DC2020/02/03

Wednesday, January 29, 2020 11:41 AM

Continue comparison of exp. Vs. imp. Solution of the heat problem.

Another advantage of explicit methods is not having to assemble the global stiffness matrix K or any other matrix that appears on the RHS (often it's only K)

$$C\dot{A} + KA = F$$

$$\tilde{C} A_{n+1} = \tilde{F}_{n+1}$$

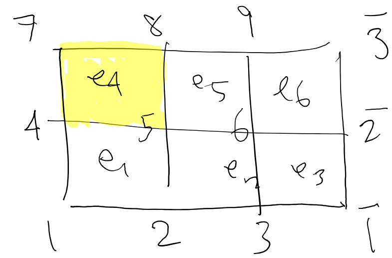
explicit solver

Solution from prev. time step

$$\begin{cases} \tilde{F}_{n+1} = \Delta t(\underline{F_n} - \underbrace{KA_n}_{\text{vector}}) + CA_n \\ \tilde{C} = C \end{cases}$$

We want to avoid computing K An because:

1. No need for memory for K (or other similar matrices on the RHS)
2. Multiplication of Kan can be expensive (if not being careful)



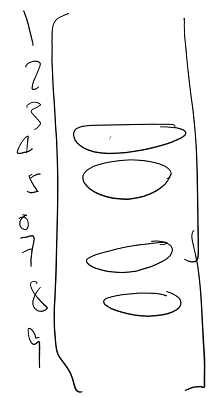
$-KA_n \longrightarrow$ element level: we need to calculate

$-k^e a^e_n$

$M^{e4} = [4 \ 5 \ 7 \ 8]$

e_4 $\quad -k^{e4}_{4\times4} \begin{bmatrix} (a_1^{e4})_n \\ (a_2^{e4})_n \\ (a_3^{e4})_n \\ (a_4^{e4})_n \end{bmatrix} = -k^{e4}_{4\times4} \begin{bmatrix} (a_4)_n \\ (a_5)_n \\ (a_7)_n \\ (a_8)_n \end{bmatrix}$

this is added to $f^{e4} = f^{e4}_n - k^{e4} a^{e4}$ $\longrightarrow$ assembled to the global system

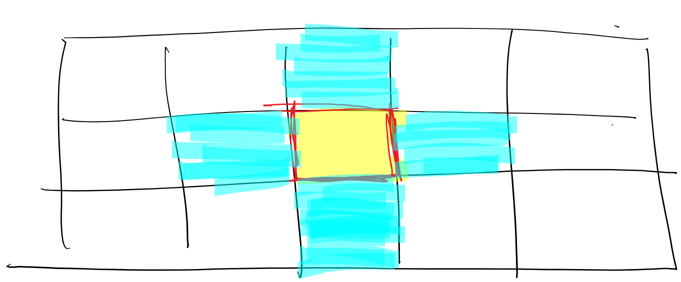


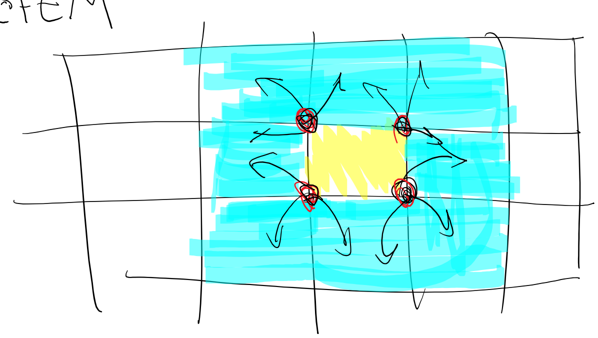
In fact, even if we had nasty nonlinear response (nonlinear elasticity, etc.) since we don't need to compute stiffness matrix, formulation and implementation of a time marching explicit method becomes quite simple.

