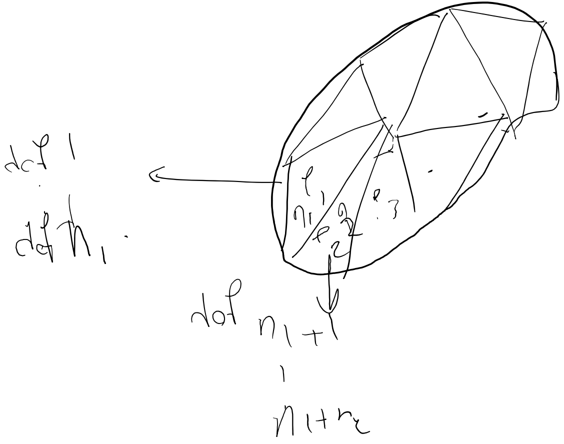
DG2020/02/05

Wednesday, February 5, 2020 11:43 AM

$$\star \quad R(\hat{T},T) = \underbrace{\int_e (\nabla \hat{T} q - \hat{T} Q)\, dv}_{\text{inside}} + \underbrace{\int_{\partial e} \hat{T} q_n^* + \varepsilon \int_{\partial e} \hat{q}_n (T^* - T)\, ds}_{\text{facet}} = 0$$

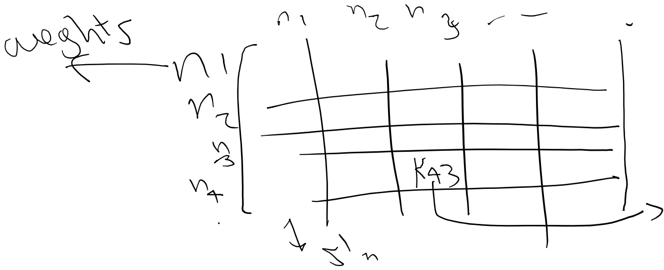
Objectives:

- The effect of epsilon on the form of stiffness matrix
- Formulas for stiffness and residual coming from different face types and interior integrals.
- Relation to interior penalty methods.



$$K \quad a = F$$

$$\Sigma n_i \times \Sigma n_i \qquad \Sigma n_i$$



weights from e_4 sh from e_3

$\hat{T}_{e_4}$ T_{e_3}

We are going to look at * and right it in the form:

$$R(\hat{T},T) = \underbrace{B(\hat{T},T)}_{\text{bilinear form}} - \underbrace{L(\hat{T}, \cancel{T})}_{\text{linear form on } \hat{T}}$$

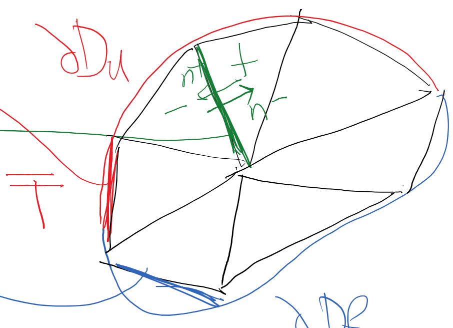
$$\downarrow \qquad\qquad \downarrow$$

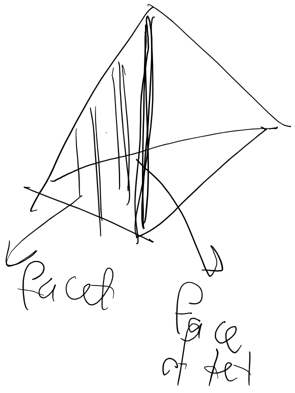
$$Ka \qquad\qquad F$$

We are going to look at this type of representation of residual for different faces types and interior of the element.

Different facet types:

1. Essential BC
2. Natural BC
3. Interior of the domain





1. Essential BC

∂H

face of ∂e

$$R_u(\hat T, T) = \int_{\partial e_u} \hat T q^*_n + \varepsilon \int_{\partial e_u} \hat q \cdot n (T^* - T)\, ds$$

facet part

on ∂e_u: $T^* = T$, $q^*_n = \vec q \cdot n$

$$= \underbrace{\int_{\partial e_u} (\hat T q \cdot n - \varepsilon \hat q \cdot n T)\, ds}_{B} + \underbrace{\varepsilon \int_{\partial e_u} \hat q \cdot n \bar T\, ds}_{L}$$

$$= B_u(\hat T, T) \longleftarrow L_u(\hat T)$$

<u>Essential BC</u>

$$R_u(\hat T, T) = \int_{\partial e_u} (\hat T q \cdot n - \varepsilon \hat q \cdot n T)\, ds$$

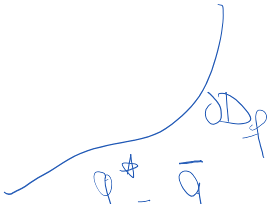
$$L_u(\hat T) = -\varepsilon \int_{\partial e_u} \hat q \cdot n \bar T\, ds$$

