}{c_b} [1 \quad \xi] \, (\underbrace{h \, d\xi}_{dv = dx})$$

$$\longrightarrow \boxed{m^e = Ch \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}}$$ we derived this before

Use this to compute Mass matrix for DG set-up below

$T^{e_1} = a_1 + a_2 x^{(1)}$    $T^{e_2} = a_3 + a_4 x^{(2)}$    $T^{e_3} = a_5 + a_6 x^{(3)}$

$e_1$    $e_2$

$x^{(1)} = \xi^{(1)}$    $x^{(2)} = \xi^{(2)}$    $x^{(3)} = \xi^{(3)}$

$$C^{e_i} = 1 \quad h^{e_i} = 1 \qquad m^{e_i} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix} \longrightarrow$$

$$M_{6\times 6} = \begin{bmatrix} 1 & 1/2 & & & & \\ 1/2 & 1/3 & & & & \\ & & 1 & 1/2 & & \\ & & 1/2 & 1/3 & & \\ & & & & 1 & 1/2 \\ & & & & 1/2 & 1/3 \end{bmatrix}$$

$$M\dot{a} + Ka = F$$

with $a = a_0$ for parabolic eqn as IC

$K, F$ given from last time

semi-discrete form for DG 3 element problem

($x \rightarrow$ discretized)

($t \rightarrow$ is not)

Some comments:

If elliptic PDE for steady-state were to be solved, we would have had Ka = F

There are many ways to discretize this problem in time. We just consider simple backward and forward Euler discretizations.

A) Backward Euler method which is an implicit time marching scheme.

In backward Euler we write the equation for the next time step: $M\dot{a}_{n+1} + K a_{n+1} = F_{n+1}$

Use $\dot{a}_{n+1} = \frac{a_{n+1} - a_n}{\Delta t}$

we obtain $M(a_{n+1} - a_n) + \Delta t K a_{n+1} = \Delta t F_{n+1}$ $\quad a_n$ is known, $a_{n+1}$ is unknown

$$\boxed{\begin{aligned} \underbrace{(M + \Delta t K)}_{\mathcal{M}_{n+1}} a_{n+1} &= \underbrace{\Delta t F_{n+1} + M a_n}_{\mathcal{F}_{n+1}} \\ \mathcal{M}_{n+1} a_{n+1} &= \mathcal{F}_{n+1} \end{aligned}}$$

(1a) Implicit-Euler time integration

B) Explicit forward Euler method: Write the equation for previous time

$$\left.\begin{aligned} &M\dot{a}_n + K a_n = F_n \\ \text{use} \quad &\dot{a}_n = \frac{a_{n+1} - a_n}{\Delta t} \end{aligned}\right\} \rightarrow \boxed{\begin{aligned} \underbrace{M}_{\mathcal{M}_{n+1}} a_{n+1} &= \underbrace{(M - \Delta t K) a_n + \Delta t F_n}_{\mathcal{F}_{n+1}} \\ \mathcal{M}_{n+1} a_{n+1} &= \mathcal{F}_{n+1} \end{aligned}}$$

$M(a_{n+1} - a_n) + \Delta t K a_n = \Delta t F_n \rightarrow$

(1b) Explicit Euler

Main differences:

1. Stiffness contributions are on the LHS for implicit method:
   a. As we will see this greatly messes up with the structure of effective M (system matrix) and make it much worse for implicit schemes even if the PDE is linear
   b. If the problem was nonlinear (MaDot + fNL(a) = F), in this case K(a) = d fNL / da is a non constant matrix. The problem remains nonlinear with implicit methods but becomes linear with the explicit one.

$M\dot{a} + f_{NL}(a) = F$

Backward Euler

$$\underbrace{\Big[M + \Delta t \underbrace{\nabla_a f_{NL}}_{K}\Big]}_{\mathcal{M}(a_{n+1})} a_{n+1} = \Delta t F_{n+1} + M a_n$$

$\mathcal{M}(a_{n+1})\, a_{n+1} \; \ldots$

Forward Euler

$M\left(\frac{a_{n+1} - a_n}{\Delta t}\right) + f_{NL}(a_n) = F_n$

For point a. We will see that DG methods have a very nice M, CFEMs can get "nice" M's but mass-lumping. In both schemes we can use this nice M to have a cheap solution for

$$\mathcal{M}_{n+1} a_{n+1} = \mathcal{F}_{n+1}$$

2. $\Delta t$ Does not appear on the LHS for explicit method. This may have some minor advantages in practice…