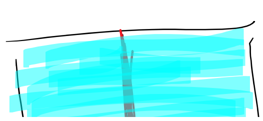
Back to comparison of DG and CFEM:

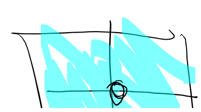
Connectivity Stencil:

CFEM

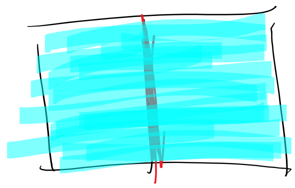


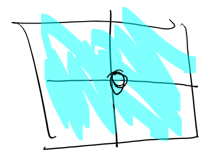






CFEM





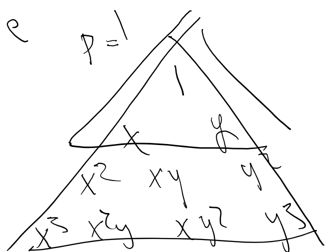
- Smaller connectivity stencil is ideal for parallel computing (because for example in domain decomposition approaches fewer PUP routines are called). PUP: pack and unpack
- Finite element methods are in general better than Finite Difference (FD) methods because no matter how high the order of accuracy is, unlike FD methods the stencil does not telescopically grow and remains within one neighbor element.
- DG methods have an advantage in this respect because their connectivity is through the edges not the nodes -> fewer between the element communications.
- In this respect, tri/tet elements gain more than square and cube elements

Note, IN HW2 if you problems with tetrahderal element average connective and dof/element you can skip it.

Average DOF ver element:

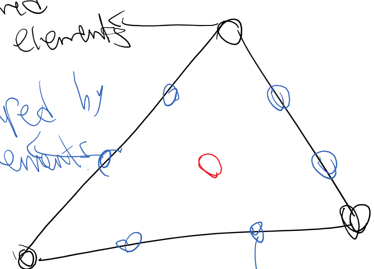
tri $p=3$ elements

Pascal triangle



10 dof for element

CFEM

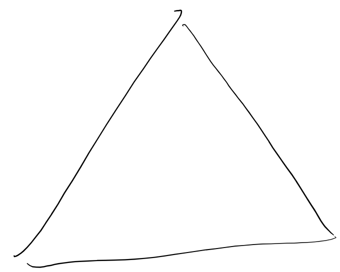


Average dof / element

$$= 3 \times \frac{1}{6} + 6 \times \frac{1}{2} + 1$$

$$= 4.5$$

DG



$10 \ \frac{\text{dof}}{\text{element}}$

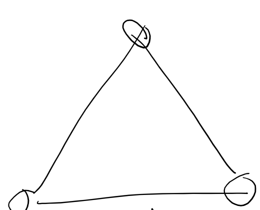
(e.g. can use monomials from Pascal triangle)

10

$p=3$

$$\text{Ratio} = \frac{10}{4.5} \simeq 2.2$$

$P=1$



$$3 \times \frac{1}{6} = .5$$

DG

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Ratio $p=1$ (

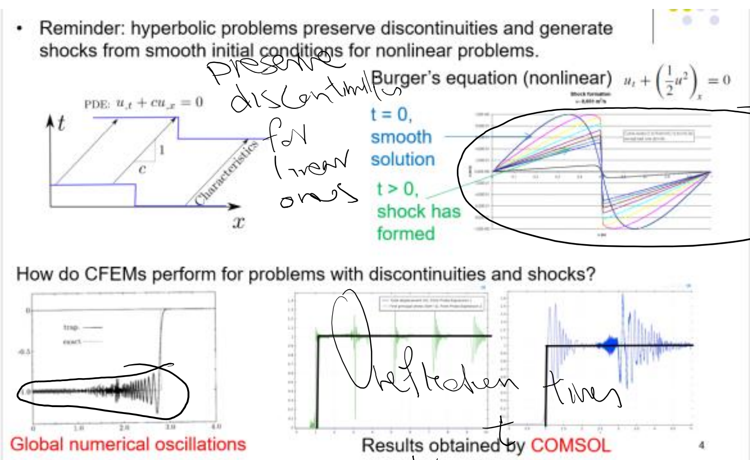
As the polynomial order increases the ratio of dofs of DF/CFEM decreases. So, DG becomes better in terms of number of unknowns as the polynomial

Ratio p=1

- As the polynomial order increases, the ratio of dofs of DF/CFEM decreases. So, DG becomes better in terms of number of unknowns as the polynomial order increases.
- Obviously, in both methods we can condense out the interior dofs (HDG for DG methods)

In any case, DG methods have more dofs, but since that ratio is more favorable at high p's, that is yet another reason why DG methods are often used with high polynomial orders.