①

2. Natural BC:

$$R_f(\hat T, T) = \int_{\partial e_f} \hat T q^*_n\, ds + \varepsilon \int_{\partial e_f} \hat q \cdot n (T^* - T)$$

natural BC



$q^*_n = \bar q$

$T^* = T$

$$= \int_{\partial e_f} \hat T \bar q\, ds$$

$$R_f(\hat T, T) = -L_f(\hat T)$$

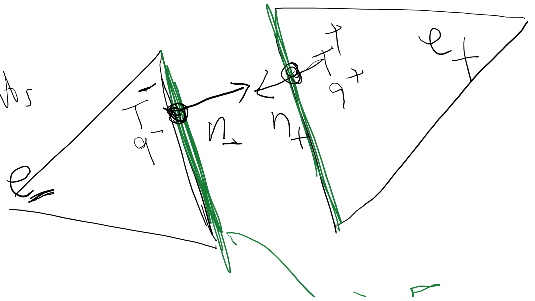
$$L_f(\hat T) = -\int_{\partial e_f} \hat T \bar q\, ds$$

note $B_f(\hat T, T) = 0$

②

3. Interior faces of the domain

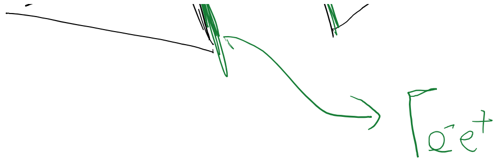
$T^\mp, q^\mp$ are interior traces of the two elements



$$R^{e-}_{\Gamma_{e^-e^+}}(\hat T^-, T) = \int \hat T^- q^*_{n^-} + \varepsilon \int \hat q^- n^- (T^* - T^-)\, ds$$

$$R_{\Gamma_{e^-e^+}}(T, T) = \int_{\Gamma_{e^-e^+}} | \, q^*_{n^-} + \varepsilon \int_{\Gamma_{e^-e^+}} q n^- (T^* - T^-) ds$$

from both elements



$$R^{e^+}_{\Gamma_{e^-e^+}}(\hat{T}^+, T) = \int_{\Gamma_{e^-e^+}} \hat{T}^+ q^{*+}_{n^+} + \varepsilon \int_{\Gamma_{e^-e^+}} \hat{q}^+ n^+ (T^{*+} - T^+) ds$$

We add these together:

$$R_{\Gamma_{e^-e^+}}(\hat{T}, T) = \int_{\Gamma_{e^-e^+}} \left(\hat{T}^- q^{*-}_{n^-} + \hat{T}^+ q^{*+}_{n^+}\right) + \varepsilon \int_{\Gamma_{e^-e^+}} \hat{q}^- \cdot n^- (T^{*-} - T^-) + \hat{q}^+ \cdot n^+ (T^{*+} - T^+) ds \Rightarrow$$

$$T^{*-} = T^{*+} = T^*$$

$$q^{*-}_{n^-} = (\vec{q}^*) \cdot n^- \qquad (q^{*-}_{n^-} = -q^{*+}_{n^+})$$

$$q^{*+}_{n^+} = (\vec{q}^*) \cdot n^+$$

3a

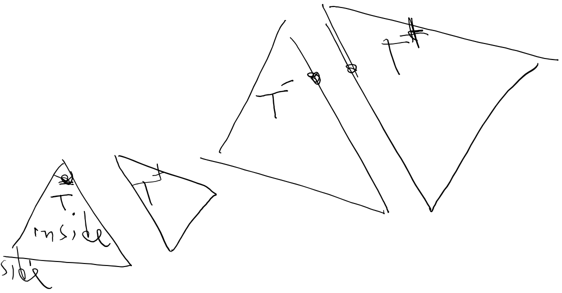
$$\boxed{R_{\Gamma_{e^-e^+}} = \int_{\Gamma_{e^-e^+}} \left(\hat{T}^- q^* \cdot n^- + \hat{T}^+ q^* \cdot n^+\right) + \varepsilon \int_{\Gamma_{e^-e^+}} \left[\hat{q}^- \cdot n^- (T^* - T^-) + \hat{q}^+ \cdot n^+ (T^* - T^+)\right] ds}$$

Notations of average and jumps for scalar and vector quantities:

T: scalar $\qquad \{T\} = \frac{1}{2}(T^- + T^+)$

Jump: different ways (Our group

$$[\phi] = \phi_{outside} - \phi_{inside}$$



_another way that is going to give the same jump value regardless of which side is in, which side is "out"

$$[[T]] = -(T^+ - T^-) \vec{n}^-$$

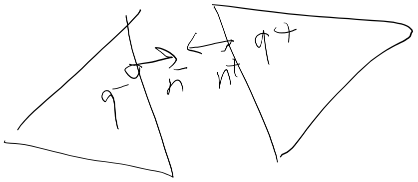
vector

$$= \; T^+(-n^-) + T n^-$$

$$\underline{T^+ n^+ + T^- n^-}$$ side does not matter

Same operations for vectors

$$\{q\} = \frac{1}{2}(q^- + q^+)$$



$$[[q]] = q^- \cdot n^- + q^+ \cdot n^+$$

$$= (q^+ - q^-)\, n^+ = -(q^+ - q^-) \cdot n^-$$

[[q]] should be zero for exact solution (net flux from-out is net flux in in plus side)

scalar T: $\{T\} = \frac{1}{2}(T^- + T^+)$ scalar $\quad [[T]] = T^- n^- + T^+ n^+$ vector

vector q: $\{q\} = \frac{1}{2}(q^- + q^+)$ vector $\quad [[q]] = q^- \cdot n^- + q^+ \cdot n^+$ scalar