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$$M\left(\frac{a_{n+1}-a_n}{\Delta t}\right)+f_{NL}(a_n)=F_n$$

$$Ma_{n+1}=\underline{Ma_n+\Delta t\left(F_n-f_{NL}(a_{n})\right)}$$

$f_{NL}$ goes to RHS

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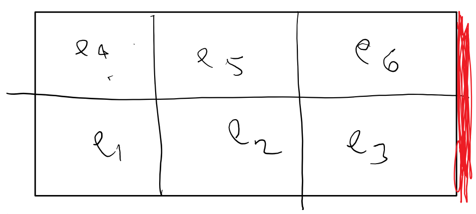
Why DG methods have an inherent advantage for an explicit solution scheme.

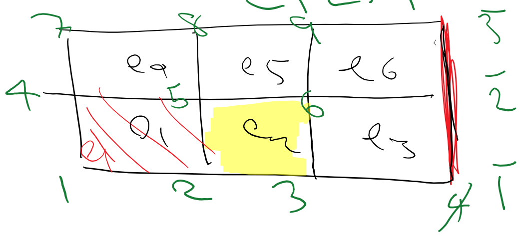
Heat conduction in 2D

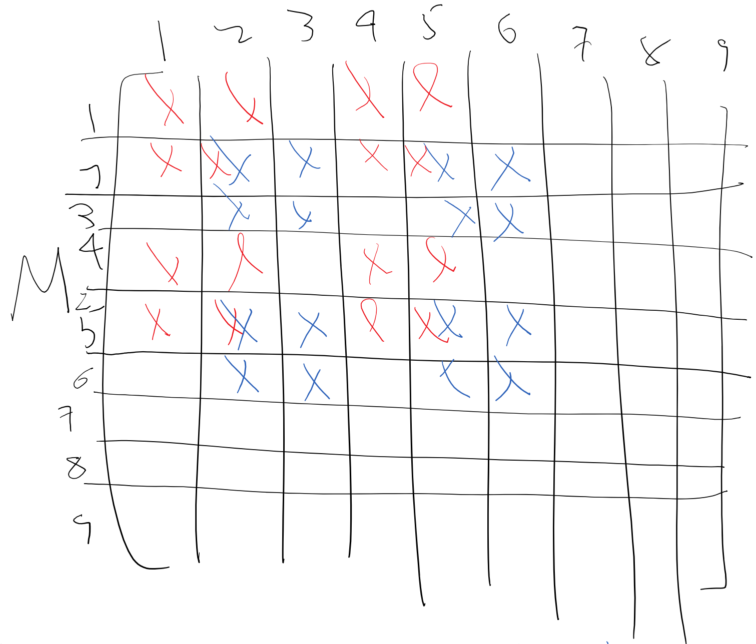



Essential BC

rest of the boundary is natural

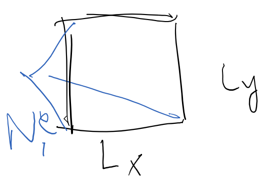
CFEM




$$M_{9\times9}=\int_D N^T_{9\times1}\,c\,N\,dV$$

$$m^e_{4\times4}=\int N^{eT}c\,N^e\,dV$$




$$M^e=\frac{cL_xL_y}{36}\begin{bmatrix}4&2&1&2\\2&4&2&1\\1&2&4&2\\2&1&2&4\end{bmatrix}$$

need to assemble these to global M

$$m^{e_2}\quad\begin{array}{c|c} & 2\ 3\ 5\ 6\\ \hline \begin{matrix}2\\3\\5\\6\end{matrix} & \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}\end{array}$$

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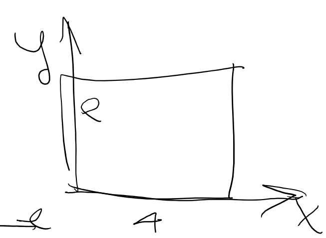
DG version for the same problem