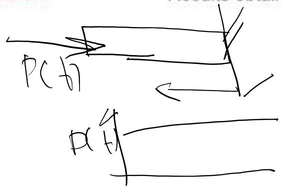
Better solution accuracy and stability for problems with high solution gradients and shocks for DG methods

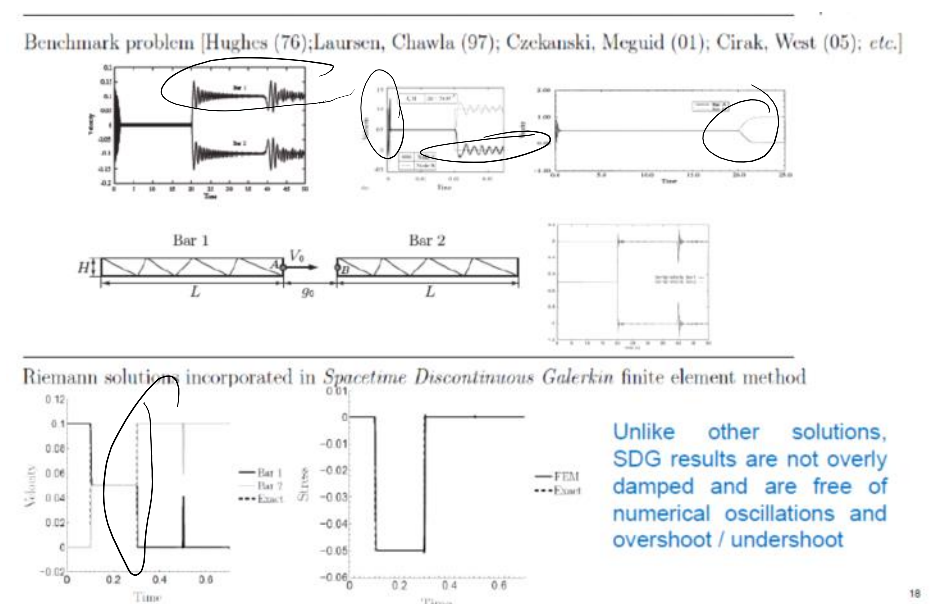


nonlinear hyperbolic PDEs generate shocks even from smooth IC

error in PDE

error in continuity





CFEM

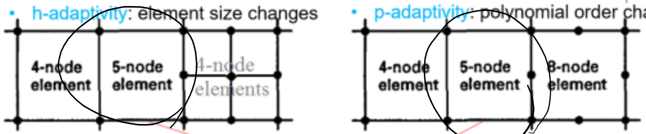
DG

DG

hp-adaptivity

CFEMs:

- h-adaptivity: element size changes
- p-adaptivity: polynomial order changes



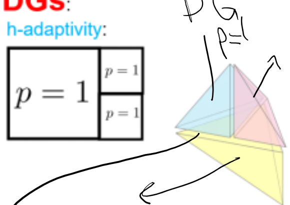
Transition elements

needed in CFEM

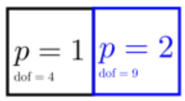
Because of strong continuity of elements transition elements are required

DGs:

- h-adaptivity:
- p-adaptivity:



p=2



Arbitrary change in size and polynomial order as jump conditions are weakly enforced

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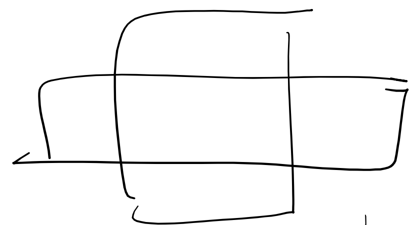
nested nonconforming nbhd



not nested

Summary of CFEMs and DG methods

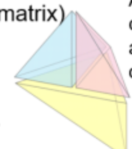
Advantages of DG methods:

1. FEM adaptivity

Resolving shocks and discontinuities for hyperbolic problems
Recovering balance laws at the element level

} no transition elements needed

2. Efficiency /dynamic problems (block diagonal "mass" matrix)
3. Parallel computing (more local communication and use of higher order elements with DG methods)
4. Superior performance for resolving discontinuities (discrete solution space better resembles the continuum solution space)
5. Can recover balance properties at the element level (vs global domain)



Arbitrary change in size and polynomial order

Disadvantages:

- Higher number of degrees of freedom:
  - Particularly important for elliptic problems (global system is solved).
  - Recently hybridizable DG methods (HDG), use Schur decomposition (static condensation) to eliminate elements internal dofs, making DG methods competitive or even better for elliptic problems as well.

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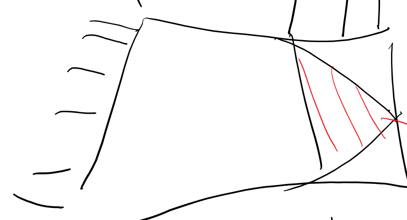
CFEM



claim balance law @ global scale

DG method we have balance laws @ element level

Another comment for HW2:

$$C\dot{u} - K\Delta u = Q$$

$\dot{u} = \frac{\partial u}{\partial t}$, $\Delta u = u_{,xx}$

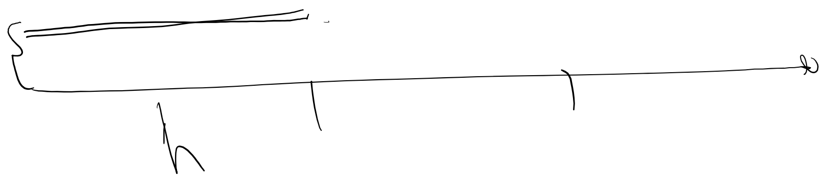
viscosity

of physical unit viscosity diffusion coefficient

$$\nu = \frac{K}{C} \rightarrow [\nu] = \frac{L^2}{T}$$

explicit scheme:

Δt_{max}



$$\underset{T}{\Delta t_{max}} = f(\underset{L}{h}, \underset{\frac{L^2}{T}}{\nu}) \propto \frac{h^2}{\nu}$$

time scale of $\underbrace{\text{parabolic}}_{\text{expl.}}$ solvers scale as $\frac{h^2}{2\nu}$

$$(\Delta t)_{max} = \mathrm{Min}\left(\Delta t^{e}_{max}\right)$$

$$\Delta t^{e}_{max} = C(p, \ldots)\, \frac{h^{e^2}}{2\nu^{e}}$$

For HW2 you'll numerically obtain the stability limit for $p = 1$, 1D problem.

Section 2: Connection of DG methods and Interior Penalty (IP) Methods
and the effect of star values / WR on stability

WR for the thermal problem:

$$R_e(\hat{T}, T) = \int_e \hat{T}\,\underbrace{(\nabla \cdot q - Q)}_{R_i}\,dV + \int_{\partial e} \hat{T}\,\underbrace{(q^*_n - q \cdot n)}_{R_f}\,ds + \varepsilon \int_{\partial e} \hat{q}_n\,\underbrace{(T^* - T)}_{R_u}\,ds = 0$$

$\hat{T}$: weight function

T: solution

int. func. for weight

notation for weight

$$\boxed{\hat{\phi} = \text{weight function of } \phi}$$

$\hat{q} = -k\nabla\hat{T}$

nicer than $q(w)$

$\varepsilon : \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}$ is a number that we choose $\longrightarrow$ results in different methods

weak formulation is obtained by IBP (Gauss theorem) applied on WR

$$\boxed{R(\hat{T}, T) = \int (\nabla \hat{T} q - \hat{T} Q)\, dv + \varepsilon \int_{\partial e} \hat{T} q_n + \int \hat{q}_{on}(T^{*} - T)\, ds = 0}$$