3a

$$\boxed{R_{\Gamma_{e^-e^+}} = \underbrace{\int_{\Gamma_{e^-e^+}} \left(\hat{T}^- q^* \cdot n^- + \hat{T}^+ q^* \cdot n^+\right)}_{I_1} + \varepsilon \int_{\Gamma_{e^-e^+}} \underbrace{\left[\hat{q}^- \cdot n^- (T^* - T^-) + \hat{q}^+ \cdot n^+ (T^* - T^+)\right]}_{I_2} ds}$$

$$I_1 = \hat{T}^- q^* \cdot n^- + \hat{T}^+ q^* \cdot n^+ = (T^- n^- + T^+ n^+) \cdot q^* = [[T]] \cdot q^*$$

$$I_2 = \hat{q}^- \cdot n^- T^* + \hat{q}^+ \cdot n^+ T^* - \underbrace{(\hat{q}^- \cdot n^- T^- + \hat{q}^+ n^+ T^+)}_{I_3}$$

$$= [[q]]\, T^* - I_3$$

$$I_3 = \hat{q}^- \cdot n^- T^- + \hat{q}^+ \cdot n^+ T^+ = \hat{q}^- \cdot n^- \left[\left(\frac{T^- + T^+}{2}\right) + \left(\frac{T^- - T^+}{2}\right)\right]$$

$I_3 = q\cdot n \quad + q\cdot n \quad = q\cdot n \left[\left(\quad\frac{}{2}\right)\left(\quad\frac{}{2}\right)\right]$

$+\hat{q}^+\cdot n^+\left[\left(\frac{T^+ + T^-}{2}\right) + \left(\frac{T^+ - T^-}{2}\right)\right]$

$$= \underbrace{\left(\hat{q}^-\cdot n^- + \hat{q}^+\cdot n^+\right)}\left(\frac{T^- + T^+}{2}\right) + \frac{\hat{q}^-}{2}\left(n^-T^- \underset{n^+}{\boxed{-n^-}}T^+\right) + \frac{\hat{q}^+}{2}\left(n^+T^+ \underset{+n^-}{-n^+}T^-\right)$$

$$= [\![\hat{q}]\!]\{\!\{T\}\!\} + \frac{\hat{q}^-}{2}\left(n^-T^- + n^+T^+\right) + \frac{\hat{q}^+}{2}\left(n^+T^+ + n^-T^-\right)$$

$$= [\![\hat{q}]\!]\{\!\{T\}\!\} + \frac{\hat{q}^-}{2}[\![T]\!] + \frac{\hat{q}^+}{2}[\![T]\!]$$

$$= [\![\hat{q}]\!]\{\!\{T\}\!\} + \underbrace{\left(\frac{\hat{q}^- + \hat{q}^+}{2}\right)}_{\{\!\{\hat{q}\}\!\}}[\![T]\!] \longrightarrow I_3 = [\![\hat{q}]\!]\{\!\{T\}\!\} + \{\!\{\hat{q}\}\!\}[\![T]\!]$$

Plug I1, I2, I3 into 3a to get:

③ $$\boxed{R_{\Gamma_{e^-e^+}}(\hat{T}, T) = \varepsilon\int_{\Gamma_{e^-e^+}} [\![\hat{q}]\!]\left(T^* - \{\!\{T\}\!\}\right) - \{\!\{\hat{q}\}\!\}[\![T]\!]\, ds + \int_{\Gamma_{e^-e^+}} [\![\hat{T}]\!]\cdot q^*\, ds}$$

general at this point

Next question $T^* = ?$ $\quad q^* = ?$

Previously, we used averages for parabolic heat conduction. Here, we are solving an elliptic PDE (CT dot term is absent from the PDE). So, we need to use elliptic fluxes.

In Arnold 2000, Arnold 20002 papers, many different formulas are given for star values of elliptic PDEs.

A general form can be given as follows:

④ $$\boxed{\begin{aligned} q^* &= \{\!\{q\}\!\} + \vec{\beta}\,[\![q]\!] + K\alpha[\![T]\!] \\ T^* &= \{\!\{T\}\!\} + \vec{\delta}\cdot[\![T]\!] \end{aligned}}$$

LDG fluxes
"Cockburn & Shu"