$$T^e=\sum_{i=1}^{4}a_i\phi_i$$

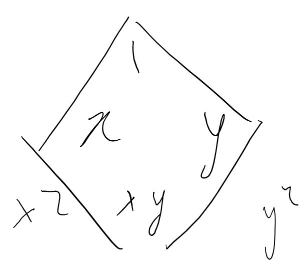
basis is formed by

$\phi_1=1$

$\phi_2=x$

$\phi_3=y$

$\phi_4=xy$




basis

$$N^e=\begin{bmatrix}1 & x & y & xy\end{bmatrix}$$

$r$ 1 $L_x$ $L_y/2$ $L_xL_y$,

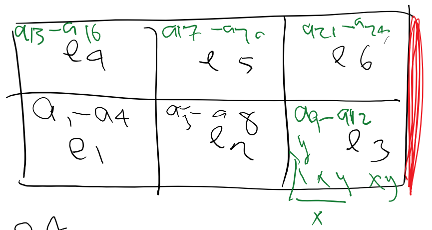
$\phi_3 = y$
$\phi_4 = xy$

$N = [1 \;\; x \;\; y \;\; xy]$

$$M^e = \int_e N^{e^T} c N^e \, dV$$

$$M^e = cL_xL_y \begin{bmatrix} 1 & L_x/2 & L_y/2 & \frac{L_xL_y}{2} \\ & L_x^2/3 & \frac{L_xL_y}{2} & \frac{L_x^2L_y}{6} \\ \text{sym} & & L_y^2/3 & \frac{L_xL_y^2}{6} \\ & & & \frac{L_x^2L_y^2}{4} \end{bmatrix}$$

need
To assemble
these to global system




Essential BC

$24 \times 24$
"Mass" matrix

$$M = \begin{array}{c|cccccc} \text{Blocks} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & M^e_{4\times4} & & & & & \\ 2 & & M^e & & & & \\ 3 & & & M^e & & & \\ 4 & & & & M^e & & \\ 5 & & & & & M^e & \\ 6 & & & & & & M^e \end{array}$$

Block-diagonal

$$M^e = \begin{bmatrix} M^{e_1^{-1}} & & & & & \\ & M^{e_2^{-1}} & & & & \\ & & M^{e_3^{-1}} & & & \\ & & & M^{e_4^{-1}} & & \\ & & & & M^{e_5^{-1}} & \\ & & & & & M^{e_6^{-1}} \end{bmatrix}$$

$\longrightarrow$ $\boxed{\text{DG} \quad M^e a^e = F^e \longrightarrow \text{Contribution from } K}$

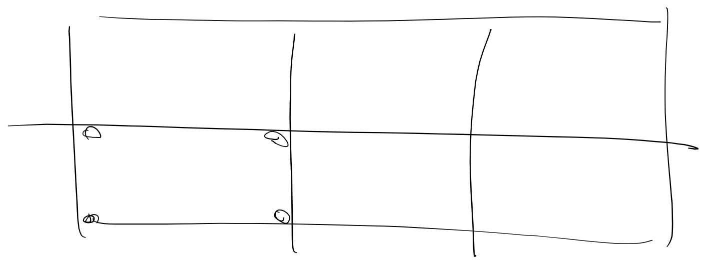
Explicit time marching

We can solve each element's unknown at the element level if DG + explicit solution scheme is used because the mass matrix is BLOCK DIAGONAL for DG methods.

Discussion points:

- If an explicit method is used only M appears to be on the LSF for its form determines the solution complexity.
  - DG methods have block diagonal mass matrix -> one element at a time solution scheme. This can make DG much more efficient even though it has more DOFs.
  - For CFEMs we have a sparse but not a block diagonal mass matrix. So we actually need to solve a big system of equation even for explicit methods

A remedy is mass lumping for CFEM methods which results in a diagonal mass matrix



$M^{ei} \longrightarrow$ diagonal $\begin{bmatrix} m_{11} & & & \\ & m_{22} & & \\ & & m_{33} & \\ & & & m_{44} \end{bmatrix}$

eg $M^e \approx \frac{C L_x L_y}{4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$ → assembly of this is a global diagonal mass matrix

So if we use "mass lumping" even for CFEMs Global M is very easy to solve (diagonal)

From the discussion above, it is clear why the majority of DG methods for hyperbolic and parabolic PDEs use explicit solvers so that they can take advantage of their block diagonal LHS matrix.

CFEM:
It can be argued that for CFEMs we get the same effect by mass lumping.
This is true and in fact almost always mass lumping is done with explicit solvers for CFEMs otherwise there is no advantage in using an explicit method (big sparse matrix like implicit solvers but without implicit solver advantages).

Especially, we not that explicit integrators tend to shorten the frequency, and mass lumping has the opposite effect. In fact, this is a "match made in heaven":) because not only mass lumping results in diagonal M in CFEM but also do this counter-acting frequency correcting effect. This improves dispersion errors.